Secantoptics of ovals and their properties

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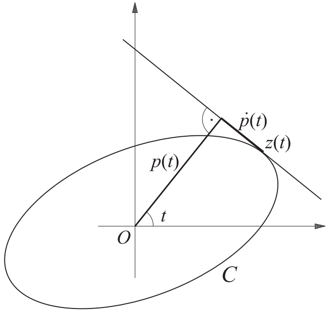
Primary information

Definition

An oval is a plane, closed curve given by the equation

$$z(t) = p(t)e^{it} + \dot{p}(t)ie^{it} \quad \text{for} \quad t \in [0, 2\pi), \tag{1}$$

where $p(t)$, called the support function of an oval is C^3 and the radius of curvature $R(t) = p(t) + \ddot{p}(t)$ is positive for all $t \in [0, 2\pi)$.



Primary information

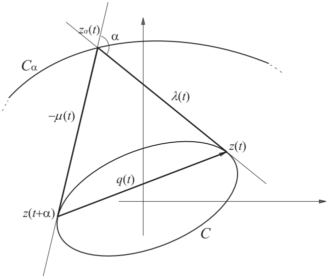
Definition [Philippe de La Hire]

An α-isoptic of a closed, convex curve is composed of those points in the plane from which the curve is seen under a fixed angle $\pi - \alpha$.

Theorem [W. Cieślak, S. Góźdź, A. Miernowski, W. Mozgawa]

The equation of an isoptic C_α of the curve C is given by

$$z_\alpha(t) = p(t)e^{it} + \{-p(t)\cot\alpha + \frac{1}{\sin\alpha}p(t+\alpha)\}ie^{it}, \qquad t \in [0, 2\pi). \qquad (2)$$

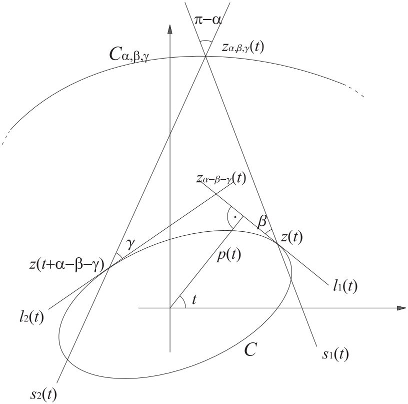


Definition of a secantoptic

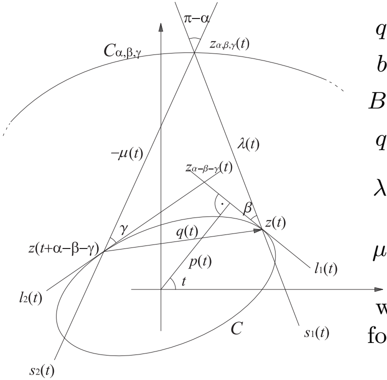
Let C be an oval and let $\beta \in [0, \pi)$, $\gamma \in [0, \pi - \beta)$ and $\alpha \in (\beta + \gamma, \pi)$ be fixed angles.

Definition

The set of intersection points $z_{\alpha,\beta,\gamma}(t)$ of $s_1(t)$ and $s_2(t)$ for $t \in [0, 2\pi]$ form a curve which we call *a secantoptic* $C_{\alpha,\beta,\gamma}$ *of an oval* C.



Definition of a secantoptic



We introduce the following notation

$$
\begin{aligned}
q(t) &= z(t) - z(t+\alpha-\beta-\gamma),\\
b(t) &= [q(t), e^{it}],\\
B(t) &= [q(t), ie^{it}],\\
q(t) &= (B(t) - ib(t))e^{it},\\
\lambda(t) &= \frac{b(t)\sin(\alpha-\beta) - B(t)\cos(\alpha-\beta)}{\sin\alpha},\\
\mu(t) &= -\frac{b(t)\sin\beta + B(t)\cos\beta}{\sin\alpha},
\end{aligned}
$$

where $[v, w] = ad - bc$
for $v = a + bi$ i $w = c + di$.

Definition of a secantoptic

Theorem

Let C be an oval and let $\beta \in [0, \pi)$, $\gamma \in [0, \pi - \beta)$ and $\alpha \in (\beta + \gamma, \pi)$ be fixed angles. Then the parametrization of secantoptic $C_{\alpha,\beta,\gamma}$ of oval C is

$$z_{\alpha,\beta,\gamma}(t) = (p(t) + \lambda(t)\sin\beta + i(\dot{p}(t) + \lambda(t)\cos\beta))e^{it} \quad for \quad t \in [0, 2\pi).$$

The equation of a secantoptic $C_{\alpha,\beta,\gamma}$ of an oval C in terms of the support function is

$$\begin{aligned} z_{\alpha,\beta,\gamma}(t) = & \Big(\sin(\alpha-\beta)(p(t)\cos\beta - \dot{p}(t)\sin\beta) + \\ & + \sin\beta(p(t+\alpha-\beta-\gamma)\cos\gamma + \dot{p}(t+\alpha-\beta-\gamma)\sin\gamma) + \\ & + i\big(-\cos(\alpha-\beta)(p(t)\cos\beta - \dot{p}(t)\sin\beta) + \\ & + \cos\beta(p(t+\alpha-\beta-\gamma)\cos\gamma + \dot{p}(t+\alpha-\beta-\gamma)\sin\gamma)\big)\Big)\frac{e^{it}}{\sin\alpha}. \end{aligned}$$

Diffeomorphism connected to secantoptics

Let C be a fixed oval. We denote by $e(C)$ the exterior of C and by ζ a half line from $z(0)$ in direction $ie^{-i\beta}$. We define a mapping

$$F_{\beta,\gamma} : (\beta + \gamma, \pi) \times (0, 2\pi) \mapsto e(C) \setminus \zeta \tag{3}$$

by the formula

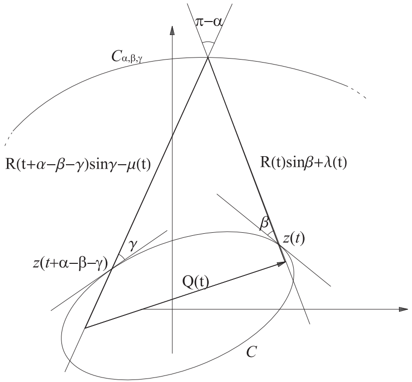
$$F_{\beta,\gamma}(\alpha, t) = z_{\alpha,\beta,\gamma}(t). \tag{4}$$

The jacobian $J(F_{\beta,\gamma})$ of $F_{\beta,\gamma}$ at (α, t) is given by

$$J(F_{\beta,\gamma}) = \frac{1}{\sin\alpha}(R(t + \alpha - \beta - \gamma)\sin\gamma - \mu(t))(R(t)\sin\beta + \lambda(t)) > 0. \tag{5}$$

Diffeomorphism connected to secantoptics

$$
\begin{aligned}
J(F_{\beta,\gamma}) &= \frac{1}{\sin\alpha}(R(t+\alpha-\beta-\gamma)\sin\gamma-\mu(t))(R(t)\sin\beta+\lambda(t)) > 0 \\
Q(t) &= (B(t)+R(t+\alpha-\beta-\gamma)\sin\gamma\sin(\alpha-\beta)-R(t)\sin^2\beta+ \\
&+ i(-b(t)-R(t+\alpha-\beta-\gamma)\sin\gamma\cos(\alpha-\beta)-R(t)\sin\beta\cos\beta))e^{it}
\end{aligned}
$$



Additional information

Definition[R. Réaumur]

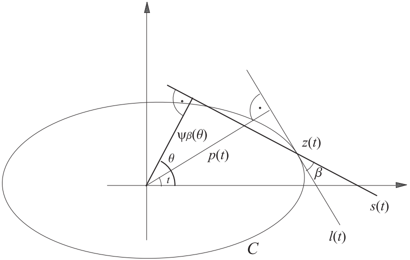
An evolutoid of angle δ of a curve $f(s)$ is the envelope of the lines making a fixed angle δ with the normal vector at $f(s)$.

The envelope Γ_β of the family of the secants of oval C obtained by rotation the tangent line $l(t)$ about the tangency point $z(t)$ through angle β can be parametrized by

$$z^\beta(t) = \psi_\beta(t)e^{it} + \dot{\psi}_\beta(t)ie^{it}, \quad (6)$$

where

$$\psi_\beta(\theta) = p(\theta - \beta)\cos\beta + \dot{p}(\theta - \beta)\sin\beta, \quad \theta \in [0, 2\pi). \quad (7)$$



Additional information

Definition[R. Langevin, G. Levitt, H. Rosenberg, Y. Martinez-Maure]

A hedgehog Γ is a curve which can be parametrized by the formula

$$z(t) = \psi(t)e^{it} + \dot{\psi}(t)ie^{it}, \tag{8}$$

where $h(\cos t, \sin t) = \psi(t)$, $h \in C^2(\mathbb{S}^1, \mathbb{R})$ is called the support function of Γ. The hedgehog is the envelope of the family of lines given by the equation

$$x\cos t + y\sin t = p(t). \tag{9}$$

Since $\psi_\beta(t)$ is at least of class $C^2(\mathbb{R})$, the curve Γ_β is a hedgehog.

Corollary

Any evolutoid of an oval is a hedgehog.

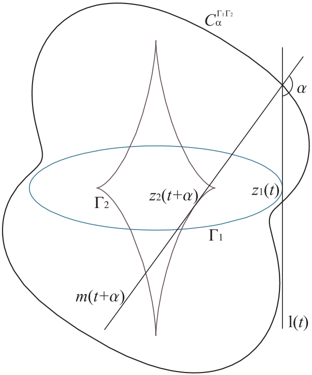
Isoptics of pairs of hedgehogs

Definition

Let

$$\Gamma_1 : \; z_1(t) = \psi_1(t)e^{it} + \dot{\psi}_1(t)ie^{it},$$
$$\Gamma_2 : \; z_2(t) = \psi_2(t)e^{it} + \dot{\psi}_2(t)ie^{it}.$$

be two hedgehogs and $\alpha \in (0, \pi)$ be fixed angle. The set of intersection points $z_\alpha^{\Gamma_1\Gamma_2}(t)$ of tangent lines $l(t)$ and $m(t+\alpha)$ for $t \in [0, 2\pi)$ form a curve which we call an *α-isoptic* $C_\alpha^{\Gamma_1\Gamma_2}$ of the pair Γ_1 and Γ_2.



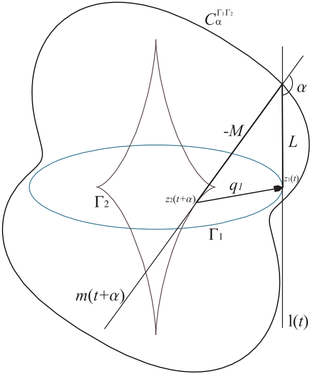
Let

$$q_1(t) = M(t)ie^{i(t+\alpha)} - L(t)ie^{it}, \tag{10}$$

where

$$L(t) = -\dot{\psi}_1(t) - \psi_1(t)\cot\alpha + \psi_2(t+\alpha)\frac{1}{\sin\alpha}, \tag{11}$$

$$M(t) = -\psi_1(t)\frac{1}{\sin\alpha} - \dot{\psi}_2(t+\alpha) + \psi_2(t+\alpha)\cot\alpha. \tag{12}$$



Theorem

Let

$$\Gamma_1 : \ z_1(t) = \psi_1(t)e^{it} + \dot{\psi}_1(t)ie^{it},$$

$$\Gamma_2 : \ z_2(t) = \psi_2(t)e^{it} + \dot{\psi}_2(t)ie^{it}.$$

be two hedgehogs and $\alpha \in (0, \pi)$ *be fixed angle. Then a parametrization of isoptic* $C_\alpha^{\Gamma_1\Gamma_2}$ *is given by*

$$z_\alpha^{\Gamma_1\Gamma_2}(t) = \psi_1(t)e^{it} + (\psi_2(t+\alpha)\frac{1}{\sin\alpha} - \psi_1(t)\cot\alpha)ie^{it}, \tag{13}$$

where $t \in [0, 2\pi)$.

Let

$$\rho_1(t) = \psi_1(t) + \dot{\psi}_2(t+\alpha)\frac{1}{\sin\alpha} - \dot{\psi}_1(t)\cot\alpha, \tag{14}$$

then

$$\dot{z}_\alpha^{\Gamma_1\Gamma_2}(t) = -L(t)e^{it} + \rho_1(t)ie^{it}. \tag{15}$$

Remark

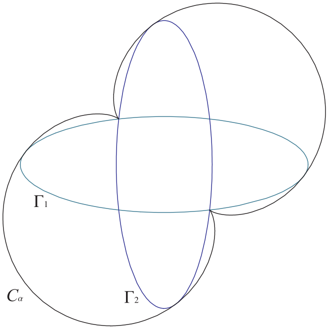
Note that

$$\left|\dot{z}_{\alpha}^{\Gamma_1\Gamma_2}(t)\right|^2 = \frac{1}{\sin^2\alpha}|q_1(t)|^2, \qquad (16)$$

hence $C_{\alpha}^{\Gamma_1\Gamma_2}$ can be a nonregular curve for $z_1(t) = z_2(t+\alpha)$ for some $t \in [0, 2\pi)$, then $|\dot{z}_{\alpha}^{\Gamma_1\Gamma_2}(t)| = 0$.

Let

$$\Gamma_1 : \frac{x^2}{9^2} + \frac{y^2}{3^2} = 1, \qquad \Gamma_2 : \frac{x^2}{3^2} + \frac{y^2}{9^2} = 1, \qquad \alpha = 1.3494818844471053.$$



Secantoptic as isoptic of pair of evolutoids

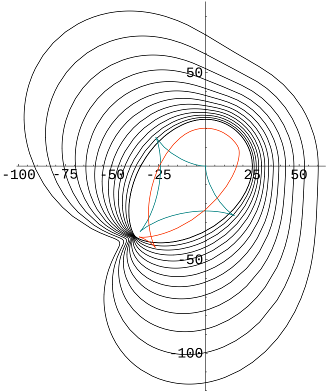
Consider two evolutoids of an oval C

$$\Gamma_{-\beta}: \ \psi_{-\beta}(t) = p(t+\beta)\cos\beta - \dot{p}(t+\beta)\sin\beta,$$

$$\Gamma_{\gamma}: \ \psi_{\gamma}(t) = p(t-\gamma)\cos\gamma + \dot{p}(t-\gamma)\sin\gamma.$$

The equation of the isoptic $C_\alpha^{\Gamma_{-\beta}\Gamma_\gamma}$, where $\beta \in [0,\pi)$, $\gamma \in [0, \pi-\beta)$ and $\alpha \in (\beta+\gamma, \pi)$ is given by

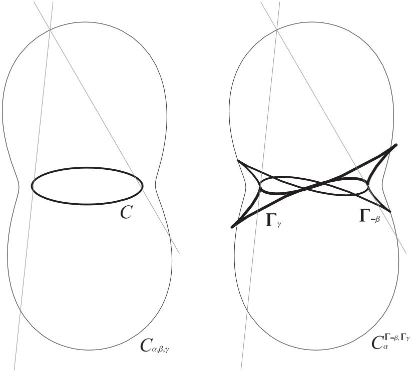
$$z_\alpha^{\Gamma_{-\beta}\Gamma_\gamma}(t) = \psi_{-\beta}(t)e^{it} + \left(\psi_\gamma(t+\alpha)\frac{1}{\sin\alpha} - \psi_{-\beta}(t)\cot\alpha\right) ie^{it}.$$



Secantoptic as isoptic of pair of evolutoids

Theorem

The isoptic $C_\alpha^{\Gamma_{-\beta}\Gamma_\gamma}$ and the secantoptic $C_{\alpha,\beta,\gamma}$ of a given oval C coincide if $\beta \in [0,\pi)$, $\gamma \in [0, \pi-\beta)$ and $\alpha \in (\beta+\gamma, \pi)$.

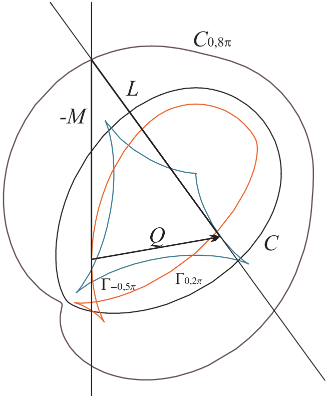


Secantoptic as isoptic of pair of evolutoids

Theorem

If $C_{\alpha,\beta,\gamma}$ *is the secantoptic of the oval* C*, then*

$$\begin{aligned} L(t) &= R(t)\sin\beta + \lambda(t), \\ M(t) &= \mu(t) - R(t+\alpha-\beta-\gamma)\sin\gamma, \\ Q(t) &= M(t)ie^{i(t+\alpha-\beta)} - L(t)ie^{i(t-\beta)} = q_1(t). \end{aligned}$$



The curvature of a secantoptic of an oval

Theorem

Let C be an oval with the support function $p(t) \in C^3$ and let $C_{\alpha,\beta,\gamma}$ be a secantoptic of C for $\alpha \in (0, \pi - \beta - \gamma)$, where $\beta \in [0, \pi)$, $\gamma \in [0, \pi - \beta)$. The curvature of the secantoptic $C_{\alpha,\beta,\gamma}$ is given by

$$\kappa(t) = \frac{\sin\alpha}{|Q(t)|^3}\left(2|Q(t)|^2 - [Q(t), \dot{Q}(t)]\right). \tag{17}$$

where $t \in [0, 2\pi)$.

Theorem

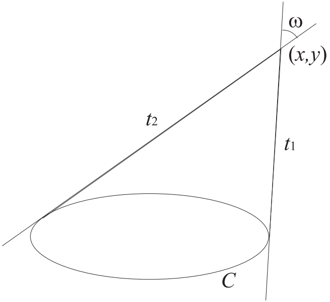
The secantoptic $C_{\alpha,\beta,\gamma}$ of an oval C is convex if and only if

$$[Q(t), \dot{Q}(t)] < 2|Q(t)|^2 \quad \text{dla} \quad t \in [0, 2\pi). \tag{18}$$

Crofton integral formula(1868)

Let Ω denote the exterior of closed, convex curve C. Then

$$\iint_{\Omega} \frac{\sin \omega}{t_1 t_2} dx dy = 2\pi^2.$$



Crofton-type integral formulas for secantoptics

Theorem

Let $\beta \in [0,\pi)$ and $\gamma \in [0,\pi-\beta)$. Consider secantoptics $C_{\alpha,\beta,\gamma}$ of an oval C, where the angle α changes in $(\beta+\gamma,\pi)$. Let Ω denotes the exterior of an oval C and let

$$\begin{aligned}\omega &= \pi-\alpha,\\ t_1 &= L(t),\\ t_2 &= -M(t).\end{aligned}$$

Then

$$\iint\limits_{\Omega} \frac{\sin\omega}{t_1 t_2}dxdy = 2\pi^2 - 2\pi(\beta+\gamma). \tag{19}$$

Let us recall, that $\int_0^{2\pi} p(t)dt$ means the length of a given convex curve C, which we denote by L_C.

Definition [Y. Martinez-Maure]

The algebraic length of a hedgehog Γ is

$$L_\Gamma = \int\limits_0^{2\pi} \psi(t)dt, \tag{20}$$

where $\psi(t)$ is the support function of Γ.

Hence, for evolutoids of an oval C we get

$$L_{\Gamma_{-\beta}} = L_C \cos\beta \qquad \text{and} \qquad L_{\Gamma_\gamma} = L_C \cos\gamma. \tag{21}$$

Crofton-type integral formulas for secantoptics

Theorem

Let $\beta \in [0, \pi)$*,* $\gamma \in [0, \pi - \beta)$ *and consider secantoptics* $C_{\alpha,\beta,\gamma}$ *of an oval* C*, for* α *in* $(\beta + \gamma, \pi)$*. Let*

$$\begin{aligned} \omega &= \pi - \alpha, \\ \tau_1 &= L(t), \\ \tau_2 &= -M(t). \end{aligned}$$

Then

$$\iint\limits_{\Omega} \frac{\sin^2 \omega}{\tau_1} dxdy = L_{\Gamma_{-\beta}}(\pi - (\beta + \gamma)) + L_{\Gamma_\gamma} \sin(\beta + \gamma)$$

and

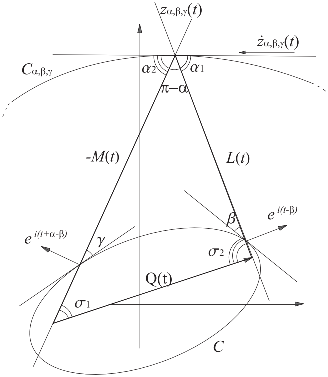
$$\iint\limits_{\Omega} \frac{\sin^2 \omega}{\tau_2} dxdy = L_{\Gamma_\gamma}(\pi - (\beta + \gamma)) + L_{\Gamma_{-\beta}} \sin(\beta + \gamma).$$

The sine theorem for secantoptics

Theorem

The secantoptic $C_{\alpha,\beta,\gamma}$ of an oval C in $z_{\alpha,\beta,\gamma}(t)$ has the following property

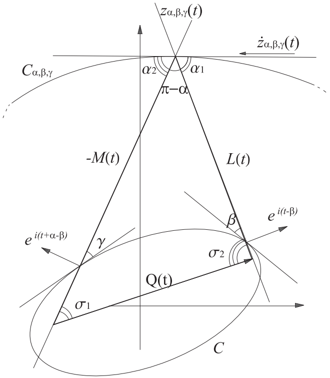
$$\frac{|Q(t)|}{\sin\alpha} = \frac{L(t)}{\sin\alpha_1} = \frac{-M(t)}{\sin\alpha_2}. \tag{22}$$

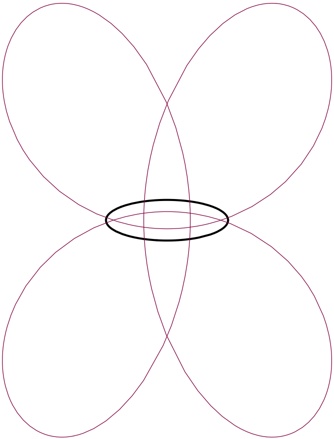


The sine theorem for secantoptics

Corollary

If α_1 and α_2 are angles formed by the tangent to the secantoptic $C_{\alpha,\beta,\gamma}$ of an oval C in $z_{\alpha,\beta,\gamma}(t)$ and lines s_1 and s_2 , and if σ_1 and σ_2 are angles formed by the vector $Q(t)$ and lines $s_1(t)$ and $s_2(t)$, then $\alpha_1 = \sigma_1$ i $\alpha_2 = \sigma_2$.





Thank you for your attention.