The Weil-Steinberg character of finite classical groups

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Let $q = p^\alpha$ be a prime power and let G be one of the following groups: $\mathrm{GL}(n,q)$, $\mathrm{U}(n,q)$, $\mathrm{Sp}(n,q)$. We write accordingly $G(m,q)$ for $\mathrm{GL}(m,q)$, $\mathrm{U}(m,q)$ or $\mathrm{Sp}(m,q)$ respectively.

- $\mathrm{U}(n,q) \leq \mathrm{GL}(n,q^2)$, isometries of the hermitean form $J_n = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}$.
- $\mathrm{Sp}(n,q) \leq \mathrm{GL}(n,q)$ ($n = 2m$ even), isometries of the symplectic form $\begin{pmatrix} & J_m \\ -J_m & \end{pmatrix}$.

Let δ be 1 in case G is symplectic or linear and 2 in case G is unitary.
Each of the above groups is a group with (split) BN-pair:

- B is the intersection of the group of upper triangular matrices of $\mathrm{GL}(n, q^\delta)$ with G
- N is the intersection of the group of monomial matrices of $\mathrm{GL}(n, q^\delta)$ with G.

The group $W = N/(B \cap N)$ is called the WEYL **group**. It is isomorphic to S_n in case G is linear and otherwise to $\mathrm{C}_2 \wr \mathrm{S}_m$, where $m = \lfloor \frac{n}{2} \rfloor$.

A subgroup of $\mathrm{GL}(n, q)$ of the form

$$L := \begin{pmatrix} \mathrm{GL}(n_1, q) & & & \\ & \mathrm{GL}(n_2, q) & & \\ & & \ddots & \\ & & & \mathrm{GL}(n_k, q) \end{pmatrix}$$

is called a Levi **subgroup**.
This Levi subgroup is contained in the **parabolic** subgroup

$$P := \begin{pmatrix} \mathrm{GL}(n_1, q) & & & * \\ & \mathrm{GL}(n_2, q) & & \\ & & \ddots & \\ & & & \mathrm{GL}(n_k, q) \end{pmatrix}.$$

The LEVI subgroup L is a complement of

$$V := \begin{pmatrix} \mathrm{I}_{n_1} & & & * \\ & \mathrm{I}_{n_2} & & \\ & & \ddots & \\ & & & \mathrm{I}_{n_k} \end{pmatrix}$$

in P, i.e. $P = L \ltimes V$ (LEVI **decomposition**).

For unitary and symplectic groups the LEVI subgroups are of the form

$$\left\{ \begin{pmatrix} A_1 & & & & & & \\ & \ddots & & & & & \\ & & A_d & & & & \\ & & & B & & & \\ & & & & J\overline{A_d}^{-\mathrm{tr}}J & & \\ & & & & & \ddots & \\ & & & & & & J\overline{A_1}^{-\mathrm{tr}}J \end{pmatrix} \mid A_i \in \mathrm{GL}(n_i, q^\delta), B \in G(k,q) \right\},$$

where $\overline{}$ is the FROBENIUS automorphism of $\mathbb{F}_{q^2}/\mathbb{F}_q$, i.e. $a \mapsto a^q$. Analogously, this group is contained in a parabolic subgroup and we have a LEVI decomposition.

Definition

Let L be a LEVI subgroup, P the corresponding parabolic subgroup.

1. The map $\mathrm{R}_L^G = \mathrm{Ind}_P^G \circ \mathrm{Infl}_L^P$ is called HARISH-CHANDRA **induction**.
2. We denote the adjoint map of R_L^G by T_L^G (**truncation**).

Remark

Let M be a LEVI *subgroup of G. The* LEVI *subgroups of M are exactly the* LEVI *subgroups of G contained in M. Define R_L^M and T_L^M analogously to the previous definition.*

Definition

Let L be a Levi subgroup. A character $\chi \in \mathrm{Irr}(L)$ is called **cuspidal** if $\mathrm{T}_M^L(\chi) = 0$ for all proper Levi subgroups M of L.

Definition

Let L be a Levi subgroup and let ϑ be a cuspidal character of L. We define

1. $\mathrm{W}_G(L) := (\mathrm{N}_G(L) \cap N)L/L$.
2. $\mathrm{W}_G(L, \vartheta) := \{w \in \mathrm{W}_G(L) \mid {}^w\vartheta = \vartheta\}$.

The latter group is called the **relative** Weyl **group** of ϑ.

Remark

Relative WEYL *groups are isomorphic to*

- *direct products of symmetric groups in case G is linear.*
- *direct products of wreath products of C_2 with symmetric groups in case G is unitary or symplectic.*

Theorem (Harish-Chandra parametrization)

Let $L \leq G$ be a LEVI *subgroup and let $\vartheta \in \mathrm{Irr}(L)$ be cuspidal. The set*

$$\mathcal{E}_\vartheta := \{\chi \in \mathrm{Irr}(G) \mid (\mathrm{T}_L^G(\chi), \vartheta) \neq 0\}$$

is called a HARISH-CHANDRA *series. Every irreducible character is contained in a unique series.*
The irreducible characters in the series $\mathcal{E}_\vartheta$ are in bijection to the irreducible characters of the relative WEYL *group of ϑ in G.*

- For $G = \mathrm{GL}(n, q)$ the WEIL character of G is the character

$$\omega\colon g \mapsto \left|\{v \in \mathbb{F}_q^n \mid gv = v\}\right|$$

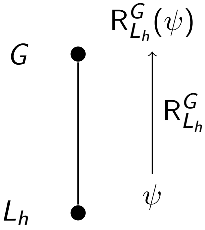
  of degree q^n. Otherwise it is a "square root" of the permutation character on the natural module.
- The STEINBERG character is an irreducible character of degree $|G|_p$ $(q = p^\alpha)$. We denote this character by St.
- The WEIL-STEINBERG character is the product of the WEIL and STEINBERG character.

Theorem (Hiss, Zalesskii)

$$\omega \cdot \mathrm{St} = \sum_{h=0}^{m} \mathrm{R}_{L_h}^{G}(\mathrm{St}_h^{-} \boxtimes \gamma'_{m-h}),$$

where $\mathrm{St}^{-} = \mathrm{St} \cdot 1^{-}$ *and*

- *(G is linear)* $m = n$, $L_h = \begin{pmatrix} \mathrm{GL}(h,q) & \\ & \mathrm{GL}(m-h,q) \end{pmatrix}$, γ'_{m-h} *is the* GELFAND-GRAEV *character.*
- *(otherwise)* $m = \lfloor \frac{n}{2} \rfloor$,
  $L_h = \{ \begin{pmatrix} A & & \\ & B & \\ & & J_h \overline{A}^{-\mathrm{tr}} J_h \end{pmatrix} \mid A \in \mathrm{GL}(h, q^{\delta}), B \in G(m-h, q) \}$ *and* γ'_{m-h} *is a "truncated"* GELFAND-GRAEV *character.*


$R^G_{L_h}(\psi)$
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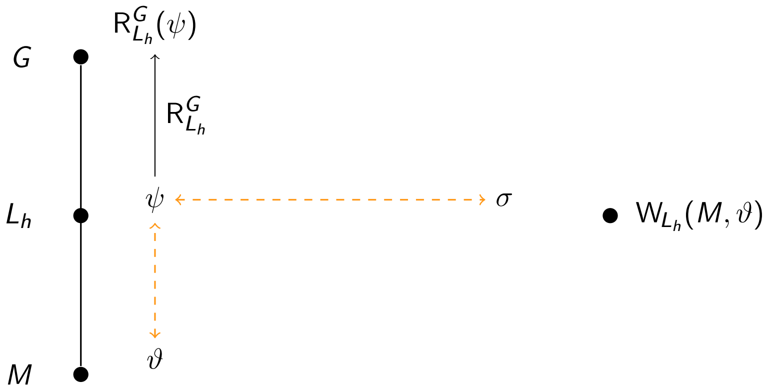
$\mathrm{R}_{L_h}^G(\psi)$
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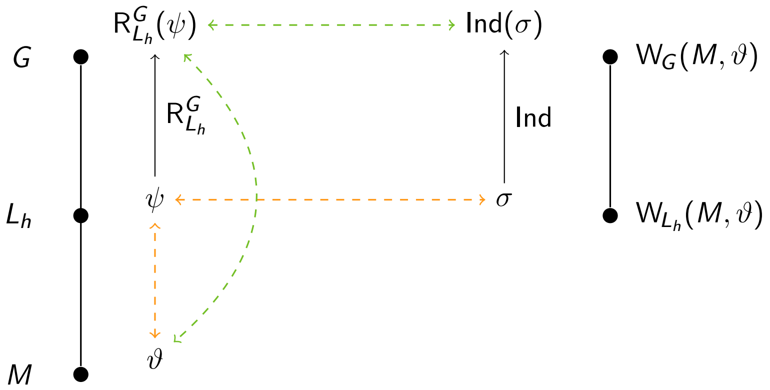
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$W_{L_h}(M, \vartheta)$

Linear:

$$\omega \cdot \mathrm{St} = \sum_{r=0}^{n} \sum_{\substack{s' \in \mathcal{S}_{n-r,q} \\ \mathrm{E}_s(-1)=0}} \sum_{i=0}^{\lfloor \frac{n-2}{2} \rfloor} (n-r-2i+1) \eta_{D_{n-r} \times M_{s'}, 1^-_{n-r} \boxtimes \tau_{s'}; \zeta^{[n-r-i,i]'} \boxtimes \varepsilon_{s'}}$$

Unitary and symplectic:

$$\omega \cdot \mathrm{St} = \sum_{h=0}^{m} \sum_{\substack{s \in \mathcal{S}_{n-2h,q} \\ \mathrm{E}_s(-1)=0}} \sum_{i=0}^{h} \eta_{D_h \times M_s, 1^-_h \boxtimes \tau_s; \xi^{\gamma_i} \boxtimes \varepsilon_s}$$