KAPITAŁ LUDZKI
NARODOWA STRATEGIA SPÓJNOŚCI

UNIWERSYTET GDAŃSKI

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FUNDUSZ SPOŁECZNY

Shell models in hydrodynamics

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PWP Interdysciplinary Doctoral Studies in Mathematical Modeling
UDA-POKL.04.01.01-00-026/13-00
Projekt jest współfinansowany przez Unię Europejską w ramach Europejskiego Funduszu Społecznego

These lectures are aimed at proving stability of some shell models; the GOY and Sabra model. These are very popular examples of simplified phenomenological models of turbulence. Although they are not based on conservations laws, they capture some essential statistical properties and features of turbulent flows, like the energy and the enstrophy cascade and the power law decay of the structure functions in some range of wave numbers, the inertial range. The readers are referred to [26], [1], [4], [10] and [11] and the references therein and to [5], [8] and [3] for some rigorous results.

In the studied models some noise is incorporated to them by only finitely many modes. To be able to show stability of models under consideration we have to develope a new criterion for asymptotic stability of Markov processes. It is possible that the simillar results may be obtained using coupling methods (see for instance [12, 15, 2]). However we provide this criterion for its simplicity. It is also interesting in itself. Moreover, we strongly believe that the criterion may be useful in the theory of SPDE's in particular with Lévy noise.

We are concerned with processes satisfying the so-called e–property (see [15, 17]). The e–property is a generalization of nonexpansiveness and it allows us to overcome some restrictiveness of the strong Feller property when applied to some SPDE's (see for instance [14]). Recall that a process is called nonexpansive if the Markov semigroup (acting on measures) corresponding to the process is nonexpansive with respect to the Wasserstein metric. A. Lasota and J. Yorke (see [18]) found a very elegant and concise condition assuring the existence and uniqueness of an invariant measure for nonexpansive Markov chains. It says that the considered chain is concentrating at some point, i.e. the chain remains in arbitrary small neighborhoods of some fixed point with positive probability independent of an initial point. The proof based on the lower bound technique was developed in [18, 16]. The result mentioned above was proved in locally compact spaces, subsequently the proof was also given in Polish spaces [23] and finally the result was extended to Markov processes [25]. Here, we prove that if a Markov process is averagely bounded and with positive probability enters into any neighborhood of a fixed point, then this process is also asymptotically stable. In particular, it admits a unique invariant measure.

The lectures are split into two parts. The first part is devoted to the theory of genaral Markov semigroups. In the second part we apply general results to Markov semigroups corresponding to the shell models. At the end of our lectures we provide literature which allows the readers to extend and improve the knowledge on topics studied here.

1 Criterion on Stability

Let (X, ρ) be a Polish space. By $B_b(X)$ we denote the space of all bounded Borel–measurable functions equipped with the supremum norm. Let $(P_t)_{t\geq 0}$ be the *Markovian semigroup* defined on $B_b(X)$. For each $t \geq 0$ we have $P_t\mathbf{1}_X = \mathbf{1}_X$ and $P_t\psi \geq 0$ if $\psi \geq 0$. Throughout this paper we shall assume that the semigroup is *Feller*, i.e. $P_t(C_b(X)) \subset C_b(X)$ for all $t > 0$. Here and in the sequel $C_b(X)$ is the subspace of all bounded continuous functions with the supremum norm $\|\cdot\|_\infty$. By $L_b(X)$ we will denote the subspace of all bounded Lipschitz functions. We shall also assume that $(P_t)_{t\geq 0}$ is *stochastically continuous*, which implies that $\lim_{t\to 0^+} P_t\psi(x) = \psi(x)$ for all $x \in X$ and $\psi \in C_b(X)$.

Let $\mathcal{M}_1$ stand for the space of all Borel probability measures on X. Denote by $\mathcal{M}_1^W$, $W \subset X$, the subspace of all Borel probability measures supported in W, i.e. $\{x \in X : \mu(B(x,r)) > 0 \text{ for any } r > 0\} \subset W$, where $B(x,r)$ denotes the ball in X with center at x and radius r. For $\varphi \in B_b(X)$ and $\mu \in \mathcal{M}_1$ we will use the notation $\langle \varphi, \mu\rangle = \int_X \varphi(x)\mu(\mathrm{d}x)$. Recall that the *total variation norm* of a finite signed measure $\mu \in \mathcal{M}_\infty - \mathcal{M}_\infty$ is given by $\|\mu\|_{TV} = \mu^+(X) + \mu^-(X)$, where $\mu = \mu^+ - \mu^-$ is the Jordan decomposition of μ.

We say that $\mu_* \in \mathcal{M}_1$ is *invariant* for $(P_t)_{t\geq 0}$ if $\langle P_t\psi, \mu_*\rangle = \langle \psi, \mu_*\rangle$ for every $\psi \in B_b(X)$ and $t \geq 0$. Alternatively, we can say that $P_t^*\mu_* = \mu_*$ for all $t \geq 0$, where $(P_t^*)_{t\geq 0}$ denotes the semigroup dual to $(P_t)_{t\geq 0}$, i.e. for a given Borel measure μ, Borel subset A of X, and $t \geq 0$ we set

$$P_t^*\mu(A) := \langle P_t\mathbf{1}_A, \mu\rangle.$$

A semigroup $(P_t)_{t\geq 0}$ is said to be *asymptotically stable* if there exists an invariant measure $\mu_* \in \mathcal{M}_1$ such that $P_t^*\mu$ converges weakly to μ_* as $t \to +\infty$ for every $\mu \in \mathcal{M}_1$. Obviously μ_* is unique.

Definition 1.1. *We say that a semigroup $(P_t)_{t\geq 0}$ has* the e–property *if the family of functions $(P_t\psi)_{t\geq 0}$ is equicontinuous at every point x of X for any bounded and Lipschitz function ψ, i.e.*

$$\forall\, \psi \in L_b(X),\, x \in X,\, \varepsilon > 0\, \exists\, \delta > 0\, \forall\, z \in B(x,\delta),\, t \geq 0 :\, |P_t\psi(x) - P_t\psi(z)| < \varepsilon.$$

Remark. One can show (see [14]) that to obtain the e–property in the case when X is a Hilbert space, it is enough to verify the above condition for every function with bounded Fréchet derivative.

Definition 1.2. *A semigroup $(P_t)_{t\geq 0}$ is called* averagely bounded *if for any $\varepsilon > 0$ and bounded set $A \subset X$ there is a bounded Borel set $B \subset X$ such that*

$$\limsup_{T\to\infty} \frac{1}{T}\int_0^T P_s^*\mu(B)\mathrm{d}s > 1 - \varepsilon \qquad \text{for } \mu \in \mathcal{M}_1^A.$$

Definition 1.3. *A semigroup $(P_t)_{t\geq 0}$ is* concentrating *at z if for any $\varepsilon > 0$ and bounded set $A \subset X$ there exists $\alpha > 0$ such that for any two measures $\mu_1, \mu_2 \in \mathcal{M}_1^A$ holds*

$$P_t^*\mu_i(B(z,\varepsilon)) \geq \alpha \text{ for } i = 1,2 \text{ and some } t > 0.$$

Proposition 1. *Let $(P_t)_{t\geq 0}$ be averagely bounded and concentrating at some $z \in X$. If $(P_t)_{t\geq 0}$ satisfies the e–property, then for any $\varphi \in L_b(X)$ and $\mu_1, \mu_2 \in \mathcal{M}_1$ we have*

$$\lim_{t\to\infty} |\langle \varphi, P_t^*\mu_1\rangle - \langle \varphi, P_t^*\mu_2\rangle| = 0. \tag{1.1}$$

Proof. First observe that to finish the proof it is enough to show that condition (1.1) holds for arbitrary Borel probability measures with bounded support. Indeed, the set of all probability measures with bounded support is dense in the space $(\mathcal{M}_1, \|\cdot\|_{TV})$. Moreover, P_t^*, $t \geq 0$, is nonexpansive with respect to the total variation norm.

Fix $\varphi \in L_b(X)$, $x_0 \in X$ and $\varepsilon \in (0, 1/2)$. Let $\mu_1, \mu_2 \in \mathcal{M}_1^{B(x_0,r_0)}$ for some $r_0 > 0$. Choose $\delta > 0$ such that

$$\sup_{t\geq 0} |P_t\varphi(x) - P_t\varphi(y)| < \varepsilon/2 \tag{1.2}$$

for $x, y \in B(z, \delta)$, by the e-property.

Since $(P_t)_{t\geq 0}$ is averagely bounded we may find $R_0 > 0$ such that

$$\limsup_{T\to\infty} \frac{1}{T} \int_0^T P_s^*\mu(B(x_0, R_0))\mathrm{d}s > 1 - \varepsilon^2/(4\|\varphi\|_\infty) \tag{1.3}$$

for any $\mu \in \mathcal{M}_1^{B(x_0,r_0)}$. Let $R > \max\{R_0, r_0\}$ satisfy

$$\limsup_{T\to\infty} \frac{1}{T} \int_0^T P_s^*\mu(B(x_0, R))\mathrm{d}s > 3/4 \tag{1.4}$$

for any $\mu \in \mathcal{M}_1^{B(x_0,R_0)}$. Since $(P_t)_{t\geq 0}$ is concentrating at z we may choose $\alpha > 0$ such that for any $\nu_1, \nu_2 \in \mathcal{M}_1^{B(x_0,R)}$ there exists $t > 0$ and the condition

$$P_t^*\nu_i(B(z, \delta)) \geq \alpha \qquad \text{for } i = 1, 2 \tag{1.5}$$

holds.

Set $\gamma := \alpha\varepsilon/2 > 0$. Let k be the minimal integer such that $4(1 - \gamma)^k\|\varphi\|_\infty \leq \varepsilon$.

We will show by induction that for every $l \leq k$, $l \in \mathbb{N}$, there exist $t_1, \ldots, t_l > 0$ and $\nu_1^i, \ldots, \nu_l^i, \mu_l^i \in \mathcal{M}_1$ such that $\nu_j^i \in \mathcal{M}_1^{B(z,\delta)}$ for $j = 1, \ldots, l$ and

$$\begin{aligned} P_{t_1+\cdots+t_l}^*\mu_i &= \gamma P_{t_2+\cdots+t_l}^*\nu_1^i + \gamma(1-\gamma)P_{t_3+\cdots+t_l}^*\nu_2^i \\ &\quad + \cdots + \gamma(1-\gamma)^{l-1}\nu_l^i + (1-\gamma)^l\mu_l^i \qquad \text{for } i = 1, 2. \end{aligned} \tag{1.6}$$

Indeed, let $t_1 > 0$ be such that

$$P_{t_1}^*\mu_i(B(z, \delta)) \geq \alpha > \gamma \qquad \text{for } i = 1, 2.$$

Set

$$\nu_1^i = \frac{P_{t_1}^*\mu_i(\,\cdot\, \cap B(z, \delta))}{P_{t_1}^*\mu_i(B(z, \delta))}, \tag{1.7}$$

$$\mu_1^i = (1-\gamma)^{-1}(P_{t_1}^*\mu_i - \gamma\nu_1^i) \qquad \text{for } i = 1, 2$$

and observe that $\mu_1^i \in \mathcal{M}_1$ and $\nu_1^i \in \mathcal{M}_1^{B(z,\delta)}$ for $i = 1, 2$. Then condition (1.6) holds for $l = 1$.

Now assume that we have done it for some l and $4(1-\gamma)^l\|\varphi\|_\infty > \varepsilon$. Then there exist $s_i > 0$ for $i = 1, 2$ such that

$$P^*_{t_1+\dots+t_l+s_i}\mu_i(X \setminus B(x_0, R_0)) < \varepsilon^2/(4\|\varphi\|_\infty)$$

for $i = 1, 2$, by (1.3). Since $(1-\gamma)^l > \varepsilon/(4\|\varphi\|_\infty)$, from the linearity of $P^*_{s_i}$ we obtain that

$$P^*_{s_i}\mu^i_l(B(x_0, R_0)) > \varepsilon \qquad \text{for } i = 1, 2.$$

Thus we may find two measures $\tilde{\mu}^1_l, \tilde{\mu}^2_l \in \mathcal{M}_1^{B(x_0,R_0)}$ such that

$$P^*_{s_i}\mu^i_l \geq \varepsilon\tilde{\mu}^i_l. \tag{1.8}$$

These measures may be defined as restriction of $P^*_{s_i}\mu^i_l$ to $B(x_0, R_0)$ respectively normed (see formula (1.7)). Further, from (1.4) it follows that

$$\begin{aligned}
&\limsup_{T\to\infty} \frac{1}{T}\int_0^T [P^*_{s+s_2}(\tilde{\mu}^1_l/2)(B(x_0, R)) + P^*_{s+s_1}(\tilde{\mu}^2_l/2)(B(x_0, R))]\mathrm{d}s \\
&= \limsup_{T\to\infty} \frac{1}{T}\int_0^T P^*_s(\tilde{\mu}^1_l/2 + \tilde{\mu}^2_l/2)(B(x_0, R))\mathrm{d}s > 3/4,
\end{aligned}$$

by the fact that $\tilde{\mu}^1_l/2 + \tilde{\mu}^2_l/2 \in \mathcal{M}_1^{B(x_0,R_0)}$. Consequently, for some $s > 0$ we have

$$P^*_{s+s_2}\tilde{\mu}^1_l(B(x_0, R)) \geq 1/2 \quad \text{and} \quad P^*_{s+s_1}\tilde{\mu}^2_l(B(x_0, R)) \geq 1/2.$$

Comparing (1.8) and the above we obtain

$$P^*_{s+s_1+s_2}\mu^i_l \geq (\varepsilon/2)\hat{\mu}^i_l$$

for some $\hat{\mu}^i_l \in \mathcal{M}_1^{B(x_0,R)}$, $i = 1, 2$, by argument similar to that in (1.8). Using it once again and taking into consideration (1.5) we obtain that there exists $t > 0$ such that

$$P^*_{t+s+s_1+s_2}\mu^i_l \geq (\alpha\varepsilon/2)\nu^i_{l+1} = \gamma\nu^i_{l+1}$$

for some $\nu^i_{l+1} \in \mathcal{M}_1^{B(z,\delta)}$ for $i = 1, 2$. Therefore, setting $t_{l+1} = t + s + s_1 + s_2$ we obtain

$$\begin{aligned}
P^*_{t_1+\dots+t_l+t_{l+1}}\mu_i &= \gamma P^*_{t_2+\dots+t_{l+1}}\nu^i_1 + \gamma(1-\gamma)P^*_{t_3+\dots+t_{l+1}}\nu^i_2 \\
&\quad + \dots + \gamma(1-\gamma)^{l-1}P^*_{t_{l+1}}\nu^i_l + \gamma(1-\gamma)^l\nu^i_{l+1} + (1-\gamma)^{l+1}\mu^i_{l+1},
\end{aligned}$$

where

$$\mu^i_{l+1} = (1-\gamma)^{-1}(P^*_{t_{l+1}}\mu^i_l - \gamma\nu^i_{l+1}) \qquad \text{for } i = 1, 2.$$

This completes the proof of condition (1.6). In turn, this and (1.2) give for $t \geq t_1 + \dots + t_k$

$$\begin{aligned}
|\langle\varphi, P^*_t\mu_1\rangle - \langle\varphi, P^*_t\mu_2\rangle| &= |\langle P_{t-(t_1+\dots t_k}\varphi, P^*_{t_1+\dots t_k}\mu_1\rangle - \langle P_{t-(t_1+\dots t_k}\varphi, P^*_{t_1+\dots t_k}\mu_2\rangle| \\
&\leq \gamma|\langle P_{t-t_1}\varphi, \nu^1_1 - \nu^2_1\rangle| + \gamma(1-\gamma)|\langle P_{t-(t_1+t_2)}\varphi, \nu^1_2 - \nu^2_2\rangle| + \dots \\
&\quad + \gamma(1-\gamma)^{k-1}|\langle P_{t-(t_1+\dots+t_k)}\varphi, \nu^1_k - \nu^2_k\rangle| + 2(1-\gamma)^k\|\varphi\|_\infty \\
&\leq (\gamma + \gamma(1-\gamma) + \dots + \gamma(1-\gamma)^{k-1}) \sup_{t\geq 0,\, x,y\in B(z,\delta)} |P_t\varphi(x) - P_t\varphi(y)| \\
&\quad + \varepsilon/2 \leq \varepsilon/2 + \varepsilon/2 = \varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, the proof is complete. □

Proposition 2. *Assume that there exists $z \in X$ such that for any $\varepsilon > 0$*

$$\limsup_{T\to\infty} \sup_{\mu\in\mathcal{M}_1} \frac{1}{T}\int_0^T P_s^*\mu(B(z,\varepsilon))\mathrm{d}s > 0. \tag{1.9}$$

If $(P_t)_{t\geq 0}$ satisfies the e–property, then it admits an invariant measure.

Proof. Assume, contrary to our claim, that $(P_t)_{t\geq 0}$ does not possess any invariant measure. From Step I of Theorem 3.1 in [17] it follows that then there exists an $\varepsilon > 0$, a sequence of compact sets $(K_i)_{i\geq 1}$, and an increasing sequence of positive reals $(q_i)_{i\geq 1}$, $q_i \to \infty$, satisfying

$$P_{q_i}^*\delta_z(K_i) \geq \varepsilon \qquad \text{for } i \in \mathbb{N}$$

and

$$\min\{\rho(x,y) : x \in K_i, y \in K_j\} \geq \varepsilon \qquad \text{for } i \neq j,\ i,j \in \mathbb{N}.$$

We will show that for every open neighborhood U of z and every $i_0 \in \mathbb{N}$ there exists $y \in U$ and $i \geq i_0$, $i \in \mathbb{N}$, such that

$$P_{q_i}^*\delta_y\left(K_i^{\varepsilon/3}\right) < \varepsilon/2,$$

where $K_i^{\varepsilon/3} = \{y \in X : \inf_{v\in K_i}\rho(y,v) < \varepsilon/3\}$.

On the contrary, suppose that there exists an open neighbourhood U of z and $i_0 \in \mathbb{N}$ such that

$$\inf\left\{P_{q_i}^*\delta_y\left(K_i^{\varepsilon/3}\right) : y \in U, i \geq i_0\right\} \geq \varepsilon/2. \tag{1.10}$$

Clearly

$$\limsup_{T\to\infty} \sup_{\mu\in\mathcal{M}_1} \frac{1}{T}\int_0^T P_s^*\mu(U)\mathrm{d}s > \alpha \tag{1.11}$$

for some $\alpha > 0$. Further, let $N \in \mathbb{N}$ satisfy $(N - i_0 + 1)\alpha\varepsilon > 2$. Choose $\gamma \in (0, \alpha\varepsilon/2)$ such that

$$(N - i_0 + 1)(\alpha\varepsilon - 2\gamma) > 2.$$

It easily follows that there exists $T_0 > 0$ such that for any $\mu \in \mathcal{M}_1$ and $T \geq T_0$ we have

$$\max_{i\leq N}\left\|\frac{1}{T}\int_0^T P_s^*\mu\,\mathrm{d}s - \frac{1}{T}\int_0^T P_{s+q_i}^*\mu\,\mathrm{d}s\right\|_{TV} < \gamma.$$

Choose $T \geq T_0$ and $\mu \in \mathcal{M}_1$ such that

$$\frac{1}{T}\int_0^T P_s^*\mu(U)\mathrm{d}s \geq \alpha, \tag{1.12}$$

by (1.11). From (1.10) and the Markov property it follows that

$$P_{s+q_i}^*\mu\left(K_i^{\varepsilon/3}\right) = \int_X P_{q_i}^*\delta_y\left(K_i^{\varepsilon/3}\right)P_s^*(\mathrm{d}y) \geq \int_U P_{q_i}^*\delta_y\left(K_i^{\varepsilon/3}\right)P_s^*(\mathrm{d}y) \geq \frac{\varepsilon}{2}P_s^*\mu(U)$$

for $i \geq i_0$ and $s \geq 0$. Consequently, we have for $i_0 \leq i \leq N$

$$\frac{1}{T}\int_0^T P_s^*\mu\left(K_i^{\varepsilon/3}\right)\mathrm{d}s \geq \frac{1}{T}\int_0^T P_{s+q_i}^*\mu\left(K_i^{\varepsilon/3}\right)\mathrm{d}s - \gamma$$
$$\geq \frac{\varepsilon}{2}\frac{1}{T}\int_0^T P_s^*\mu(U)\mathrm{d}s - \gamma \geq \frac{\varepsilon}{2}\alpha - \gamma,$$

by (1.12). From this and the fact that $K_i^{\varepsilon/3} \cap K_j^{\varepsilon/3} = \emptyset$ for $i \neq j$ we obtain

$$\frac{1}{T}\int_0^T P_s^*\mu\left(\bigcup_{i=i_0}^N K_i^{\varepsilon/3}\right)\mathrm{d}s = \sum_{i=i_0}^N \frac{1}{T}\int_0^T P_s^*\mu\left(K_i^{\varepsilon/3}\right)\mathrm{d}s$$
$$\geq (N - i_0 + 1)(\varepsilon\alpha - 2\gamma)/2 > 1,$$

which is impossible.

Now analogously as in the proof of Theorem 3.1 in [17], Step III, we define a sequence of Lipschitzian functions $(f_n)_{n\geq 1}$, a sequence of points $(y_n)_{n\geq 1}$, $y_n \to z$ as $n \to \infty$, two increasing sequences of integers $(i_n)_{n\geq 1}$, $(k_n)_{n\geq 1}$, $i_n < k_n < i_{n+1}$ for $n \in \mathbb{N}$, and a sequence of reals $(p_n)_{n\geq 1}$ such that

$$f_n|_{K_{i_n}} = 1, \qquad 0 \leq f_n \leq \mathbf{1}_{K_{i_n}^{\varepsilon/3}}, \qquad \mathrm{Lip}\, f_n \leq 3/\varepsilon, \tag{1.13}$$

$$\left| P_{p_n}\left(\sum_{i=1}^n f_i\right)(z) - P_{p_n}\left(\sum_{i=1}^n f_i\right)(y_n)\right| > \frac{\varepsilon}{4}, \tag{1.14}$$

$$P_{p_n}^*\delta_u\left(\bigcup_{i=k_n}^\infty K_i^{\varepsilon/3}\right) < \frac{\varepsilon}{16} \quad \text{for } u \in \{z, y_n\} \tag{1.15}$$

for every $n \in \mathbb{N}$. From (1.13)-(1.15) it follows (see the proof of Theorem 3.1 in [17], Step III, once again) that

$$|P_{p_n}f(z) - P_{p_n}f(y_n)| > \frac{\varepsilon}{8}$$

for $n \in \mathbb{N}$ and $f := \sum_{n=1}^\infty f_n \in L_b(X)$. Since $y_n \to z$ as $n \to \infty$, this contradicts the assumption that the family $\{P_t f : t \geq 0\}$ is equicontinuous in z. The proof is complete. □

Theorem 1. *Let $(P_t)_{t\geq 0}$ be averagely bounded and concentrating at some $z \in X$. If $(P_t)_{t\geq 0}$ satisfies the e–property, then it is asymptotically stable.*

Proof. Fix $x \in X$. Since $(P_t)_{t\geq 0}$ is averagely bounded there is $R > 0$ such that

$$\limsup_{T\to\infty} \frac{1}{T}\int_0^T P_s^*\delta_x(B(x,R))\mathrm{d}s > \frac{1}{2}.$$

Let $(T_n)_{n\geq 1}$ be an increasing sequence of reals such that $T_n \to \infty$ as $n \to \infty$ and

$$\frac{1}{T_n}\int_0^{T_n} P_s^*\delta_x(B(x,R))\mathrm{d}s > \frac{1}{2} \qquad \text{for } n \in \mathbb{N}.$$

Set $\mu_n = \frac{1}{T_n}\int_0^{T_n} P_s^*\delta_x\,\mathrm{d}s$, $n \in \mathbb{N}$, and observe that there are $\mu_n^R \in \mathcal{M}_1^{B(x,R)}$ such that

$$\mu_n \geq \frac{1}{2}\mu_n^R \qquad \text{for } n \in \mathbb{N}.$$

Indeed, we may define μ_n^R by the formula $\mu_n^R = \mu_n(\,\cdot \cap B(x,R))/\mu_n(B(x,R))$ for $n \in \mathbb{N}$. Further, observe that, by concentrating at z, for fixed $\varepsilon > 0$ there is $\alpha > 0$ such that we have

$$P_{s_n}^*\mu_n^R(B(z,\varepsilon)) \geq \alpha$$

for some $s_n > 0$, $n \in \mathbb{N}$. Hence

$$P_{s_n}^*\mu_n(B(z,\varepsilon)) \geq \frac{1}{2}\alpha \qquad \text{for } n \in \mathbb{N},$$

by linearity of $(P_t^*)_{t\geq 0}$. Consequently,

$$\frac{1}{T_n}\int_0^{T_n} P_s^*(P_{s_n}^*\delta_x)(B(z,\varepsilon))\mathrm{d}s \geq \frac{1}{2}\alpha \qquad \text{for } n \in \mathbb{N},$$

and condition (1.9) in Proposition 2 is satisfied. Now Proposition 2 implies the existence of an invariant measure. Further, from Proposition 1 it follows that for any $f \in L_b(X)$ and $\mu \in \mathcal{M}_1$

$$\langle \varphi, P_t^*\mu\rangle \to \langle \varphi, \mu_*\rangle$$

as t tends to $+\infty$. Application of the Alexandrov theorem finishes the proof (see [?]). □

2 The models

2.1 GOY and Sabra shell models and functional setting

Let $u = (u_{-1}, u_0, u_1, \ldots)$ be an infinite sequence of complex valued functions on $[0,\infty)$ satisfying the following equations for $n = 1, 2, \ldots$

$$\mathrm{d}u_n(t) + \nu k_n^2 \nu_n(t)\mathrm{d}t + [B(u,u)]_n \mathrm{d}t = \sigma_n \mathrm{d}w_n \tag{2.1}$$

with the initial conditions

$$u_{-1}(t) = u_0(t) = 0 \quad \text{and} \quad u_n(0) = \xi_n.$$

Here $k_n = k_0 2^n$, $k_0 > 1$ and $\nu > 0$. Moreover $(w_n(t))_{n\geq 1}$ denotes a sequence of independent Brownian motions on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. It is assumed that

$\sigma_n \in \mathbb{C}$ and there is $n_0 \in \mathbb{N}$ such that $\sigma_n = 0$ for $n \geq n_0$. Further B is a bilinear operator which will be defined later on.

Let H be the set of all sequences $u = (u_1, u_2, \ldots)$ of complex numbers such that $\sum_n |u_n|^2 < \infty$. We consider H as a *real* Hilbert space endowed with the inner product $(\cdot,\cdot)$ and the norm $|\cdot|$ of the form

$$(u,v) = \mathrm{Re} \sum_{n\geq 1} u_n v_n^*, \quad |u|^2 = \sum_{n\geq 1} |u_n|^2, \tag{2.2}$$

where v_n^* denotes the complex conjugate of v_n. The space H is separable. Let $A : \mathrm{D}(A) \subset H \to H$ be the non-bounded linear operator defined by

$$(Au)_n = k_n^2 u_n, \quad n = 1, 2, \ldots, \qquad \mathrm{D}(A) = \Big\{ u \in H \, : \, \sum_{n\geq 1} k_n^4 |u_n|^2 < \infty \Big\}.$$

The operator A is clearly self-adjoint, strictly positive definite since $(Au, u) \geq k_0^2 |u|^2$ for $u \in \mathrm{D}(A)$. For any $\alpha > 0$, set

$$\mathcal{H}_\alpha = \mathrm{D}(A^\alpha) = \{ u \in H \, : \, \sum_{n\geq 1} k_n^{4\alpha} |u_n|^2 < +\infty \}, \ \|u\|_\alpha^2 = \sum_{n\geq 1} k_n^{4\alpha} |u_n|^2 \text{ for } \ u \in \mathcal{H}_\alpha.$$

Obviously $\mathcal{H}_0 = H$. Define

$$V := \mathrm{D}(A^{\frac{1}{2}}) = \Big\{ u \in H \, : \, \sum_{n\geq 1} k_n^2 |u_n|^2 < +\infty \Big\}$$

and set

$$\mathcal{H} = \mathcal{H}_{\frac{1}{4}}, \ \|u\|_{\mathcal{H}} = \|u\|_{\frac{1}{4}}.$$

Then V is a Hilbert space for the scalar product $(u,v)_V = \mathrm{Re}(\sum_n k_n^2 \, u_n \, v_n^*)$, $u, v \in V$ and the associated norm is denoted by

$$\|u\|^2 = \sum_{n\geq 1} k_n^2 \, |u_n|^2.$$

The adjoint of V with respect to the H scalar product is $V' = \{ (u_n) \in \mathbb{C}^{\mathbb{N}} \, : \, \sum_{n\geq 1} k_n^{-2} \, |u_n|^2 < +\infty \}$ and $V \subset H \subset V'$ is a Gelfand triple. Let $\langle u \, , \, v \rangle_{V',V} = \mathrm{Re} \left(\sum_{n\geq 1} u_n \, v_n^* \right)$ denote the duality between $u \in V'$ and $v \in V$.

Set $u_{-1} = u_0 = 0$, let a, b be real numbers and let $B : H \times V \to H$ (or $B : V \times H \to H$) denote the bilinear operator defined by

$$[B(u,v)]_n = i \left(a k_{n+1} u_{n+1}^* v_{n+2}^* + b k_n u_{n-1}^* v_{n+1}^* - a k_{n-1} u_{n-1}^* v_{n-2}^* - b k_{n-1} u_{n-2}^* v_{n-1}^* \right)$$

for $n = 1, 2, \ldots$ in the GOY shell model (see, e.g. [26]) or

$$[B(u,v)]_n = i \left(a k_{n+1} u_{n+1}^* \, v_{n+2} + b k_n u_{n-1}^* v_{n+1} + a k_{n-1} u_{n-1} v_{n-2} + b k_{n-1} u_{n-2} v_{n-1} \right),$$

in the Sabra shell model introduced in [20].

Obviously, there exists $C>0$ such that

$$|B(u,v)| \le C\|u\||v| \qquad \text{for } u\in V \text{ and } v\in H. \tag{2.3}$$

Note that B can be extended as a bilinear operator from $H\times H$ to V' and that there exists a constant $C>0$ such that given $u,v\in H$ and $w\in V$ we have

$$|\langle B(u,v)\,,\,w\rangle_{V',V}| + |\big(B(u,w)\,,\,v\big)| + |\big(B(w,u)\,,\,v\big)| \le C\,|u|\,|v|\,\|w\|. \tag{2.4}$$

An easy computation proves that for $u,v\in H$ and $w\in V$ (resp. $v,w\in H$ and $u\in V$),

$$\langle B(u,v)\,,\,w\rangle_{V',V} = -\big(B(u,w)\,,\,v\big) \text{ (resp. } \big(B(u,v)\,,\,w\big) = -\big(B(u,w)\,,\,v\big)\text{).}$$

Hence $(B(v,u),u)=0$ for $u\in H$ and $v\in V$. Furthermore, $B:V\times V\to V$ and $B:\mathcal{H}\times\mathcal{H}\to H$; indeed, for $u,v\in V$ (resp. $u,v\in\mathcal{H}$) we have

$$\begin{aligned}\|B(u,v)\|^2 &= \sum_{n\ge1} k_n^2\,|B(u,v)_n|^2 \ \le C\,\|u\|^2 \sup_n k_n^2 |v_n|^2 \le C\,\|u\|^2\,\|v\|^2,\\ |B(u,v)| &\le C\,\|u\|_{\mathcal{H}}\,\|v\|_{\mathcal{H}}.\end{aligned}$$

2.2 Well-posedness

Consider the abstract equation on H of the form

$$\mathrm{d}u(t) = [-\nu Au(t) + B(u(t),u(t))]\,\mathrm{d}t + Q\mathrm{d}W(t), \quad t\ge0 \tag{2.5}$$

with the initial condition $u(0)=\xi\in H$, where $Q=(q_{i,j})_{i,j\in\mathbb{N}}$ is some matrix with $\mathrm{Tr}(QQ^*)<\infty$ and $W(t)=(w_n(t))_{n\ge1}$ is a cylindrical Wiener noise on some filtered space $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\ge0},\mathbb{P})$.

Definition 2.1. *A stochastic process $u(t,\omega)$ is a generalized solution in $[0,T]$ of the system* (2.5) *if*

$$u(\cdot,\omega)\in C([0,T];H)\cap L^2(0,T;\mathcal{H})$$

for $\mathbb{P}$-a.e. $\omega\in\Omega$, u is progressively measurable in these topologies and equation (2.5) *is satisfied in the integral sense*

$$\begin{aligned}&(u(t),\varphi) + \int_0^t \nu(u(s),A\varphi)\mathrm{d}s + \int_0^t \left(B\left(u(s),\varphi\right),u(s)\right)\mathrm{d}s\\ &= (\xi,\varphi) + (QW(t),\varphi)\end{aligned}$$

for all $t\in[0,T]$ and $\varphi\in D(A)$.

Theorem 2. *Let us assume that the initial condition ξ is an $\mathcal{F}_0$-random variable with values in H. Then there exists a unique solution $(u(t))_{t\geq 0}$ to equation* (2.5)*. Moreover, if $\mathbb{E}|\xi|^2 < +\infty$, then*

$$\mathbb{E}|u(t)|^2 + \int_0^t 2\nu\mathbb{E}\|u(s)\|^2\mathrm{d}s = \mathbb{E}|\xi|^2 + \mathrm{Tr}(QQ^*)t \tag{2.6}$$

for any $t \geq 0$.

Proof. We will prove well–posedness using a pathwise argument (for similar results see [3] and the references therein). Let us introduce the Ornstein-Uhlenbeck process solution of

$$\begin{cases} \mathrm{d}z(t) + \nu Az(t)\mathrm{d}t = Q\mathrm{d}W, \\ z(0) = 0. \end{cases} \tag{2.7}$$

The above equation has a unique progressively measurable solution such that $\mathbb{P}$-a.s.

$$z \in C([0,T];\mathcal{H})$$

(for more details see [6]). Set $v = u - z$. Then for $\mathbb{P}$-a.e. $\omega \in \Omega$

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}v(t) + \nu Av(t) - B(v(t)+z(t), v(t)+z(t)) = 0, \\ v(0) = \xi, \end{cases} \tag{2.8}$$

is a deterministic system. The existence and uniqueness of global weak solutions v follow from the Galerkin approximation procedure and then passing to the limit using the appropriate compactness theorems. We omit the details which can be found in [3] and the references therein. Instead, we present the formal computations which lead to the basic a priori estimates, this is in order to stress the role played by z. Using equation (2.8) and various properties of the nonlinear operator B, we have

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|v(t)|^2 + \nu\|v(t)\|^2 &\leq |(B(v(t)+z(t), z(t)), v(t))| \\ &\leq C\|v(t)\||v(t)+z(t)||z(t)| \\ &\leq \frac{\nu}{2}\|v(t)\|^2 + C(\nu)\left(|v(t)|^2|z(t)|^2 + |z(t)|^4\right). \end{aligned}$$

Using Gronwall's Lemma and the fact that $\|z\|_{C([0,T];\mathcal{H})} \leq C(\omega)$, we have

$$\sup_{0\leq t\leq T}|v(t)|^2 \leq C(|\xi|, T, C(\omega)).$$

Again, using the above inequality in the previous estimate, we obtain that

$$\int_0^T \|v(s)\|^2\mathrm{d}s \leq C(|\xi|, T, C(\omega)).$$

Then, by classical arguments, see [27], $v \in C([0,T];H) \cap L^2(0,T;\mathrm{D}(A^{1/2}))$. Therefore $u = v + z \in C([0,T];H) \cap L^2(0,T;\mathrm{D}(A^{1/4}))$ $\mathbb{P}$-a.s.

To finish the proof observe that condition (2.6) follows from Itô's formula. □

The uniqueness of solutions is established in the following theorem.

Theorem 3. *Let* $\left(u^{(1)}(t)\right)_{t\geq 0}$, $\left(u^{(2)}(t)\right)_{t\geq 0}$, *be two continuous adapted solutions of* (2.5) *in* H, *with the initial conditions* $u_0^{(1)}$ *and* $u_0^{(2)}$ *as above. Then there is a constant* $C(\nu) > 0$, *depending only on* ν, *such that* $\mathbb{P}$*-a.s.*

$$\left|u^1(t) - u^2(t)\right|^2 \leq \mathrm{e}^{C(\nu)\int_0^t \left|u^1(s)\right|^2 ds} \left|u_0^1 - u_0^2\right|^2 \quad t \geq 0.$$

Proof. Let us put $u(t) = u^1(t) - u^2(t)$. Then u is the solution of the following equation

$$\mathrm{d}u + \nu A u \mathrm{d}t - \left(B(u^1, u^1) - B(u^2, u^2)\right) \mathrm{d}t = 0.$$

Using again the properties of operator B, we obtain

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}|u|^2 + \nu\|u\|^2 &\leq |(B(u, u^1), u)| \\ &\leq \frac{\nu}{2}\|u\|^2 + C(\nu)|u|^2|u^1|^2. \end{aligned}$$

Hence, by the Gronwall lemma, we obtain that

$$|u(t)|^2 \leq |u(0)|^2 \mathrm{e}^{C(\nu)\left(\int_0^T |u^1(s)|^2 ds\right)},$$

which finishes the proof. □

2.3 Stability of the model

Let a diagonal matrix $Q = (q_{i,j})_{i,j\in\mathbb{N}}$ be such that there is $n_0 \in \mathbb{N}$ and $q_{n,n} = 0$ for $n \geq n_0$. Consider the equation on H of the form

$$\mathrm{d}u(t) = [-\nu A u(t) + B(u(t), u(t))]\mathrm{d}t + Q\mathrm{d}W(t) \quad t \geq 0, \tag{2.9}$$

where $(W(t))_{t\geq 0}$ is a certain cylindrical Wiener process on a filtered space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$.

By Theorem 2 for every $x \in H$ there is a unique continuous solution $(u^x(t))_{t\geq 0}$ in H, hence the transition semigroup is well defined. From Theorem 3 we obtain that the solution satisfies the Feller property, i.e. for any $t \geq 0$ if $x_n \to x$ in H, then $\mathbb{E}f(u^{x_n}(t)) \to \mathbb{E}f(u^x(t))$ for any $f \in C_b(H)$. Set

$$P_t f(x) = \mathbb{E}f(u^x(t)) \quad \text{for any } f \in C_b(H).$$

Obviously $(P_t)_{t\geq 0}$ is stochastically continuous. First note that $DP_t f(x)[v]$, the value of the Frechet derivative $DP_t f(x)$ at $v \in H$, is equal to $\mathbb{E}\left\{Df(u^x(t))[U(t)]\right\}$, where $U(t) := \partial u^x(t)[v]$ and

$$\partial u^x(t)[v] := \lim_{\eta\downarrow 0} \frac{1}{\eta}\left(u^{x+\eta v}(t) - u^x(t)\right)$$

and the limit is in $L^2(\Omega,\mathcal{F},\mathbb{P};H)$ (see [14] also [13]). The process $U=(U(t))_{t\ge 0}$ satisfies the linear evolution equation

$$\begin{aligned}\frac{\mathrm{d}U(t)}{\mathrm{d}t} &= -\nu AU(t)+B(u^x(t),U(t))+B(U(t),u^x(t)),\\ U(0) &= v.\end{aligned} \tag{2.10}$$

Suppose that $\mathcal{X}$ is a certain Hilbert space and $\Phi\colon H\to\mathcal{X}$ a Borel measurable function. Given an $(\mathcal{F}_t)_{t\ge0}$-adapted process $g\colon[0,\infty)\times\Omega\to H$ satisfying $\mathbb{E}\int_0^t|g(s)|^2\mathrm{d}s<\infty$ for each $t\ge 0$ we denote by $\mathcal{D}_g\Phi(u^x(t))$ the Malliavin derivative of $\Phi(u^x(t))$ in the direction of g; that is the $L^2(\Omega,\mathcal{F},\mathbb{P};\mathcal{X})$-limit, if exists, of

$$\mathcal{D}_g\Phi(u^x(t)):=\lim_{\eta\downarrow 0}\frac{1}{\eta}\left[\Phi(u^x_{\eta g}(t))-\Phi(u^x(t))\right],$$

where $u^x_g(t)$, $t\ge 0$, solves the equation

$$\mathrm{d}u^x_g(t)=\left[-\nu Au^x_g(t)+B(u^x_g(t),u^x_g(t))\right]\mathrm{d}t+Q\left(\mathrm{d}W(t)+g(t)\mathrm{d}t\right),\qquad u^x_g(0)=x.$$

In particular, one can easily show that when $\mathcal{X}=H$ and $\Phi=I$, where I is the identity operator, the Malliavin derivative of $u^x(t)$ exists and the process $D(t):=\mathcal{D}_gu^x(t)$, $t\ge0$, solves the linear equation

$$\begin{aligned}\frac{\mathrm{d}D}{\mathrm{d}t}(t) &= -\nu AD(t)+B(u^x(t),D(t))+B(D(t),u^x(t))+Qg(t),\\ D(0) &= 0.\end{aligned} \tag{2.11}$$

Directly from the definition of the Malliavin derivative we conclude the *chain rule:* suppose that $\Phi\in C^1_b(H;\mathcal{X})$ then

$$\mathcal{D}_g\Phi(u^x(t))=D\Phi(u^x(t))[D(t)].$$

(Here $C^1_b(H;\mathcal{X})$ denotes the space of all bounded continuous functions $\Phi: H\to\mathcal{X}$ with continuous and bounded first derivative with the natural norm. In the case when $\mathcal{X}=\mathbb{R}$ we simply write $C^1_b(H)$.) In addition, the *integration by parts formula* holds, see Lemma 1.2.1, p. 25 of [21]. Indeed, suppose that $\Phi\in C^1_b(H)$. Then

$$\mathbb{E}[\mathcal{D}_g\Phi(u^x(t))]=\mathbb{E}\left[\Phi(u^x(t))\int_0^t(g(s),\mathrm{d}W(s))\right]. \tag{2.12}$$

Lemma 1. *Let* $\eta\in(0,\nu/(2\max q^2_{i,i})]$. *Then we have*

$$\mathbb{E}(\exp(\eta|u^x(t)|^2+\eta\nu\int_0^t\|u^x(s)\|^2\mathrm{d}s))\le 2\exp(\eta(\mathrm{Tr}\,Q^2)t+\eta|x|^2).$$

Proof. Fix $\eta \in (0, \nu/(2\max q_{i,i}^2)]$. Let $M(t) = \eta \int_0^t (u^x(s), Q\mathrm{d}W(s))$ and let $N(t) = M(t) - \eta\nu \int_0^t \|u^x(s)\|^2 \mathrm{d}s$. Set $\alpha = \nu/\max q_{i,i}^2$. Then we have $\nu\|u^x(s)\|^2 \geq \alpha |Qu^x(s)|^2$. Now observe that $N(t) \leq M(t) - (\alpha/\eta)\langle M\rangle(t)$, where $\langle M\rangle(t)$ denotes the quadratic variation of the continous L^2–martingale M with the filtration generated by the noise. Hence by a standard variation of the Kolmogorov–Doob martingale inequality (see [22]) we have

$$\mathbb{P}(N(t) \geq K) \leq \exp(-\alpha K/\eta)$$

and consequently we obtain

$$\mathbb{P}(\exp N(t) \geq \exp K) \leq \exp(-\alpha K/\eta) \leq \exp(-2K)$$

for any $K > 0$. An easy observation that if some positive random variable, say X, satisfies the condition $\mathbb{P}(X \geq C) \leq C^{-2}$ for every $C > 0$, then $\mathbb{E}X \leq 2$ gives

$$\mathbb{E}(\exp(\eta|u^x(t)|^2 + \eta\nu \int_0^t \|u^x(s)\|^2 \mathrm{d}s - \eta(\mathrm{Tr}\, Q^2)t - \eta|x|^2)) \leq 2,$$

by Itô's formula. This completes the proof. □

The crucial role in our consideration is played by the following lemma. The idea of its proof is taken from [13].

Lemma 2. *Let $(P_t)_{t\geq 0}$ correspond to problem* (2.9). *If Q satisfies the condition:*

$$q_{1,1}, \ldots, q_{N_*,N_*} \neq 0 \quad \text{for } N_* > \log_2(2C^2 \max q_{i,i}^2/\nu^3 + \mathrm{Tr}\, Q^2/(2\max q_{i,i}^2))/2, \tag{2.13}$$

where $C > 0$ is given by (2.3), then for any $f \in C_b^1(H)$ and $R > 0$ there exists a constant $C_0 > 0$ such that

$$\sup_{t\geq 0} \sup_{|x|\leq R} \sup_{|v|\leq 1} |DP_t f(x)[v]| \leq C_0 \|f\|_{C_b^1(H)}. \tag{2.14}$$

Proof. Fix $N_* > \log_2(2C^2 \max q_{i,i}^2/\nu^3 + \mathrm{Tr}\, Q^2/(2\max q_{i,i}^2))/2$. The proof will be split into three steps.

Step I: Let $g : [0,\infty) \times \Omega \to H$ be a measurable function such that $\mathbb{E}\int_0^t |g(s)|^2 \mathrm{d}s < \infty$ for any $t \geq 0$. Let $\omega_t(x) := \mathcal{D}_g u^x(t)$ and $\rho_t(v,x) := \partial u^x(t)[v] - \mathcal{D}_g u^x(t)$. Then,

$$\begin{aligned} DP_t f(x)[v] &= \mathbb{E}\{ Df(u^x(t))[\omega_t(x)]\} + \mathbb{E}\{Df(u^x(t))[\rho_t(v,x)]\} \\ &= \mathbb{E}\{\mathcal{D}_g f(u^x(t))\} + \mathbb{E}\{Df(u^x(t))[\rho_t(v,x)]\} \\ &\overset{(2.12)}{=} \mathbb{E}\left\{ f(u^x(t)) \int_0^t (g(s), \mathrm{d}W(s))\right\} + \mathbb{E}\{Df(u^x(t))[\rho_t(v,x)]\}. \end{aligned}$$

We have

$$\left|\mathbb{E}\left\{ f(u^x(t)) \int_0^t (g(s), \mathrm{d}W(s))\right\}\right| \leq \|f\|_{L^\infty} \left(\mathbb{E}\int_0^t |g(s)|^2 \mathrm{d}s\right)^{1/2}$$

and

$$|\mathbb{E}\,\{Df(u^x(t))[\rho_t(v,x)]\}| \leq \|f\|_{C_b^1(H)}\mathbb{E}\,|\rho_t(v,x)| \leq \|f\|_{C_b^1(H)}(\mathbb{E}\,|\rho_t(v,x)|^2)^{1/2}.$$

Step II: Let $\xi(t) = (\xi_1(t), \xi_2(t), \ldots) : [0,\infty) \to H$ be a solution to the following system:

$$\frac{\mathrm{d}\xi_i(t)}{\mathrm{d}t} = -\frac{\xi_i(t)}{2\sqrt{\sum_{i=1}^{N_*}\xi_i^2(t)}} \qquad \text{for } i = 1, \ldots, N_*$$

$$\frac{\mathrm{d}\xi_i(t)}{\mathrm{d}t} = -\nu k_i^2\xi_i(t) + [B(u^x(t),\xi(t)) + B(\xi(t),u^x(t))]_i \qquad \text{for } i \geq N_* + 1.$$

with $\xi(0) = v$. We assume also that $\xi_i(t)/2\sqrt{\sum_{i=1}^{N_*}\xi_i^2(t)} = 0$ if $\sqrt{\sum_{i=1}^{N_*}\xi_i^2(t)} = 0$ (see [13]). Observe that $\xi_1(t), \xi_2(t), \ldots, \xi_{N_*}(t) = 0$ for $t \geq 2$.

Now we choose $g : [0,+\infty) \times \Omega \to H$ to be given by the formulae:

$$g_i(t) = \frac{1}{q_{i,i}}\left(-\nu k_i^2\xi_i(t) + [B(u^x(t),\xi(t)) + B(\xi(t),u^x(t))]_i - \frac{\xi_i(t)}{2\sqrt{\sum_{i=1}^{N_*}\xi_i^2(t)}}\right)$$

for $i = 1, \ldots, N_*$ and $g_i(t) = 0$ for $i \geq N_* + 1$.

It is easy to see that $\rho_t = \xi(t)$ for any $t \geq 0$. Indeed, observe that

$$\frac{\mathrm{d}\xi(t)}{\mathrm{d}t} + Qg(t) = -\nu A\xi(t) + B(u^x(t),\xi(t)) + B(\xi(t),u^x(t))$$

and

$$\xi(0) = v.$$

On the other hand, subtracting equation (2.10) from (2.11) we obtain the equation for ρ_t. Since ρ_t and $\xi(t)$ solve the same equation with the same initial condition $\rho_0 = \xi(0) = v$, we obtain $\rho_t = \xi(t)$ for $t \geq 0$.

Step III: To show (2.14) it is enough to prove that

$$\sup_{|x|\leq R}\sup_{|v|\leq 1}\mathbb{E}\int_0^\infty |g(s)|^2\mathrm{d}s < \infty$$

and

$$\sup_{t\geq 0}\sup_{|x|\leq R}\sup_{|v|\leq 1}\mathbb{E}|\xi(t)|^2 < \infty.$$

We know that $\sum_{i=1}^{N_*}|\xi_i(t)|^2 \leq |v|^2 \leq 1$ for $t \geq 0$. In particular $\xi_i(t) = 0$ for $t \geq 2$ and $i = 1, \ldots, N_*$. Let $\zeta(t) = (\xi_{N_*+1}(t), \xi_{N_*+2}(t), \ldots)$. It is easy to see that ζ satisfies the inequality

$$\frac{\mathrm{d}|\zeta(t)|^2}{\mathrm{d}t} \leq -\nu k_{N_*}^2|\zeta(t)|^2 + 2C\|u^x(t)\||\zeta(t)|^2 + 2\tilde{C}\|u^x(t)\||\zeta(t)| \quad \text{for } t \geq 0, \tag{2.15}$$

where $\tilde{C}$ is some positive constant dependent only on C. Choose $\varepsilon > 0$ and $\gamma \in (0,1)$ such that

$$-\nu k_{N_*}^2 + \varepsilon + 2C^2 \max q_{i,i}^2/\nu^2 + \nu \operatorname{Tr} Q^2/(2\gamma \max q_{i,i}^2) < 0.$$

From equation (2.15) we derive

$$\frac{\mathrm{d}|\zeta(t)|^2}{\mathrm{d}t} \leq (-\nu k_{N_*}^2 + 2C\|u^x(t)\| + \varepsilon)|\zeta(t)|^2 + C(\varepsilon)\|u^x(t)\|^2$$

and using Gronwall's lemma we obtain

$$\begin{aligned} |\zeta(t)|^2 &\leq \left(|v|^2 + C(\varepsilon)\int_0^t \|u^x(s)\|^2 \mathrm{d}s\right) e^{(-\nu k_{N_*}^2+\varepsilon)t + 2C\int_0^t \|u^x(s)\|\mathrm{d}s} \\ &\leq e^{(-\nu k_{N_*}^2+\varepsilon)t}\left[1 + C(\varepsilon)\int_0^t \|u^x(s)\|^2 \mathrm{d}s\right] e^{2C\int_0^t \|u^x(s)\|\mathrm{d}s}. \end{aligned}$$

Hence we obtain that there exist constant $A > 0$ (independent of $t \geq 0$, $v \in B(0,1)$ and $x \in B(0,R)$) such that

$$|\zeta(t)|^2 \leq A \exp(\gamma(-\nu k_{N_*}^2 + \varepsilon + 2C^2 \max q_{i,i}^2/\nu^2)t) \exp\left(\nu/(2\max q_{i,i}^2)\int_0^t \|u^x(s)\|^2 \mathrm{d}s\right)$$

for all $t \geq 0$, by the fact that $-\nu k_{N_*}^2 + \varepsilon + 2C^2 \max q_{i,i}^2/\nu^2 < 0$. Thus

$$\begin{aligned} &\sup_{|x|\leq R, |v|\leq 1, t\geq 0} \mathbb{E}|\zeta(t)|^2 \\ &\leq A \exp(\gamma(-\nu k_{N_*}^2 + \varepsilon + 2C^2 \max q_{i,i}^2/\nu^2)t)\mathbb{E}\left(\exp\left(\nu/(2\max q_{i,i}^2)\int_0^t \|u^x(s)\|^2 \mathrm{d}s\right)\right). \end{aligned}$$

Using Lemma 1 we obtain

$$\sup_{t\geq 0, |x|\leq R, |v|\leq 1} \mathbb{E}|\zeta(t)|^2 \leq \tilde{A} \exp(\gamma(-\nu k_{N_*}^2 + \varepsilon + 2C^2 \max q_{i,i}^2/\nu^2 + \nu/(2\gamma \max q_{i,i}^2) \operatorname{Tr} Q^2)t)$$

for some $\tilde{A} > 0$. On the other hand, by the definition of N_*, k_n and the choice of ε, γ we have

$$-\nu k_{N_*}^2 + \varepsilon + 2C^2 \max q_{i,i}^2/\nu^2 + \nu/(2\gamma \max q_{i,i}^2) \operatorname{Tr} Q^2 < 0.$$

Now we must evaluate

$$\mathbb{E}\int_0^t |g(s)|^2 \mathrm{d}s \leq 2 \sup_{0\leq s\leq 2} \mathbb{E}|g(s)|^2 + \mathbb{E}\int_2^t |g(s)|^2 \mathrm{d}s.$$

The first term on the right side of the above inequality is bounded uniformly in $|x| \leq R$ and $|v| \leq 1$. Further, for $s \geq 2$ we have

$$|g(s)| \leq \tilde{C}\|u^x(s)\||\zeta(s)|$$

and

$$
\begin{aligned}
&\mathbb{E}\int_2^t |g(s)|^2 \mathrm{d}s \le \tilde{C}^2 \mathbb{E}\int_2^\infty \|u^x(s)\|^2 |\zeta(s)|^2 \mathrm{d}s \\
&\le \hat{C}\,\mathbb{E}[\int_2^\infty \|u^x(s)\|^2 \exp(\gamma(-\nu k_{N_*}^2 + \varepsilon + 2C^2 \max q_{i,i}^2/\nu^2)s) \\
&\times \exp(\nu/(2\max q_{i,i}^2) \int_0^s \|u^x(r)\|^2 \mathrm{d}r)\mathrm{d}s] \\
&\le \hat{C}\mathbb{E}[\int_2^\infty \exp(\gamma(-\nu k_{N_*}^2 + \varepsilon + 2C^2 \max q_{i,i}^2/\nu^2)s) \\
&\times \exp(\nu/(2\max q_{i,i}^2)|u^x(s)|^2 + \nu/(2\max q_{i,i}^2) \int_0^s \|u^x(r)\|^2 \mathrm{d}r)\mathrm{d}s] \\
&\le \hat{C} \int_2^\infty [\exp(\gamma(-\nu k_{N_*}^2 + \varepsilon + 2C^2 \max q_{i,i}^2/\nu^2)s) \\
&\times \mathbb{E}\exp(\nu/(2\max q_{i,i}^2)|u^x(s)|^2 + \nu/(2\max q_{i,i}^2) \int_0^s \|u^x(r)\|^2 \mathrm{d}r)]\mathrm{d}s \\
&\le C' \int_2^\infty \exp(\gamma(-\nu k_{N_*}^2 + \varepsilon + 2C^2 \max q_{i,i}^2/\nu^2 + \nu \operatorname{Tr} Q^2/(2\gamma \max q_{i,i}^2))s)\mathrm{d}s,
\end{aligned}
$$

for any $x \in B(0, R)$, where the constant C' depends only on R. Using again the assumption on N_* we obtain

$$
\sup_{|x|\le R, |v|\le 1} \mathbb{E}\int_2^\infty |g(s)|^2 \mathrm{d}s < \infty.
$$

This completes the proof. □

Lemma 3. *(Average boundedness) Let* $(P_t)_{t\ge0}$ *correspond to problem* (2.5). *Then* $(P_t)_{t\ge0}$ *is averagely bounded.*

Proof. Fix an $\varepsilon > 0$ and let $r > 0$ be given. If $x \in B(0, r)$, then

$$
\begin{aligned}
&\frac{1}{T}\int_0^T P_s^*\delta_x(H \setminus B(0,R))\mathrm{d}s = \frac{1}{T}\int_0^T \mathbb{P}(|u^x(s)| > R)\mathrm{d}s \le \frac{1}{T}\int_0^T \mathbb{P}(\|u^x(s)\| > R)\mathrm{d}s \\
&= \frac{1}{T}\int_0^T \mathbb{P}(\|u^x(s)\|^2 > R^2)\mathrm{d}s \le \frac{1}{T}\int_0^T \frac{\mathbb{E}\|u^x(s)\|^2}{R^2}\mathrm{d}s \\
&= \frac{1}{\nu R^2}\frac{1}{T}\int_0^T \nu\mathbb{E}\|u^x(s)\|^2 \mathrm{d}s \le \frac{1}{\nu R^2}(\operatorname{Tr} Q^2 + |x|^2/T) \le \frac{1}{\nu R^2}(\operatorname{Tr} Q^2 + r^2/T)
\end{aligned}
$$

for arbitrary $R > 0$, by (2.6). Hence there is $R_0 > 0$ such that

$$
\liminf_{T\to+\infty} \frac{1}{T}\int_0^T P_s^*\delta_x(B(0,R_0))\mathrm{d}s > 1 - \varepsilon.
$$

On the other hand, by Fatou's lemma we have

$$\liminf_{T\to+\infty}\frac{1}{T}\int_0^T P_s^*\mu(B(0,R_0))\mathrm{d}s \geq \int_H \left(\liminf_{T\to+\infty}\frac{1}{T}\int_0^T P_s^*\delta_x(B(0,R_0))\mathrm{d}s\right)\mu(\mathrm{d}x)$$
$$\geq \int_H (1-\varepsilon)\mu(\mathrm{d}x) = 1-\varepsilon$$

for any $\mu \in \mathcal{M}_1^{B(0,r)}$. The proof is complete. □

Lemma 4. *(Concentrating at* 0*) Let* $(P_t)_{t\geq 0}$ *correspond to problem* (2.5)*. Then* $(P_t)_{t\geq 0}$ *is concentrating at* 0*.*

Proof. Consider first the deterministic equation

$$\mathrm{d}v^x(t) = [-\nu A v^x(t) + B(v^x(t), v^x(t))]\mathrm{d}t$$

with the initial condition $v^x(0) = x$. Then

$$\frac{1}{2}\frac{\mathrm{d}|v^x(t)|^2}{\mathrm{d}t} \leq -\nu k_0 |v^x(t)|^2$$

and consequently

$$|v^x(t)|^2 \to 0 \qquad \text{as } t \to +\infty$$

uniformly on bounded sets. Further, fix $\varepsilon > 0$ and $r > 0$. Let $t_0 > 0$ be such that $v^x(t_0) \in B(0, \varepsilon/2)$ for all $x \in B(0,r)$. We may show (see Theorem 8 in [3]) that the process corresponding to the considered model is stochastically stable (see also [14]), i.e. there exists $\eta > 0$ and the set $F_\eta = \{\omega \in \Omega : \sup_{0\leq t\leq t_0} |QW(t)(\omega)| \leq \eta\}$ such that

$$|u^x(t_0)(\omega) - v^x(t_0)| \leq \varepsilon/2 \qquad \text{for any } \omega \in F_\eta.$$

Since the process is degenerate, we have $\alpha := \mathbb{P}(F_\eta) > 0$. Consequently, we obtain

$$P_{t_0}^*\delta_x(B(0,\varepsilon)) \geq \mathbb{P}(\{\omega \in \Omega : u^x(t_0)(\omega) \in B(0,\varepsilon)\}) \geq \mathbb{P}(F_\eta) = \alpha$$

for arbitrary $x \in B(0,r)$. Since

$$P_{t_0}^*\mu(B(0,\varepsilon)) = \int_H P_{t_0}^*\delta_x(B(0,\varepsilon))\mu(\mathrm{d}x),$$

we obtain $P_{t_0}^*\mu(B(0,\varepsilon)) \geq \alpha$ for any $\mu \in \mathcal{M}_1^{B(0,r)}$. But $\varepsilon > 0$ and $r > 0$ were arbitrary and hence the concentrating property follows. □

We may formulate the main theorem of this part of our paper.

Theorem 4. *The semigroup* $(P_t)_{t\geq 0}$ *corresponding to problem* (2.5) *with* Q *satisfying condition (2.13) is asymptotically stable. In particular, it admits a unique invariant measure.*

Proof. From Lemma 2 it follows that the semigroup $(P_t)_{t\geq 0}$ satisfies the e–property. It is also averagely bounded and concentrating at 0, by Lemmas 3 and 4. Application of Theorem 1 finishes the proof. □

Remark: Observe that condition (2.13) implies that the system with not too much noise is stable even when the noise is added to the first mode only.

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