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Shell models in hydrodynamics

Tomasz Szarek

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Basic definitions

Let H be a separable Hilbert space. Let $\mathcal{B}(H)$ be the space of all Borel subsets of H and let $B_b(H)$ (resp. $C_b(H)$) be the Banach space of all bounded, measurable (resp. continuous) functions on H equipped with the supremum norm $\|\cdot\|_\infty$. We denote by $C_b^1(H)$ the space of all bounded differentiable functions on H with bounded derivative.

Let $Z = ((Z^x(t))_{t\geq 0}, x \in H)$ be a Markov family taking values in H and let $(P_t)_{t\geq 0}$ be its transition semigroup.

We shall assume that the semigroup $(P_t)_{t\geq 0}$ is *Feller*, i.e. $P_t(C_b(H)) \subset C_b(H)$ and that the Markov family is *stochastically continuous*, which implies that: $\lim_{t\to 0+} P_t\psi(x) = \psi(x)$ for all $x \in H$ and $\psi \in C_b(H)$.

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We say that a transition semigroup $(P_t)_{t\geq 0}$ has the *e-property* at $x \in H$ if the family of functions $(P_t\psi)_{t\geq 0}$ is equicontinuous at x for any $\psi \in C_b^1(H)$.

The semigroup $(P_t)_{t\geq 0}$ has the e-property if the above condition holds at any $x \in H$.

Let $(P_t^*)_{t\geq 0}$ be the dual semigroup defined on the space of all Borel probability measures $\mathcal{M}_1$ by the formula

$$P_t^*\mu(B) := \int_H P_t \mathbf{1}_B \mathrm{d}\mu \quad \text{for } B \in \mathcal{B}(H).$$

Recall that $\mu_* \in \mathcal{M}_1$ is *invariant* for the semigroup $(P_t)_{t\geq 0}$ (or the Markov family $(Z^x(t))_{t\geq 0}$) if $P_t^*\mu_* = \mu_*$ for all $t \geq 0$.

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Stabilty results

A semigroup $(P_t)_{t\geq 0}$ is called *averagely bounded* if

$$\forall\ \epsilon > 0, r > 0 \,\exists\ R > 0 \,\forall\ \mu \in \mathcal{M}_1^{B(0,r)}$$

$$\limsup_{T\to\infty} \frac{1}{T}\int_0^T P_s^*\mu(B(0,R))\mathrm{d}s > 1-\epsilon.$$

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A semigroup $(P_t)_{t\geq 0}$ is *concentrating* at z if

$$\forall\ \varepsilon > 0, r > 0\ \exists\ \alpha > 0\ \forall\ \mu_1, \mu_2 \in \mathcal{M}_1^{B(0,r)}\ \exists\ t > 0$$

$$P_t^* \mu_i(B(z, \varepsilon)) \geq \alpha \text{ for } i = 1, 2.$$

A semigroup $(P_t)_{t\geq 0}$ is *asymptotically stable* if there exists an invariant measure $\mu_* \in \mathcal{M}_1$ and $\text{w-lim}_{t\to\infty} P_t^* \mu = \mu_*$ as $t \to \infty$ for any $\mu \in \mathcal{M}_1$.

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Stabilty results

Theorem 1. (H. Bessaih, R. Kapica and T.S., 2010)

Let $(P_t)_{t\geq 0}$ be averagely bounded and concentrating at some $z \in H$. If $(P_t)_{t\geq 0}$ satisfies the e–property at z, then it is asymptotically stable.

The stochastic Goy model

Let $\{u_n\}_{n\geq -1}$ be an infinite sequence of real valued functions satisfying

$$\mathrm{d}u_n(t)+\nu k_n^2 u_n(t)\mathrm{d}t = [k_{n-1}u_{n-1}^2(t)-k_n u_n(t)u_{n+1}(t)]\mathrm{d}t+\sigma_n \mathrm{d}w_n$$

for $n \geq 1$ with the initial conditions

$$u_{-1}(t) = u_0(t) = 0 \quad \text{and} \quad u_n(0) = \xi_n.$$

Here $k_n = k_0 2^n$, $k_0 > 1$ and $\nu > 0$.
Moreover $(w_n(t))_{n\geq 1}$ denotes a sequence of independent Brownian motions on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. we will assume that there exists n_0 such that $\sigma_n \neq 0$ for $n = 1, \ldots, n_0$ and $\sigma_n = 0$ for $n > n_0$.

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The stochastic Goy model

Let us introduce the following space

$$H = \{u = (u_1, u_2, \dots) \in \mathbb{R}^\infty : \sum_{n=1}^{\infty} |u_n|^2 < \infty\}.$$

It is a real Hilber space with the inner product

$$\langle u, v \rangle_H = \sum_{n=1}^{\infty} u_n v_n.$$

We introduce now the Hilbert subspace

$$V = \{u \in H : \sum_{n=1}^{\infty} k_n^2 |u_n|^2 < \infty\}.$$

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We introduce the bilinear operator $B(\cdot,\cdot) : H \times V \to H$ defined by the formula

$$B(u,v)_n = k_{n-1}u_{n-1}v_{n-1} - k_n u_n v_{n+1}.$$

We check that $B(u,v) \in H$ for $(u,v) \in V \times H$. Moreover, there is a constant $C > 0$ such that

$$\|B(u,v)\|_H \leq C\|u\|_H\|v\|_V \quad \text{and} \quad \|B(u,v)\|_H \leq C\|u\|_V\|v\|_H$$

for u and v in the proper spaces.
Finally, we have $\langle B(u,v), v\rangle = 0$ for $u \in H, v \in V$.

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Let

$$D(A) = \{u \in H : \sum_{n=1}^{\infty} k_n^4 |u_n|^2 < \infty\}.$$

On this space we define the linear operator $A : D(A) \subset H \to H$ as

$$(Au)_n = k_n^2 u_n, \qquad \text{for any } u \in D(A).$$

The operator A is selfadjoint and strictly positive definite

$$\langle Au, u \rangle_H \geq k_0 \|u\|_H^2, \qquad \text{for any } u \in D(A).$$

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A general model

Consider the equation

$$\mathrm{d}u(t) = [-\nu Au(t) + B(u(t), u(t))]\mathrm{d}t + Q\mathrm{d}W(t), \quad (1)$$

where $(W(t))_{t\geq 0}$ is a certain cylindrical Wiener process on a filtered space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. Further, $Q = (q_{i,j})_{i,j\geq 1}$ is a diagonal matrix and there exists n_0 such that $q_{i,i} \neq 0$ for $i \leq n_0$ and $q_{i,i} = 0$ for $i > n_0$.

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where $(W(t))_{t\geq 0}$ is a certain cylindrical Wiener process on a filtered space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. Further, $Q = (q_{i,j})_{i,j\geq 1}$ is a diagonal matrix and there exists n_0 such that $q_{i,i} \neq 0$ for $i \leq n_0$ and $q_{i,i} = 0$ for $i > n_0$.

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A general model

We impose an initial condition given by an $\mathcal{F}_0-$ measurable random vector $u_0 : \Omega \to H$. As usual, we interpret the equation in the integral sense

$$\langle u(t), \varphi \rangle_H + \int_0^t \nu \langle u(s), A\varphi \rangle_H \mathrm{d}s - \int_0^t \langle B(u(s), \varphi), u(s) \rangle_H \mathrm{d}s$$
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A general model

We set

$$P_t f(x) = \mathbb{E} f(u^x(t)) \quad \text{for any } f \in B_b(H),$$

where $(u^x(t))_{t\geq 0}$ is the solution to (1) with $u^x(0) = x$.[1]
It is easy to see that $(P_t)_{t\geq 0}$ satisfies the Feller property.

[1] The existence of solution was proved by D. Barbato, M. Barsanti, H. Bessaih, F. Flandoli, (2006), *Some rigorous results on a stochastic Goy model,* Journal of Statistical Physics, Vol. 125, No 3, 677-716

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E-property

The crucial role is played by the following lemma proved using the Malliavin calculus.

Lemma

Let $(P_t)_{t\geq 0}$ correspond to problem (1). If

$$n_0 > \log_2(2C^2 \max \sigma_i^2/\nu^3 + \mathrm{Tr}\, Q^2/(2 \max \sigma_i^2))/2,$$

then the semigroup $(P_t)_{t\geq 0}$ satisfies the e–property. In particular, for any $\psi \in C_b^1(H)$ and $R > 0$ there exists a positive constant C such that

$$\sup_{t\geq 0} \sup_{\|x\|_H \leq R} \|DP_t f(x)\|_H \leq C\|f\|_{C_b^1(H)}.$$

(Here Df denotes the Fréchet derivative of a given function $f \in C_b^1(H)$).

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Boundedness of $(P_t)_{t\geq 0}$

It may be shown that for any $r > 0$ there exists $C(r) > 0$ such that

$$\frac{1}{T}\mathbb{E}\int_0^T \|u^x(s)\|_H^2 \mathrm{d}s \leq C(r) \quad \text{for } x \in B(0,r).$$

Fix $r > 0$. By Tshebishev's inequality for any $\epsilon > 0$ there exists $R_0 > 0$ such that

$$\liminf_{T\to\infty} \frac{1}{T}\int_0^T P_s^*\delta_x(B(0,R))\mathrm{d}s > 1-\epsilon$$

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The Fatou lemma gives

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Concentrating of $(P_t)_{t\geq 0}$ at 0

We consider the deterministic problem

$$\mathrm{d}v^x(t) = -\nu A v^x(t)\mathrm{d}t + B(v^x(t), v^x(t))\mathrm{d}t \qquad \text{for } t \geq 0$$

with the initial condition $v^x(0) = x$. We easily check that $\{0\}$ is its global attractor.

We show that the family $(u^x(t))_{t\geq 0}$, $x \in H$ is *stochastically stable*, i.e.,

$$\forall\, \varepsilon,\ R,\ t > 0: \qquad \inf_{x\in B(0,R)} \mathbb{P}\left(\|u^x(t) - v^x(t)\|_H < \varepsilon\right) > 0.$$

Putting together two above facts we easily obtain that $(P_t)_{t\geq 0}$ is concentrating at 0.

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Stability of $(P_t)_{t\geq 0}$

Since $(P_t)_{t\geq 0}$ satisfies the e-property and is averagely bounded and concentrating at 0, from Theorem 1 follows:

Theorem 2. (H. Bessaih, R. Kapica and T.S., 2010)

The semigroup $(P_t)_{t\geq 0}$ corresponding to problem (1) is asymptotically stable provided

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