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Markov operators and their stability

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1 Introduction

Let (X, ρ) be a Polish space. Let $\mathcal{B}(X)$ be the space of all Borel subsets of X and let $B(X)$ (resp. $C(X)$) be the Banach space of all bounded, measurable (resp. continuous) functions on X equipped with the supremum norm $\|\cdot\|_\infty$. We denote by $\mathrm{Lip}_b(X)$ the space of all bounded Lipschitz continuous functions on X.

By $\mathbf{1}_A$ for $A \in \mathcal{B}(X)$ we denote the indicator of the set A.

By $\mathcal{M}$ and $\mathcal{M}_1$ we denote the family of Borel measures such that $\mu(X) < \infty$ for $\mu \in \mathcal{M}$ and $\mu(X) = 1$ for $\mu \in \mathcal{M}_1$. For any Borel set $A \subset X$ by $\mathcal{M}_1^A$ we will denote the subset of $\mathcal{M}_1$ such that $\mu \in \mathcal{M}_1^A$ if $\mu(X \setminus A) = 0$.

An operator $P : \mathcal{M} \to \mathcal{M}$ will be called a *Markov operator* if it satisfies the following two conditions:

- positive linearity: $P(\lambda_1\mu_1 + \lambda_2\mu_2) = \lambda_1 P\mu_1 + \lambda_2 P\mu_2$ for $\lambda_1, \lambda_2 \geq 0$; $\mu_1, \mu_2 \in \mathcal{M}$;
- preservation of the measure: $P\mu(X) = \mu(X)$ for $\mu \in \mathcal{M}$.

2 Nonnegative functionals

By $C^*(X)$ and $C^*_+(X)$ we denote the adjoint space and the subspace of nonnegative functionals, respectively. Our goal is to write a decomposition formula for $\varphi \in C^*_+(X)$.

We say that $\varphi \in C^*_+(X)$ is a Riesz functional if there exists a Borel measure μ such that

$$\varphi(f) = \int_X f(x)\mu(\mathrm{d}x) \qquad \text{for } f \in C(X).$$

We say that $\varphi_* \in C^*_+(X)$ is a Banach functional if for every Borel measure μ the condition

$$\varphi_*(f) \geq \int_X f(x)\mu(dx) \quad \text{for } f \in C_+(X) \tag{1}$$

implies $\mu(X) = 0$.

We may formulate the following decomposition theorem for $\varphi \in C^*_+(X)$ (see [3]).

Theorem 1. *For every functional $\varphi \in C^*_+(X)$ there exists a Riesz functional φ_0 and a Banach functional φ_* such that*

$$\varphi = \varphi_0 + \varphi_*. \tag{2}$$

This decomposition is unique, namely the measure μ corresponding to φ_0 is given by the formula

$$\mu(K) = \inf\{\varphi(f) : f \in C(X),\ f \geq \mathbf{1}_K\}, \tag{3}$$

where $K \subset X$ is an arbitrary compact set.

Let φ and ψ belong to $C_+^*(X)$. We write $\psi \leq \varphi$ if $\psi(f) \leq \varphi(f)$ for $f \in C_+(X)$. Using formula (3) it is easy to derive the following

Corollary 1. *Let $\varphi \in C_+^*(X)$ be given and let φ_0 be the Riesz functional appearing in decomposition (2). If $\psi \in C_+^*(X)$ is a Riesz functional and $\psi \leq \varphi$, then $\psi \leq \varphi_0$.*

Lemma 1. *Let $\varphi \in C_+^*(X)$ and let $K \subset X$ be a compact set. Then $\varphi_K : C(K) \to \mathbb{R}$ defined by the formula*

$$\varphi_K(f) = \inf\{\varphi(g) : g \in C_+(X), g(x) \geq f(x) \text{ for } x \in K\} \quad \text{for } f \in C_+(K) \tag{4}$$

and

$$\varphi_K(f) = \varphi_K(f^+) - \varphi_K(f^-) \quad \text{for } f \in C(K) \tag{5}$$

belongs to $C_+^*(K)$.

Proof. It is easy to see that $\varphi_K \geq 0$. let $f_1, f_2 \in C_+(K)$. Then

$$\begin{aligned}
\varphi_K(f_1 + f_2) &= \inf\{\varphi(g) : g \in C_+(X), g(x) \geq f_1(x) + f_2(x) \text{ for } x \in K\} \\
&\leq \inf\{\varphi(g_1 + g_2) : g_1, g_2 \in C_+(X), g_1(x) \geq f_1(x) \text{ and } g_2(x) \geq f_2(x) \text{ for } x \in K\} \\
&= \inf\{\varphi(g_1) : g_1 \in C_+(X), g_1(x) \geq f_1(x) \text{ for } x \in K\} \\
&+ \inf\{\varphi(g_2) : g_2 \in C_+(X), , g_2(x) \geq f_2(x) \text{ for } x \in K\} = \varphi_K(f_1) + \varphi_K(f_2).
\end{aligned}$$

To show the inverse inequality take $g \in C_+(X)$ such that $g(x) \geq f_1(x) + f_2(x)$ for $x \in K$. Let $\tilde{f}_1, \tilde{f}_2$ be the extension of f_1, f_2 to X, respectively. Set

$$\tilde{g}_1 = \min(\tilde{f}_1, g) \quad \text{and} \quad \tilde{g}_2 = g - \tilde{g}_1.$$

Obviously $\tilde{g}_1, \tilde{g}_2 \in C_+(X)$. Further, since $g(x) \geq f_1(x) + f_2(x) \geq f_1(x)$ for $x \in K$ we have $\tilde{g}_1(x) \geq f_1(x)$ for $x \in K$. On the other hand, we have $\tilde{g}_1 \leq \tilde{f}_1$, therefore $\tilde{g}_1(x) = f_1(x)$ for $x \in K$. Finally, we obtain that $\tilde{g}_2(x) = g(x) - \tilde{g}_1(x) \geq f_1(x) + f_2(x) - f_1(x) = f_2(x)$ for $x \in K$. Consequently,

$$\begin{aligned}
\varphi_K(f_1 + f_2) &= \inf\{\varphi(g) : g \in C_+(X), g(x) \geq f_1(x) + f_2(x) \text{ for } x \in K\} \\
&\geq \inf\{\varphi(\tilde{g}_1 + \tilde{g}_2) : \tilde{g}_1, \tilde{g}_2 \in C_+(X), \tilde{g}_1(x) \geq f_1(x) \text{ and } \tilde{g}_2(x) \geq f_2(x) \text{ for } x \in K\} \\
&= \inf\{\varphi(\tilde{g}_1) + \varphi(\tilde{g}_2) : \tilde{g}_1, \tilde{g}_2 \in C_+(X), \tilde{g}_1(x) \geq f_1(x) \text{ and } \tilde{g}_2(x) \geq f_2(x) \text{ for } x \in K\} \\
&= \inf\{\varphi(\tilde{g}_1) : \tilde{g}_1 \in C_+(X), \tilde{g}_1(x) \geq f_1(x) \text{ for } x \in K\} \\
&+ \inf\{\varphi(\tilde{g}_2) : \tilde{g}_2 \in C_+(X), \tilde{g}_2(x) \geq f_2(x) \text{ for } x \in K\} = \varphi_K(f_1) + \varphi_K(f_2).
\end{aligned}$$

The equality $\varphi_K(f_1 + f_2) = \varphi_K(f_1) + \varphi_K(f_2)$ for $f_1, f_2 \geq 0$ easily extends to arbitrary $f_1, f_2 \in C(K)$. We also easily check that $\varphi_K(\lambda f) = \lambda\varphi_K(f)$ for any $f \in C(X)$ and $\lambda \in \mathbb{R}$. We are done. □

Our starting point is the following generalization of the Riesz representation theorem.

Theorem 2. *A functional $\varphi \in C^*_+(X)$ is a Riesz functional if and only if the following condition (R) is satisfied: for every $\delta > 0$ there is a compact set $K \subset X$ such that*

$$\varphi(f) \leq \delta \qquad \text{for } f \in C_+(X),\ \ f \leq \mathbf{1}_{X\setminus K}. \tag{6}$$

The proof may be found in [2] .

Now having a functional $\varphi \in C^*_+(X)$ we are going to define a maximal Riesz functional φ_0 such that $\varphi \geq \varphi_0$.

Let $\varepsilon > 0$ be given. We say that a closed set $A \subset X$ belongs to the class $\mathcal{F}_\varepsilon$ if A can be covered by a finite family of balls with radius ε. Further let

$$C_\varepsilon(X) = \{f \in C(X) : \operatorname{supp} f \in \mathcal{F}_\varepsilon\},$$

where $\operatorname{supp} f = \{x \in X : f(x) \neq 0\}$. Evidently $C_\varepsilon(X)$ is a linear space.

If $\varphi \in C^*_+(X)$ is given, we define a family of functionals φ_ε, $\varepsilon \geq 0$, by the formulas

$$\varphi_\varepsilon(f) = \sup\{\varphi(g) : g \in C_\varepsilon(X),\ 0 \leq g \leq f\} \quad \text{for } f \in C_+(X),\, \varepsilon > 0, \tag{7}$$

$$\varphi_0(f) = \lim_{\varepsilon \to 0} \varphi_\varepsilon(f) = \inf\nolimits_{\varepsilon > 0} \varphi_\varepsilon(f) \quad \text{for } f \in C_+(X) \tag{8}$$

and

$$\varphi_\varepsilon(f) = \varphi_\varepsilon(f^+) - \varphi_\varepsilon(f^-) \quad \text{for } f \in C(X),\, \varepsilon \geq 0. \tag{9}$$

It is not difficult to verify that $\varphi_\varepsilon \in C^*_+(X)$ for $\varepsilon \geq 0$.

Theorem 3. *Let $\varphi \in C^*_+(X)$ be given. The functional φ_0 defined by formulas (7)–(9) satisfies the (R) property and consequently there exists a Borel measure μ_0 such that*

$$\varphi_0(f) = \int_X f(x)\mu_0(dx) \qquad \text{for } f \in C(X).$$

Proof. We will break up the proof of Theorem 3 into three steps.

Step I. Fix an $\varepsilon > 0$. Using the functional φ_ε we may define a set function μ_ε by the formula

$$\mu_\varepsilon(A) = \inf\{\varphi_\varepsilon(f) : f \in C(X),\ \ f \geq \mathbf{1}_A\}. \tag{10}$$

It is easy to verify that for an arbitrary finite sequence $E_1, \dots, E_n$ of subsets of X the condition of subadditivity is satisfied, i.e.

$$\mu_\varepsilon(\bigcup_{i=1}^n E_i) \leq \sum_{i=1}^n \mu_\varepsilon(E_i). \tag{11}$$

We claim that μ_ε has also the following property: For every $\varepsilon > 0$ and $\delta > 0$ there exists a set $A \in \mathcal{F}_\varepsilon$ such that

$$\mu_\varepsilon(X \setminus A) \leq \delta.$$

In fact, according to the definition of $\varphi_\varepsilon(\mathbf{1}_X)$ there exists a function $g \in C_\varepsilon$ such that

$$\varphi_\varepsilon(\mathbf{1}_X) - \varphi_\varepsilon(g) \le \delta \quad \text{and} \quad 0 \le g \le \mathbf{1}_X.$$

Define $A = \operatorname{supp} g$ and $f_0 = \mathbf{1}_X - g$. Then $f_0 \ge \mathbf{1}_{X\setminus A}$ and consequently

$$\begin{aligned}\mu_\varepsilon(X \setminus A) &= \inf\{\varphi_\varepsilon(f) : f \ge \mathbf{1}_{X\setminus A}\} \le \varphi_\varepsilon(f_0) \\ &= \varphi_\varepsilon(\mathbf{1}_X) - \varphi_\varepsilon(g) \le \delta.\end{aligned}$$

Step II. Fix $\delta > 0$. We are going to define the set K which appears in (6). Define

$$\eta = \delta(1 + \|\varphi_0\|)^{-1},$$

where $\|\cdot\|$ denotes the norm in $C^*(X)$. Set

$$\varepsilon_n = n^{-1}, \quad \eta_n = \eta \cdot 2^{-n}$$

and write $\mu_n = \mu_{\varepsilon_n}$. According to the Step I for every n there exists a set A_n such that

$$\mu_n(X \setminus A_n) \le \eta_n \quad \text{and} \quad A_n \in \mathcal{F}_{\varepsilon_n}.$$

Define

$$K_n = \bigcap_{i=1}^{n} A_i \quad \text{and} \quad K = \bigcap_{n=1}^{\infty} K_n.$$

Observe that $K_n = K_{n-1} \cap A_n$. Since K_{n-1} is closed and $A_n \in \mathcal{F}_{\varepsilon_n}$, we have also $K_n \in \mathcal{F}_{\varepsilon_n}$. If $K_n \ne \emptyset$ for every n, then according to the Kuratowski theorem K is a compact nonempty set. Moreover for every open set $G \supset K$ there is a number n_0 such that

$$K_n \subset G \qquad \text{for } n \ge n_0. \tag{12}$$

If $K_n = \emptyset$ for some $n = n_0$, then inclusion (12) is trivial.

Step III. Now having δ and K we are going to verify condition (6). Let $f \in C(X)$ be such that $0 \le f \le \mathbf{1}_{X\setminus K}$ and let $\overline{f} = f - \eta$. Evidently the set

$$G = \{x \in X : \overline{f}(x) < 0\}$$

is open and contains K. Thus according to (12) we can find an integer n such that $K_n \subset G$. Since $\overline{f} \le \mathbf{1}_{X\setminus G}$, we have

$$\varphi_{\varepsilon_n}(\overline{f}) \le \inf\{\varphi_{\varepsilon_n}(h) : h \in C(X), h \ge 1_{X\setminus G}\} = \mu_n(X \setminus G). \tag{13}$$

Moreover

$$\begin{aligned}\mu_n(X \setminus G) \le \mu_n(X \setminus K_n) &= \mu_n(\bigcup_{i=1}^{n}(X \setminus A_i)) \\ &\le \sum_{i=1}^{n} \mu_n(X \setminus A_i) \le \sum_{i=1}^{n} \mu_i(X \setminus A_i) \le \sum_{i=1}^{n} \eta_i = \eta.\end{aligned} \tag{14}$$

Finally, using (13) and (14) we obtain

$$\begin{aligned}\varphi_0(f) &= \varphi_0(\overline{f}) + \eta\varphi_0(\mathbf{1}_X) \le \varphi_{\varepsilon_n}(\overline{f}) + \eta\|\varphi_0\| \\ &\le \mu_n(X \setminus G) + \eta\|\varphi_0\| \le \eta(1 + \|\varphi_0\|) = \delta,\end{aligned}$$

which completes the proof. □

Proof of Theorem 1. Define the functional φ_0 by formulas (7)–(9) and write $\varphi_* = \varphi - \varphi_0$. We are going to show that φ_* is a Banach functional. Assume that for some measure μ condition (1) is satisfied. From the definition of φ_ε it follows that

$$\lim_{\varepsilon\to 0} \|\varphi_\varepsilon - \varphi_0\| = \lim_{\varepsilon\to 0} \varphi_\varepsilon(\mathbf{1}_X) - \varphi_0(\mathbf{1}_X) = 0.$$

Fix a compact set $K \subset X$ and a number $\eta > 0$. Choose $\varepsilon > 0$ such that $\|\varphi_\varepsilon - \varphi_0\| < \eta$. Since $\varphi_* = \varphi - \varphi_0$ from (1) it follows that

$$\varphi(f) - \varphi_\varepsilon(f) \ge \int_X f(x)\mu(dx) - \eta \quad \text{for } f \in C(X),\ 0 \le f \le 1.$$

Since K is a compact set there exists a continuous function f_ε such that $\mathbf{1}_K \le f_\varepsilon \le 1$ and $f_\varepsilon \in C_\varepsilon(X)$. In this case $\varphi_\varepsilon(f_\varepsilon) = \varphi(f_\varepsilon)$ and consequently

$$0 \ge \int_X f_\varepsilon(x)\mu(dx) - \eta \ge \mu(K) - \eta.$$

Since η was an arbitrary positive number, this gives $\mu(K) = 0$. The measure μ is equal to zero on compact sets and according to the Ulam theorem $\mu \equiv 0$.

In order to verify (6) again fix a compact set $K \subset X$ and a number $\eta > 0$. As before choose $\varepsilon > 0$ such that $\|\varphi_\varepsilon - \varphi_0\| < \eta$ and define the corresponding function f_ε. We have

$$\varphi(f_\varepsilon) = \varphi_\varepsilon(f_\varepsilon) \le \varphi_0(f_\varepsilon) - \eta = \int_X f_\varepsilon(x)\mu(dx) - \eta.$$

Since $\varepsilon > 0$ is arbitrary, this implies

$$\inf\{\varphi(f) : f \in C(X),\ f \ge \mathbf{1}_K\} \le \mu(K) - \eta. \tag{15}$$

On the other hand it is obvious that

$$\varphi(f) \ge \varphi_0(f) \ge \mu(K) \quad \text{for } f \in C(X),\, f \ge \mathbf{1}_K. \tag{16}$$

Since the positive number η was arbitrary, conditions (15) and (16) imply (6). Thus according to the Ulam theorem (see [1]) the measure μ is uniquely defined. This in turn implies that decomposition (2) is unique. □

3 Invariant measures and stability

We start with the definition of concentrating Markov operators.

Definition 1. *An operator P is called concentrating if for every $\varepsilon > 0$ there exist $C \in \mathcal{B}(X)$ with* $\operatorname{diam} C \le \varepsilon$ *and $\alpha > 0$ such that*

$$\liminf_{n\to\infty} P^n\mu(C) > \alpha \qquad \text{for } \mu \in \mathcal{M}_1. \tag{17}$$

By $\mathcal{C}_\varepsilon$, $\varepsilon > 0$, we will denote the family of all Borel sets C for which there exists a finite set $\{x_1, \dots, x_n\} \subset X$ such that $C \subset \bigcup_{i=1}^n B(x_i, \varepsilon)$. We set

$$\mathcal{N}(C, \varepsilon) := \{x \in X : d(x, C) = \inf\{\rho(x, y) : y \in C\} < \varepsilon\}$$

for any subset $C \subset X$.

In the space $\mathcal{M}_{sig} := \mathcal{M} - \mathcal{M}$ we introduce the *Wasserstein norm*

$$\|\mu - \nu\|_W = \sup\left\{\left|\int_X f \mathrm{d}(\mu - \nu)\right| : \|f\|_\infty \le 1, \operatorname{Lip} f \le 1\right\}$$

for $\mu, \nu \in \mathcal{M}$.

We have:

- Convergence in the Wasserstein norm is equivalent to the weak convergence, i.e.

$$\lim_{n\to\infty} \|\mu_n - \mu\| = 0 \ \text{ iff } \ \lim_{n\to\infty} \int_X f(x)\mu_n(\mathrm{d}x) = \int_X f(x)\mu(\mathrm{d}x)$$

  for any $f \in C(X)$.

- The space $(\mathcal{M}_1, \|\cdot\|)$ is a complete and separable space (a Polish space).

An operator P is called *nonexpansive* if

$$\|P\mu_1 - P\mu_2\|_W \le \|\mu_1 - \mu_2\|_W \quad \text{for any } \mu_1, \mu_2 \in \mathcal{M}_1.$$

Lemma 2. *If $\|\mu_1 - \mu_2\|_W \le \varepsilon^2$ for $\mu_1, \mu_2 \in \mathcal{M}_1$ and some $\varepsilon > 0$, then*

$$\mu_1(\mathcal{N}(C, \varepsilon)) \ge \mu_2(C) - \varepsilon \qquad \text{for every Borel set } C \subset X.$$

The proof is left as an exercise for the reader.

We are in a position to prove a very simple criterion for the existence of an invariant measure.

Proposition 1. *Let P be a nonexpansive Markov operator. Suppose that there exists a measure $\mu \in \mathcal{M}_1$ such that for every $\varepsilon > 0$ there is a set $C_\varepsilon \in \mathcal{C}_\varepsilon$ satisfying $P^n\mu(C_\varepsilon) \ge 1 - \varepsilon$ for $n \in \mathbb{N}$. Then P has an invariant distribution.*

Proof. Let $\mu \in \mathcal{M}_1$ be as in the statement of the lemma. Fix $\varepsilon > 0$. Let $C_{\varepsilon/2^k} \in \mathcal{C}_{\varepsilon/2^k}$, $k \geq 1$, be such that

$$P^n\mu(C_{\varepsilon/2^k}) \geq 1 - \frac{\varepsilon}{2^k} \qquad \text{for } n \in \mathbb{N}.$$

Without loss of generality we may assume that C_ε are closed. Define $K = \bigcap_{k=1}^{\infty} C_{\varepsilon/2^k}$ and observe that K is compact. Further, we have

$$\begin{aligned} P^n\mu(X \setminus K) =& P^n\mu(X \setminus \bigcap_{k=1}^{\infty} C_{\varepsilon/2^k}) = P^n\mu(\bigcup_{k=1}^{\infty}(X \setminus C_{\varepsilon/2^k})) \\ &\leq \sum_{k=1}^{\infty} P^n\mu(X \setminus C_{\varepsilon/2^k}) \leq \sum_{k=1}^{\infty} \frac{\varepsilon}{2^k} = \varepsilon \qquad \text{for } n \in \mathbb{N}. \end{aligned} \tag{18}$$

Set

$$\bar{\mu}_n = \frac{\mu + P\mu + \cdots + P^{n-1}\mu}{n} \qquad \text{for } n \in \mathbb{N}. \tag{19}$$

By (18) the sequence $(\bar{\mu}_n)_{n\geq 1}$ is tight. From the Prokhorov theorem (see [1]) it folows that there exists a subsequence of integers $(m_n)_{n\geq 1}$ and a measure $\bar{\mu} \in \mathcal{M}_1$ such that $\bar{\mu}_{m_n} \to \bar{\mu}$ as $n \to \infty$. Since P is nonexpansive, we obtain $P\bar{\mu}_{m_n} \to P\bar{\mu}$. Finally, from (19) it follows that $\|P\bar{\mu}_{m_n} - \bar{\mu}_{m_n}\|_W \to 0$ as $n \to \infty$ and consequently $P\bar{\mu} = \bar{\mu}$. □

Now we may establish the main result of this section (see [4]).

Theorem 4. *Let P be a nonexpansive and concentrating Markov operator. Then P is asymptotically stable.*

Proof. The proof will be given in two steps. First we will show that P admits an invariant measure. Then we prove that P is asymptotically stable.

Step I. By Proposition 1 it is enough to show that for every $\varepsilon > 0$ and $\mu \in \mathcal{M}_1$ there exists $C_\varepsilon \in \mathcal{C}_\varepsilon$ such that $P^n\mu(C_\varepsilon) \geq 1 - \varepsilon$ for $n \in \mathbb{N}$. To do this fix $\varepsilon > 0$ and $\mu \in \mathcal{M}_1$. Set $\bar{\varepsilon} = \varepsilon^2/16$. Let $\alpha > 0$ and $A \subset X$ be chosen for $\bar{\varepsilon}$ in Definition 1. If $P^n\mu(A) \geq \alpha/2$, then

$$P^n\mu \geq (\alpha/2)\nu_n, \tag{20}$$

, where $\nu_n \in \mathcal{M}_1^A$ is of the form

$$\nu_n(B) = \frac{P^n\mu(B \cap A)}{P^n\mu(A)}.$$

Define

$$\delta = \sup\{\gamma \geq 0 : \exists \quad C_{\varepsilon/2} \in \mathcal{C}_{\varepsilon/2} \quad \liminf_{n\to\infty} P^n\mu(C_{\varepsilon/2}) \geq \gamma\}.$$

Let $\gamma \geq 0$ and $C_{\varepsilon/2} \in \mathcal{C}_{\varepsilon/2}$ be such that $0 \leq \delta - \gamma < \alpha\varepsilon/8$ and

$$\liminf_{n\to\infty} P^n\mu(C_{\varepsilon/2}) \geq \gamma.$$

We are now in a position to show that

$$P^n\nu(\mathcal{N}(C_{\varepsilon/2},\varepsilon/2)) \geq 1-\frac{\varepsilon}{2} \quad \text{for } n\in\mathbb{N} \text{ and } \nu\in\mathcal{M}_1^A. \tag{21}$$

On the contrary, suppose that for some $\nu_0\in\mathcal{M}_1^A$ and $n_0\in\mathbb{N}$

$$P^{n_0}\nu_0(\mathcal{N}(C_{\varepsilon/2},\varepsilon/2)) < 1-\frac{\varepsilon}{2}.$$

By the Ulam theorem (see [1]), this implies that there exists a compact set $K\subset X\setminus \mathcal{N}(C_{\varepsilon/2},\varepsilon/2)$ such that $P^{n_0}\nu_0(K)\geq\varepsilon/2$. Since P is nonexpansive, we have

$$\|P^{n_0}\nu_0 - P^{n_0}\nu\| \leq \|\nu_0-\nu\| \leq \operatorname{diam} A \leq \frac{\varepsilon^2}{16}$$

for $\nu\in\mathcal{M}_1^A$. Now Lemma 2 shows that $P^{n_0}\nu(\mathcal{N}(K,\varepsilon/4))\geq\varepsilon/4$. Putting $B=\mathcal{N}(K,\varepsilon/4)$ we obtain that $B\in\mathcal{C}_{\varepsilon/2}$ and consequently $B\cup C_{\varepsilon/2}\in\mathcal{C}_{\varepsilon/2}$. Applying (20) we see that

$$P^{n+n_0}\mu(B)\geq(\alpha/2)P^{n_0}\nu_n(B)\geq\alpha\varepsilon/8$$

for every sufficiently large n. Since $B\cap C_{\varepsilon/2}=\emptyset$, we see that

$$\begin{aligned}\liminf_{n\to\infty} P^n\mu(B\cup C_{\varepsilon/2}) &\geq \liminf_{n\to\infty} P^n\mu(B) + \liminf_{n\to\infty} P^n\mu(C_{\varepsilon/2})\\ &\geq \gamma+\alpha\varepsilon/8>\delta,\end{aligned}$$

which contradicts the definition of δ. Thus (21) holds. Put $C=\mathcal{N}(C_{\varepsilon/2},\varepsilon/2)$ and note that $C\in\mathcal{C}_\varepsilon$.

We will define by an induction argument a sequence of integers $(n_k)_{k\geq1}$ and two sequences of distributions $(\mu_k)_{k\geq1}$, $(\nu_k)_{k\geq1}$. If $k=0$ we define $n_0=0$ and $\mu_0=\nu_0=\mu$. If $k\geq1$ is fixed and n_{k-1}, μ_{k-1}, ν_{k-1} are given we choose, according to (17), an integer n_k such that

$$P^{n_k}\mu_{k-1}(A)\geq\frac{\alpha}{2} \tag{22}$$

and we define

$$\nu_k(B)=\frac{P^{n_k}\mu_{k-1}(B\cap A)}{P^{n_k}\mu_{k-1}(A)}, \tag{23}$$

$$\mu_k(B)=\frac{1}{1-\alpha/2}\{P^{n_k}\mu_{k-1}(B)-\frac{\alpha}{2}\nu_k(B)\}. \tag{24}$$

Observe that $\nu_k\in\mathcal{M}_1^A$. Using equation (24) it is easy to verify by induction that

$$\begin{aligned}P^{n_1+\ldots+n_k}\mu &= \frac{\alpha}{2}P^{n_2+\ldots+n_k}\nu_1+\frac{\alpha}{2}\left(1-\frac{\alpha}{2}\right)P^{n_3+\ldots+n_k}\nu_2\\ &\quad+\ldots+\frac{\alpha}{2}\left(1-\frac{\alpha}{2}\right)^{k-1}\nu_k+\left(1-\frac{\alpha}{2}\right)^k\mu_k.\end{aligned} \tag{25}$$

Let $k \in \mathbb{N}$ be such that

$$\left(1-(1-\alpha/2)^k\right)(1-\varepsilon/2) \geq 1-\varepsilon.$$

Since $\nu_i \in \mathcal{M}_1^A$, $i=1,2,\ldots,k$, (22) and (23) hold, we have

$$\begin{aligned} P^n\mu(C) &\geq \frac{\alpha}{2}P^{n-n_1}\nu_1(C) + \frac{\alpha}{2}\left(1-\frac{\alpha}{2}\right)P^{n-n_1-n_2}\nu_2(C) \\ &+ \ldots + \frac{\alpha}{2}\left(1-\frac{\alpha}{2}\right)^{k-1}P^{n-n_1-\ldots-n_k}\nu_k(C) \\ &\geq \left(1-(1-\alpha/2)^k\right)(1-\varepsilon/2) \geq 1-\varepsilon \end{aligned}$$

for $n \geq n_1+\ldots+n_k$. By the Ulam theorem, we can find a compact set $K \subset X$ such that

$$P^n\mu(K\cup C) \geq 1-\varepsilon \qquad \text{for } n \in \mathbb{N}.$$

Since $K \cup C \in \mathcal{C}_\varepsilon$, Proposition 1 shows that P has an invariant distribution.

Step II. We turn our attention to stability. To do this fix $\mu_1,\mu_2 \in \mathcal{M}_1$ and $\varepsilon > 0$. According to the concentrating property of P we may choose $A \in \mathcal{B}_b(X)$ with $\operatorname{diam} A \leq \varepsilon$ and $\alpha > 0$ such that (17) holds. As in Step I we define by induction a sequence of integers $(n_k)_{k\geq 1}$ and four sequences of distributions $(\mu_i^k)_{k\geq 1}$, $(\nu_i^k)_{k\geq 1}$, $i=1,2$, such that (23), (24) hold. Then, by (25), we have

$$\begin{aligned} P^{n_1+\ldots+n_k}\mu_i &= \alpha P^{n_2+\ldots+n_k}\nu_i^1 + \alpha\,(1-\alpha)\,P^{n_3+\ldots+n_k}\nu_i^2 \\ &+ \ldots + \alpha\,(1-\alpha)^{k-1}\,\nu_i^k + (1-\alpha)^k\,\mu_i^k \qquad \text{for } k \in \mathbb{N} \text{ and } i=1,2. \end{aligned}$$

Since P is nonexpansive and $\|\nu_1^k - \nu_2^k\|_W \leq \operatorname{diam} A \leq \varepsilon$, we have

$$\begin{aligned} \|P^{n_1+\ldots+n_k}\mu_1 - P^{n_1+\ldots+n_k}\mu_2\|_W &\leq \alpha\|\nu_1^1-\nu_2^1\|_W + \alpha(1-\alpha)\|\nu_1^2-\nu_2^2\|_W \\ &+ \ldots + \alpha(1-\alpha)^{k-1}\|\nu_1^k-\nu_2^k\|_W \\ &+ (1-\alpha)^k\|\mu_1^k-\mu_2^k\|_W \leq \varepsilon + 2(1-\alpha)^k. \end{aligned}$$

Since $\varepsilon > 0$, $k \in \mathbb{N}$, $\mu_1,\mu_2 \in \mathcal{M}_1$ were arbitrary and P was nonexpansive, this finishes the proof. □

We say that a Markov operator P satisfies the *e–property* at $z \in X$ if it is a Feller operator with dual U and for any bounded Lipschitz function f we have

$$\lim_{\rho(x,z)\to 0} \sup\{|U^n f(x) - U^n f(z)| : n \geq 1\} = 0.$$

The following theorem was proved in [5].

Theorem 5. *Let $P : \mathcal{M} \to \mathcal{M}$ be a Markov operator. Assume that there exists $z \in X$ such that for every $\varepsilon > 0$ we have*

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} P^i \delta_x(B(z,\varepsilon)) > 0 \tag{26}$$

for some measure $x \in X$. If P satisfies the e–property at z, then P admits an invariant measure.

Proof. To finish the proof it suffices to show that for every $\varepsilon > 0$ there exists a compact set $K \subset X$ such that

$$\liminf_{n\to\infty} P^n \delta_z(K^\varepsilon) \geq 1 - \varepsilon, \tag{27}$$

where $K^\varepsilon = \{x \in X : \inf_{y\in K} \rho(x,y) < \varepsilon\}$, by Proposition 1.

Assume, contrary to our claim, that (27) does not hold for some $\varepsilon > 0$. By Ulam's theorem (see [1]) there exist a sequence of compact sets $(K_i)_{i\geq 1}$ and a sequence of integers $(q_i)_{i\geq 1}$ satisfying

$$P^{q_i}\delta_z(K_i) > \varepsilon$$

and

$$\min\{\rho(x,y) : x \in K_i, y \in K_j\} \geq \varepsilon/3 \quad \text{for } i,j \in \mathbb{N},\ i \neq j. \tag{28}$$

We first show that for every open set O containing z and $j \in \mathbb{N}$ there exist $y \in O$ and $i \geq j$ such that

$$P^{q_i}\delta_y(K_i^{\varepsilon/12}) < \varepsilon/2.$$

On the contrary, suppose that there exist an open set O' containing z and $i_0 \in \mathbb{N}$ such that

$$\inf\{P^{q_i}\delta_y(K_i^{\varepsilon/12}) : y \in O',\ \ i \geq i_0\} \geq \varepsilon/2. \tag{29}$$

Let $x \in X$ be such that condition (26) holds with O' in place of O. Let $\alpha > 0$ be such that

$$\limsup_{n\to\infty} \left(\frac{1}{n} \sum_{i=1}^{n} P^i \delta_x(O') \right) > \alpha.$$

By (28), (29) and the Chapman–Kolmogorov equation we obtain

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} P^i \delta_x \left(\bigcup_{j=i_0}^{N} K_j^{\varepsilon/12} \right) > (N - i_0)\alpha\varepsilon/2$$

for every $N \geq i_0$, which is impossible.

Now we will define by induction a sequence of Lipschitz continuous functions $(\tilde{f}_n)_{n\geq 1}$, a sequence of points $(y_n)_{n\geq 1}$, $y_n \to z$ as $n \to \infty$, and three increasing sequences of integers $(i_n)_{n\geq 1}$, $(k_n)_{n\geq 1}$, $(m_n)_{n\geq 1}$, $i_{n+1} > k_n > i_n$ for $n \in \mathbb{N}$, such that

$$\tilde{f}_{n|K_{i_n}} = 1 \quad \text{and} \quad 0 \leq \tilde{f}_n \leq \mathbf{1}_{K_{i_n}^{\varepsilon/12}}, \tag{30}$$

$$|U^{m_n}(\sum_{i=1}^{n}\tilde{f}_i)(z) - U^{m_n}(\sum_{i=1}^{n}\tilde{f}_i)(y_n)| > \varepsilon/4 \tag{31}$$

and

$$P^{m_n}\delta_u\left(\bigcup_{i=k_n}^{\infty} K_i^{\varepsilon/12}\right) < \varepsilon/16 \quad \text{for } u = z,\ y_n, \quad n \in \mathbb{N}. \tag{32}$$

Let $n = 1$. From what has already been proved, it follows that there exist $y_1 \in B(z,1)$ and $i_1 \in \mathbb{N}$ such that

$$P^{q_{i_1}}\delta_{y_1}(K_{i_1}^{\varepsilon/12}) < \varepsilon/2.$$

Set $m_1 = q_{i_1}$ and let $k_1 > i_1$ be such that

$$P^{m_1}\delta_u\left(\bigcup_{i=k_1}^{\infty} K_i^{\varepsilon/12}\right) < \varepsilon/16 \quad \text{for } u = z,\ y_1.$$

Let $\tilde{f}_1$ be an arbitrary Lipschitz function satisfying

$$\tilde{f}_{1|K_{i_1}} = 1 \quad \text{and} \quad 0 \le \tilde{f}_1 \le \mathbf{1}_{K_{i_1}^{\varepsilon/12}}. \tag{33}$$

Thus

$$|U^{m_1}\tilde{f}_1(z) - U^{m_1}\tilde{f}_1(y_1)| \ge P^{m_1}(z, K_{i_1}) - P^{m_1}(y_1, K_{i_1}^{\varepsilon/12}) > \varepsilon/2.$$

If $n \ge 2$ is fixed and $\tilde{f}_1, \ldots, \tilde{f}_{n-1};\ y_1, \ldots, y_{n-1};\ i_1, \ldots, i_{n-1};\ k_1, \ldots, k_{n-1};\ m_1, \ldots, m_{n-1}$ are given we choose $\sigma < n^{-1}$ such that

$$\left|U^m\left(\sum_{i=1}^{n-1}\tilde{f}_i\right)(z) - U^m\left(\sum_{i=1}^{n-1}\tilde{f}_i\right)(y)\right| < \varepsilon/8 \tag{34}$$

for $y \in B(z,\sigma)$ and $m \in \mathbb{N}$. Similarly as in the first part, we may choose $y_n \in B(z,\sigma)$ and $i_n > k_{n-1}$ such that

$$P^{q_{i_n}}\delta_{y_n}(K_{i_n}^{\varepsilon/12}) < \varepsilon/2.$$

Set $m_n = q_{i_n}$ and let $\tilde{f}_n$ be an arbitrary Lipschitz function satisfying condition (30). Let $k_n > i_n$ be such that

$$P^{m_n}\delta_u\left(\bigcup_{i=k_n}^{\infty} K_i^{\varepsilon/12}\right) < \varepsilon/16 \quad \text{for } u = z,\ y_n.$$

From this, (34) and the definition of $\tilde{f}_n$ we have

$$\begin{aligned}
&\left|U^{m_n}\left(\sum_{i=1}^{n}\tilde{f}_i\right)(z) - U^{m_n}\left(\sum_{i=1}^{n}\tilde{f}_i\right)(y_n)\right| \\
&\ge \left|U^{m_n}\tilde{f}_n(z) - U^{m_n}\tilde{f}_n(y_n)\right| \\
&- \left|U^{m_n}\left(\sum_{i=1}^{n-1}\tilde{f}_i\right)(z) - U^{m_n}\left(\sum_{i=1}^{n-1}\tilde{f}_n\right)(y_n)\right| \\
&> \varepsilon/2 - \varepsilon/8 > \varepsilon/4.
\end{aligned}$$

We now define $f = \sum_{i=1}^{\infty} \tilde{f}_i$. By (28) and (30) f is a Lipschitz continuous function and $\|f\|_\infty \le 1$. Finally, by (31) and (32) we have

$$|U^{m_n} f(z) - U^{m_n} f(y_n)| > \varepsilon/8 \qquad \text{for } n \in \mathbb{N}$$

and since $y_n \to z$ as $n \to \infty$, this contradicts the assumption that $\{U^n f \colon n \in \mathbb{N}\}$ is equicontinuous in z. □

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