Shell models - exercises with hints

Tomasz Szarek

PWP Interdysciplinary Doctoral Studies in Mathematical Modeling
UDA-POKL.04.01.01-00-026/13-00
Projekt jest współfinansowany przez Unię Europejską w ramach Europejskiego Funduszu Społecznego

Exercise 1: (Gronwall's inequality) Let $f : [0, \infty) \to [0, \infty)$ be a continuous function such that

$$f(t) \leq A + B \int_0^t f(s)\mathrm{d}s \qquad \text{for } t \geq 0 \text{ and some } A, B > 0. \tag{1}$$

Then we have

$$f(t) \leq A \exp(Bt) \qquad \text{for } t \geq 0.$$

Solution: Consider the function $\varphi : [0, \infty) \to [0, \infty)$ defined by the formula:

$$\varphi(t) = \exp(-Bt) \int_0^t f(s)\mathrm{d}s \qquad \text{for all } t \geq 0.$$

Obviously φ is differentiable and

$$\varphi'(t) = -B \exp(-Bt) \int_0^t f(s)\mathrm{d}s + \exp(-Bt)f(t) \leq A \exp(-Bt),$$

by equation (1).

Consequently, integrating both sides of the above inequality and taking into account that $\varphi(0) = 0$ we obtain

$$\varphi(t) \leq -A/B(\exp(-Bt) - 1)$$

and

$$B \int_0^t f(s)\mathrm{d}s \leq -A(1 - \exp(Bt)).$$

This gives

$$A + B \int_0^t f(s)\mathrm{d}s \leq A \exp(Bt)$$

and consequently

$$f(t) \leq A \exp(Bt) \qquad \text{for } t \geq 0.$$

□

Exercise 2: Let $\mathbf{W} = \{W(h) : h \in H\}$ denote an isonormal Gaussian process associated with the Hilbert space H with the scalar product $\langle \cdot, \cdot \rangle_H$ defined on some probability space $(\Omega, \mathcal{F}, P)$, i.e. $\mathbf{W}$ is a centered Gaussian family of random variables such that $\mathbf{E}(W(h)W(g)) = \langle h, g \rangle_H$ for all $h, g \in H$. Prove that the mapping $h \to W(h)$ is linear.

Solution: For any $\lambda, \mu \in \mathbb{R}$ and $h, g \in H$ we have

$$\begin{aligned} \mathbf{E}((W(\lambda h + \mu g) - \lambda W(h) - \mu W(g))^2) &= \|\lambda h + \mu g\|_H^2 \\ &+ \lambda^2 \|h\|_H^2 + \mu^2 \|h\|_H^2 - 1\lambda \langle \lambda h + \mu g, \rangle_H \\ &- 2\mu \langle \lambda h + \mu g, g \rangle_H + 2\lambda\mu \langle h, g \rangle_H = 0. \end{aligned}$$

Hence $W(\lambda h + \mu g) = \lambda W(h) = \mu W(g)$ and we are done. □

Exercise 3: Let $H_n(x)$, $n \geq 1$, denote the nth *Hermite polynomial*, which is defined by

$$H_n(x) = \frac{(1)^n}{n!} e^{x^2/2} \frac{\mathrm{d}^n}{\mathrm{d}x^n}(e^{-x^2/2}), \quad \text{for } n \geq 1$$

and $H_0(x) = 1$. Show the following properties

$$H_n'(x) = H_{n-1}(x), \quad n \geq 1, \tag{2}$$

$$(n+1)H_{n+1}(x) = nH_n(x) - H_{n-1}(x), \quad n \geq 1, \tag{3}$$

$$H_n(-x) = (-1)^n H_n(x), \quad n \geq 1. \tag{4}$$

Solution: Observe that the coefficients of the expansion in powers of t of the function $F(x,t) = \exp(tx - t^2/2)$. In fact, we have

$$\begin{aligned} F(x,t) &= \exp(x^2/2 - (x-t)^2/2) \\ &= e^{x^2/2} \sum_{n=0}^{\infty} \frac{t^n}{n!} \left(\frac{\mathrm{d}^n}{\mathrm{d}x^n}(e^{-(x-t)^2/2} \right)_{|t=0} \\ &= \sum_{n=0}^{\infty} t^n H_n(x). \end{aligned}$$

Then (2) and (3) follow from $\frac{\partial F}{\partial x} = tF$, respectively, and $\frac{\partial F}{\partial t} = (x-t)F$, and (refe7) is a consequence of $F(-x,t) = F(x,-t)$. □

Exercise 4: Let X, Y be two random variables with joint Gaussian distribution such that $\mathbf{E}(X) = \mathbf{E}(Y) = 0$ and $\mathbf{E}(X^2) = \mathbf{E}(Y^2) = 1$. Then we have

$$\mathbf{E}(H_n(X)H_m(Y)) = 0 \qquad \text{if } n \neq m$$

and

$$\mathbf{E}(H_n(X)H_n(Y)) = \frac{1}{n!}(\mathbf{E}(XY))^n.$$

Solution: From the fact that X, Y are two random variables with joint Gaussian distribution we obtain for $s, t \in \mathbb{R}$

$$\mathbf{E}(\exp(sX - s^2/2)\exp(tY - t^2/2)) = \exp(stE(XY)).$$

at $s = t = 0$ in both sides of the above equality yields

$$\mathbf{E}(n!m!H_n(X)H_m(Y)) = \begin{cases} 0, & \text{if } n \neq m; \\ n!(\mathbf{E}(XY))^n, & \text{if } n = m. \end{cases}$$

□

Exercise 5: Let $\mathbf{W} = \{W(h) : h \in H\}$ denote an isonormal Gaussian process associated with the Hilbert space H with the scalar product $\langle \cdot, \cdot \rangle_H$ defined on some probability space $(\Omega, \mathcal{F}, P)$. Let

$$F = f(W(h_1), \ldots, W(h_n)),$$

where f belongs to the set of all infinitely continuously differentiable functions $f : \mathbb{R}^n \to \mathbb{R}$ such that f and all of its partial derivarives have polynmial growth ($f \in C_p^\infty(\mathbb{R}^n)$). The derivative of F is the H–valued random variable given by

$$DF = \sum_{i=1}^{n} \partial_i f(W(h_1), \ldots, W(h_n)) h_i.$$

Prove that

$$\mathbf{E}(\langle DF, h \rangle_H) = \mathbf{E}(FW(h)). \tag{5}$$

Solution: First notice that we can normalize equation (5) and assume that the norm of h is one. There exist orthonormal elements of H, $e_1, \ldots, e_n$, such that $h = e_1$ and F is a smooth random variable of the form

$$F = f(W(e_1), \ldots, W(e_n)),$$

where $f \in C_p^\infty(\mathbb{R}^n)$. Let $\phi(x)$ denote the density of the standard normal distribution on $\mathbf{R}^n$, that is,

$$\phi(x) = (2\pi)^{-n/2} \exp\left(\sum_{i=1}^{n} x_i^2/2 \right).$$

Then we have

$$\begin{aligned} \mathbf{E}(\langle DF, h \rangle_H) &= \int_{\mathbb{R}^n} \partial_1 f(x) \phi(x) \mathrm{d}x = \int_{\mathbb{R}^n} f(x) \phi(x) x_1 \mathrm{d}x \\ &= \mathbf{E}(FW(e_1)) = \mathbf{E}(FW(h)), \end{aligned}$$

which completes exercise 5. $\square$