Shell models - the list of exercises

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Exercise 1. Prove that if X_t is a continuous non random function, then the stochastic integral $I_t(X) = \int_0^t X_s \mathrm{d}W_s$ is a Gaussian process.

Exercise 2. Let $\{W_t : t \geq 0\}$ be a one-dimensional Brownian motion relative to a filtration $\{\mathcal{F}_t : t \geq 0\}$. Let τ be a stopping time with respect to the given filtration with $\mathbf{E}\tau < \infty$. Prove the Wald Identities:

$$\mathbf{E}W_\tau = 0 \quad \text{and} \quad \mathbf{E}W_\tau^2 = \mathbf{E}\tau.$$

Exercise 3. Find the stochastic integral $\int_0^t W_s^n \mathrm{d}W_s$, where $\{W_t : t \geq 0\}$ is a one-dimensional Brownian motion.

Exercise 4. Let $\{W_t : t \geq 0\}$ be a -ne-dimensional Brownian motion. Prove that the process

$$X_t = (1-t)\int_0^t \frac{\mathrm{d}W_s}{1-s} \qquad 0 \leq t < 1$$

is the solution of the stochastic differential equation

$$\mathrm{d}X_t = \frac{X_t}{t-1}\mathrm{d}t + \mathrm{d}W_t, \quad 0 \leq t < 1,\ X_0 = 0.$$

Exercise 5: For every $n \geq 1$ let us define the Hermite polynomial $H_n(\lambda, x)$ by

$$H_n(\lambda, x) = \lambda^{n/2} H_n(x/\sqrt{\lambda}), \quad \text{where } x \in \mathbb{R} \text{ and } \lambda > 0.$$

Check that

$$\exp(tx - t^2\lambda/2) = \sum_{n=0}^{\infty} t^n H_n(\lambda, x).$$

Exercise 6: Let $\{W_t : t \geq 0\}$ be a one-dimensional Brownian motion. Show that the process $\{H_n(t, W_t) : t \geq 0\}$ is a martingale.

Exercise 7: Deduce the following explicit expression for the Hermite polynomials

$$H_n(x) = \sum_{k=0}^{[n/2]} \frac{(-1)^k x^{n-2k}}{k!(n-2k)!2^k}.$$

Exercise 8: Show that if Y is a random variable with distribution $N(0, \sigma^2)$, then

$$\mathbf{E}(H_{2m}(Y)) = \frac{(\sigma^2 - 1)^m}{2^m m!},$$

and $\mathbf{E}(H_n(Y)) = 0$ if n is odd.

Exercise 9: Let F and G be smooth random variables, and let $h \in H$ for some Hilbert space H with the scalar product $\langle \cdot, \cdot \rangle_H$. Show that

$$\mathbf{E}(G\langle DF, h\rangle_H) = \mathbf{E}(-F\langle DG, h\rangle_H + FGW(h)).$$