Karl Weierstraß
(1815 – 1897)

Karl Weierstrass and some basic notions of the calculus

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Being here in Braniewo where Weierstrass was once a teacher, I can’t resist to start with a little story from that time. The director of the Catholic Gymnasium in Braunsberg where Weierstrass was teaching during the years 1848 until 1855 reports:

One morning, when Weierstrass had to teach the highest class („Prima“), he didn’t show up. The director looked for him and found him sitting in his apartment totally absorbed in his work. He had been working all night and had not realized that morning had begun. He told the director that he cannot interrupt his work, for he is on the track of an important discovery that will surprise the experts.

As it is told, the director gave the lessons instead of the teacher. Perhaps Weierstrass was working on Abelian functions. Actually, his first publication on the theory of those functions appeared in the so-called *Schulprogramm* (school prospectus) of the Catholic Gymnasium in Braunsberg (printed in 1849)

Jahresbericht

über das

Königl. Katholische Gymnasium

zu

BRAUNSBERG

in dem Schuljahre 1848/49

mit welchem zu der

Oeffentlichen Prüfung am 2. August

und zu den

Schlussfeierlichkeiten am 3. August

ergebenst einladet

der Direktor der Anstalt

Dr. Ferd. Schultz.

Vorangeht die Abhandlung: Beitrag zur Theorie der Abel'schen Integrale, von dem Lehrer der Mathematik und Physik Herrn Weierstrass.

Braunsberg,
gedruckt bei C. A. Heyne.

Abelian functions became the life-long focus of Weierstrass's research. These functions are periodic functions of complex variables, the first non-trivial case of which are the elliptic functions. The theory of Abelian functions was the most impressive achievement of nineteenth-century analysis.
That *Schulprogramm* is a scarce document. It's not very surprising, that the paper published at such a place had gone unnoticed.

In my talk today, I would like to speak about some aspects in the work of Weierstrass concerning the foundation of analysis. The main topics will be the notion of real numbers and the well-known theorem of Bolzano-Weierstrass.

According to the available archive documents, Weierstrass presents the theorem of Bolzano-Weierstrass for the first time in his introduction to the theory of analytic functions, a lecture course held in the winter semester 1863/64. There exist notes that Hermann Amandus Schwarz (1843 – 1921) took during the lectures (Schwarz became successor on Weierstrass's Berlin chair in 1892). Here we read: *I need a lemma that is indispensable for finer mathematical investigations*. Apparently, this is a quotation of what Weierstrass's actually had said during the lecture. And a little bit later: *If there is an infinite sequence of quantities*

lying in a finite region, then there is necessarily at least one point in the neighborhood of which lie infinitely many others.
This is the first appearance of the theorem of Bolzano-Weierstrass, although not written down correctly by Schwarz. Of course, the correct version is that in every neighborhood of that point lie infinitely many other points. Later on in his lectures, Weierstrass stresses the central importance of this theorem even more. So, for instance, in his lectures in 1886:

Now we come to the development of a theorem that is not only one of the most important for the theory of numbers but it is the necessary basis for the most related investigations at all.

In order to prove such a theorem, an exact foundation of the real number system has to be available. But such a theory was not established at that time. And therefore, logically, we find in these lectures also the first introduction to the concept of a real number.

Weierstrass's construction is based – one could say, quite Weierstrass-like – on the concept of infinite series (Weierstrass's function theory can be described as a theory of power series).
A first, still as a quite vague hint, this is contained already a little bit earlier in his lectures on differential calculus during the summer 1861. Schwarz attended these lectures and we have his handwritten notes, which show that Weierstrass probably had not yet developed a theory of irrational numbers but that he had some ideas on that matter. For we read in the notes of Schwarz:

There are quantities which cannot represented by the unit and the parts of the unit (Weierstrass refers here to the irrational numbers)*; for those one applies infinite series.*

By the way, in this lecture Weierstrass uses the existence of a supremum for sets with an upper bound as an evident fact without giving any further explanation.
After an attack of faintness during a lecture in December 1861 he was only able to continue teaching after a year-long break. From then on he remained sitting during lectures, with students responsible for writing on the board.
It is quite possible that Weierstrass developed his theory of irrational numbers during this break.

Now we come to his construction of a real number system in his lectures 1863/64. Weierstrass presupposes the notion of natural numbers. As a first step he explains what positive rational numbers should be. He considers the commutative ring generated by the unit 1 and by the so-called parts $\frac{1}{n}$ of the unit for arbitrary natural numbers *n* which one has to consider simply formally as elements e_n with property $n \cdot e_n = 1$. Then the rational numbers are defined as the elements of that ring. But that's, of course, not enough, for he must define the equality of two such sums. They are equal, if they can be transformed into each other by substitutions of $\frac{1}{n}$ by $\frac{1}{mn} + \frac{1}{mn} + ... + \frac{1}{mn}$ (m terms) and vice versa substitutions (therefore as a special case, the unit 1 by $\frac{1}{m} + \frac{1}{m} + ... + \frac{1}{m}$) for arbitrary natural numbers m,n. In our terminology, the rational numbers are equivalence classes with respect to the transformations mentioned. It's obvious how to define the order relation for them.
Now comes the crucial step in the construction of irrational numbers. We confine ourselves to the positive real numbers. In what follows, I slightly deviate from Weierstrass's original version, but only with respect to the shape not with respect to the contents.

Weierstrass considers infinite series $\sum_{n=1}^{\infty} a_n \frac{1}{n}$, all a_n are natural numbers or 0. In order to define a number, one has to suppose that all partial sums lie below a fixed bound. Now the equality of two such sums is yet to be defined. For any rational number r we let $r < \sum_n a_n \frac{1}{n}$ if r is less than a certain partial sum: $r < \sum_{n=1}^{k} a_n \frac{1}{n}$ (note that for rational numbers this makes sense). Then, an order relation can be introduced: We let $\sum_n a_n \frac{1}{n} \le \sum_n b_n \frac{1}{n}$, if for any rational number r with $r < \sum_n a_n \frac{1}{n}$ it follows that $r < \sum_n b_n \frac{1}{n}$. Now, an equality can be defined: $\sum_n a_n \frac{1}{n} = \sum_n b_n \frac{1}{n}$, if $\sum_n a_n \frac{1}{n} \le \sum_n b_n \frac{1}{n}$ and $\sum_n b_n \frac{1}{n} \le \sum_n a_n \frac{1}{n}$.

In our terminology, the positive real numbers are defined as equivalence classes with respect to the equality relation.

The addition of the new objects is defined component-wise $\sum a_n \frac{1}{n} + \sum b_n \frac{1}{n} = \sum (a_n + b_n) \frac{1}{n}$ and the multiplication is defined by the sum of all products of terms involved $\sum a_n b_m \frac{1}{n \cdot m}$ (such that $\frac{1}{n} \cdot \frac{1}{m} = \frac{1}{n \cdot m}$)

One must prove that these definitions make sense, that means they lead to real numbers again and are independent of the representatives.

The important aspect of completeness, that means, that any new formation of infinite series formed by the new objects will produce no "new numbers", is not formulated explicitly. But it is expressed, for instance, in the following theorem equivalent to the completeness property (the quotation is from the lecture in 1874):

The sum formed by an infinite number of terms has a finite value [that means is a real number] *if the sum of arbitrarily many* [one has to add: finitely] *lies below a bound independent of the number of terms*.

Such an infinite sum is defined by $\sum_{i=1}^{\infty} \alpha_i = \sum_{n=1}^{\infty} \left(\sum_{i=1}^{\infty} a_{in} \right) \frac{1}{n}$ with $\alpha_i = \sum_{i=1}^{\infty} a_{in} \frac{1}{n}$, if $\sum_{i=1}^{\infty} a_{in}$ is finite ($= s_n$ that means almost all terms are equal to 0) and $\sum_{n=1}^{\infty} s_n \frac{1}{n}$ is a bounded series, such that it defines a real number.

As a consequence, it follows from that theorem at once that any bounded, monotone increasing sequence of real numbers converges, a statement, again equivalent to the completeness property.
By the way, these results are obtained without using the Bolzano-Weierstrass theorem (which appears much later in the course in 1874).

In Weierstrass's lectures the construction of all real numbers, including the negative ones, corresponds to the formation of pairs as it is the usual procedure today.

The question may arise, why not give the definition of real numbers simply as the limits of sequences of rational numbers. Let Weierstrass answer this (the quotation is from his lecture in 1886, June 8):

Usually, the value of infinite series is defined as the limit of the partial sums: $\lim\limits_{n=\infty} s_n$. *According to the arithmetical point of view that we are preferring, this is not permissible. We do not assume the existence of a limit, but consider the limit notion as something that has to be defined.*

And one day later he says: *If we start with rational numbers then it makes no sense to define the irrational numbers as limits of the rationals because initially we do not know whether other than rational numbers do exist.* […] *However, once the concept of real numbers is established, then one can consider the irrational numbers as limits of the rationals. For we can always from a number formed by infinitely many elements remove elements* [a finite number of which is meant] *such that the remainder is smaller than any arbitrary little quantity* δ *; hence, there are infinitely many rational numbers lying as close to the irrationals as ever one wishes.*

Weierstrass' student, Georg Cantor (1845 – 1918; he came to Berlin in the autumn of 1863 and remained until the summer of 1866), remarks in 1883 in a publication:

It has to be emphasized as essential that the number to be defined cannot be considered from the beginning as the sum $\sum a_\nu$ *of the infinite sequence* (a_ν)*. I think, Weierstrass was the first who avoided this logical error generally adopted in the past.*

In his lecture in the summer of 1874 Weierstrass says:

If the infinite sum is of finite value [that means it defines a real number], *then it can always divided in two quantities, one of which consists of a finite number of terms and the* *other of an infinite number of terms such that the latter is smaller than an arbitrary given small quantity.*

In our terminology, this is nothing else but $\sum_{n=1}^{\infty} a_n \frac{1}{n} = \lim_{k\to\infty} \sum_{n=1}^{k} a_n \frac{1}{n}$.

Weierstrass never published his theory of irrational numbers. The only sources to get to know his method are the lecture notes of his students, as far as the manuscripts have survived the times, or publications of them. He presented his construction as a part of his lecture course “Introduction to the theory of analytic functions”. Weierstrass gave that course in a cycle of two years, from 1863/64 until 1884/85, and also in his lecture “Selected chapters of function theory” in the summer of 1886. The treatment of the concept of real numbers occupies in those lectures different, sometimes considerable space. For instance, in the summer of 1874 it amounts to 85 handwritten pages of a total of 700 pages for the whole manuscript. This shows the importance that Weierstrass attached to these basic concepts. In his lecture in the summer of 1874 he says:

The reason for the main difficulties in higher analysis is the vague and not sufficiently detailed presentation of the arithmetical basic notions and operations.

Let me add at this point some more general remarks about Weierstrass’s lectures. What he offered was often not published. Both students from all regions of Germany and already educated mathematicians (as for instance the Swedish Gösta Mittag-Leffler) attended his lectures, sometimes more than 250 students. An impressive participation! From some of these lectures notes and detailed handwritten manuscripts have survived. To write a copy and to sell it was an option for the poorer students to get some money. These texts have contributed to the further spreading of Weierstrass’s mathematics.

One of the first publications in which theorems of Weierstrass’s lectures are discussed is a paper of Eduard Heine (1821 – 1881), professor in Halle. You know his name from the Heine-Borel theorem. He suggested to the young Georg Cantor to study trigonometric series, which, finally, led Cantor to the development of his set theory. Heine was in contact with Weierstrass. In a paper “Elements of the theory of functions” dated October 1871 Heine says:

The development of the theory of functions is essentially obstructed by the fact that some of the elementary theorems in that, though they are proven by an astute researcher [referring to Weierstrass] *are questioned, such that the results of an investigation are not*

accepted everywhere, if they are based on these indispensable fundamental theorems. […] *Their truth is related to the not completely established definition of irrational numbers* […].
He then continues:

[…] *to construct the general numbers* [that's Heine's name for real numbers] *with the help of those especially appropriate sequences which are denoted here as "Zahlenreihen" (number sequences)* […] *is a lucky further stage of the original kind of invention using the multiples of certain quantities in an infinite number.*

It is remarkable that Heine considers Cantor's idea to construct real numbers by means of fundamental sequences as a further development of Weierstrass's original number concept. Heine points out that his theorems (for instance the intermediate value theorem for continuous functions) already are all proven by Weierstrass by means of his number concept but that he will now give proofs using Cantor's number concept.

As it is well-known, besides Weierstrass and Cantor, Richard Dedekind (1831 – 1916) independently developed a quite different approach to real numbers. He had found his concept of "cuts" during his lectures on the calculus in 1858 (published only in 1872). Cantors's starting point were uniqueness theorems for trigonometric series. He presented his number concept for the first time in a lecture on differential calculus in summer 1870 (published in 1872 too).
The first person to publish a theory of real numbers was the French mathematician Charles Méray in 1869. His concept is based – like Cantor's – on fundamental sequences, but his paper had gone largely unnoticed.

It would be interesting to know Weierstrass's reaction to Heine's or Cantor's constructions. Although the Berlin Academy of Sciences keeps the correspondence between Weierstrass and Schwarz with hundreds of letters, unfortunately, in that very period of time there are considerable gaps in this correspondence so that we cannot say anything about that.

Now I would like to come back to the theorem of Bolzano-Weierstrass.
In our days this theorem appears in the beginning of every lecture on differential calculus. The meaning is immediately clear. Why waste time for a proof?
Weierstrass answered this. He points out in a lecture dated June 9th, 1886:

The beginner will feel inclined to think that such a theorem is something obvious. In fact, let us assume for a moment, that x' [elements of an infinite set of real numbers] *lie* *between the bounds a and b. Then it is evident, if one refers to the idea of a straight line, that, if in the finite interval* [*a,b*] *lie infinitely many points that there is at least one point where points of the given set accumulate in infinity. However, we will provide here an exact proof of the theorem.*

Two days later he again comments the necessity of a proof:

If we fix on the straight line AB a unit measure, then each point C is represented by a number quantity. But, on the other hand, it is in advance not clear that to each given number quantity corresponds a certain point. At least for a moment, it cannot be excluded the possibility that the set of all positive real numbers is something more than the set of the points.

How did Weierstrass prove the theorem for the first time in 1863/64?

At first he points out for what purpose he will use the theorem:

I would like to prove the theorem mentioned yesterday: That two functions coincide, if they coincide for infinitely many points.

This statement is known as the uniqueness theorem for power series. Then follow the words mentioned already at the beginning: *I need a lemma that is indispensable for finer mathematical investigations*.

It’s not so easy to decipher the text because of the permanent abbreviations and gaps. But for notes written during a lecture that is quite understandable. However, the contents can quite probably be reconstructed as follows:

It is assumed that in the square defined by the points 0, 1, 1+*i*, *i* of the complex plane lie infinitely many points. Bisecting the sides yields four new squares. In at least one of them must lie infinitely many points again. Let x_1 denote its lower left corner.

We have $x_1 = \frac{a_1}{2}$ for some $a_1 \in \{0,1,1+i,i\}$. Bisecting that square produces four new squares again. Analogously, we obtain a square with lower left corner x_2 containing infinitely many of the given points. It follows $x_2 = \frac{a_1}{2} + \frac{a_2}{4}$ for some $a_2 \in \{0,1,1+i,i\}$.

Repeating this procedure yields a sequence of complex numbers (namely the corners) $x_n = \frac{a_1}{2} + \frac{a_2}{4} + ... + \frac{a_n}{2^n}$ with $a_n \in \{0,1,1+i,i\}$. Since $|a_n| \le \sqrt{2}$ the sequence converges (it is absolutely convergent). Here we reach the point where Weierstrass’s number concept is to be applied (in a somewhat broader context for complex numbers instead of the real ones).

Then it is shown that the limit of the corners has the desired property (namely, to be an accumulation point). The lemma is done.
That’s the first proof of the theorem of Bolzano-Weierstrass at all.

Note that the theorem is not stated for subsets of real numbers as it is common today in the introductory lectures on analysis. The same situation we find in the notes of Wilhelm Killing (1847 – 1923) of Weierstrass’s lecture in 1868 (thus, I can mention here the second renowned teacher of Braunsberg). Of course, that is due to the fact that Weierstrass discusses in these lectures only power series for complex variables. Weierstrass included the treatment of functions of one real variable in his lectures in the summer of 1874. Consequently, the theorem of Bolzano-Weierstrass is then stated and proved for bounded sets of real numbers. It is applied in the proofs of the known statements for continuous functions of one real variable. However, in that version the theorem is already discussed in the 1870 correspondence between Cantor and Schwarz. I will come back to this point later on.

Bernard Bolzano

(1781 – 1848)

I would like to come to Bolzano’s contributions. Bernard Bolzano was a Bohemian philosopher, logician and Catholic priest. He studied also mathematics in Prague and published mathematical papers.

What part has Bolzano in the theorem of Bolzano-Weierstrass?

His relevant paper was published in Prague in 1817: *Purely analytical proof of the theorem, that between each two roots which guarantee an opposing result* [in sign], *at least one real root of the equation lies.*

With respect to the theorem of Bolzano-Weierstrass the most important statement is the following

Lehrsatz. If M *is a property of real numbers that does not hold for all* x *, and there exists a number* u *such that all numbers* $x < u$ *have property* M *, then there exists a* <u>*largest*</u> U *such that all numbers* $x < U$ *have property* M *.*

At first sight, perhaps, it looks quite different from the theorem of Bolzano-Weierstrass. But actually, both theorems are equivalent. I leave the proof as a nice exercise to the interested student.

I would like to say some words about Bolzano's proof of the *Lehrsatz*.

In §7 a criterion for convergence of sequences of functions is stated, that is exactly what we call the Cauchy convergence criterion for sequences of functions (as it occurs in his "Cours d'analyse", Paris 1821). By the way, Bolzano formulates his criterion not for sequences of real numbers. As opposed to Cauchy, Bolzano proves the theorem (we have to say, he tries to prove). He argues: *under this assumption* [the Cauchy condition] *it is possible to determine that quantity* [that means the limit] *as precisely as ever one wishes.* Thus, Bolzano concludes the existence of the limit from the fact that this quantity can be calculated with arbitrary exactness. The gap in his deduction is that the number concept is not discussed (and that's the only gap).

In §9 he applies the Cauchy criterion to infinite series.

In §12 the *Lehrsatz* is stated. Bolzano's proof runs as follows. He constructs a sequence of intervals $[u_n, U_n]$ such that property M holds for all $x < u_n$, but M doesn't hold for all $x < U_n$. Let $[u, u + D]$ be a starting interval (there is one by assumption). Bisecting the interval generates at least one new interval with the analogous property with respect to M . Repeating that procedure leads to an infinite sequence (u_n) such that the u_n are the partial sums of the infinite series $U = u + \frac{D}{2^{m_1}} + \frac{D}{2^{m_2}} + ...$ and the powers of 2 in the denominators are strictly monotonic increasing. That series is convergent by the Cauchy criterion for infinite series of §9. The number U has the desired property stated in the *Lehrsatz*.

In §13 Bolzano underlines that for sets bounded from above a maximum does not necessarily exist.

For what purpose does Bolzano use his *Lehrsatz*?
He told it already in the paper's title: the existence of zeroes for continuous functions. In fact, in §15 he proves a more general theorem: Let f, g be continuous functions on the closed interval $[a,b]$, $f(a) < g(a), f(b) > g(b)$, then there exists a $x \in [a,b]$ such that $f(x) = g(x)$. Especially, as stated in the title of his paper, if $f(a)$ and $f(b)$ differ in sign, then there is a $x \in [a,b]$ such that $f(x) = 0$.
To prove the theorem the set $\{u \mid \text{ for all } x < u : f(a+x) < g(a+x)\}$ is considered. It has an upper bound. Hence the *Lehrsatz* applies to it. It follows that $a + U$ has the required property where U is from the *Lehrsatz*.

Why is Bolzano interested in these matters?
His interest has a general background. In his 1817 paper he discusses that point in some detail. He criticizes references made to geometrical imagination in mathematical proofs. As an example he mentions Gauss's first proof of the so-called fundamental theorem of algebra, where Gauss made extensive use of geometric representations, reducing the problem to the proof that there exist points of intersection of certain algebraic curves in the plane. Bolzano says: […] *a geometrical proof is in fact a logical circular reasoning. Even though the geometrical truth to which the reference is made is most evident such that there is no need to make it* <u>*sure*</u> *nevertheless it requires a* <u>*proof*</u>*.*
At that time, such a point of view is exotic. His view on the fundamental concepts of analysis is sharper than that of his contemporaries.

It is not clear whether Weierstrass was aware of Bolzano's paper when he discussed his lemma in the lectures in 1863/64. In any case, Bolzano's name doesn't appear in the notes of Schwarz. The same is true for the notes of Killing in the summer of 1868. As an exception, in his last lecture on the number concept in summer 1886, Bolzano is mentioned a few times. Anyway, the correspondence between Cantor an Schwarz shows clearly that Weierstrass knew Bolzano's paper at least in 1869.
According to the notes of Killing, Weierstrass's formulation of the lemma in the summer of 1868 is a little bit closer to Bolzano's *Lehrsatz*:

If a function has a property on a bounded region infinitely often, then there exists at least one point such that for every neighborhood of it the function has the required property.

Weierstrass brings forward only a very short argument in that lecture without any reference to his number concept: namely, let the region be contained in a square. Repeated bisection (as described above) allows "*to approach the point as close as ever we wish".* That's all. This reminds of Bolzano's argument, who thought the existence of his "*largest U*" is proved by saying that it is possible to determine that quantity with arbitrary accuracy.

It is remarkable that, in fact, Bolzano uses the theorem of Bolzano-Weierstrass in the form as it is familiar to us. Up to the year 1930 this was completely unknown and could, of course, not have played any role in giving the theorem that name and was of no influence to the generation of Weierstrass and Cantor at all.
But, in which form does the theorem occur in Bolzano's work?
Bolzano wrote the manuscript around 1830, entitled *"Functionenlehre"* (theory of functions). He considers a function which is not bounded on a closed interval $[a,b]$. He shows that the function cannot be continuous (§20). His proof goes as follows: There must exist for all natural numbers n points $x_n \in [a,b]$ such that $f(x_n) > n$. Then he deduces that there is a point c such that in any arbitrary small neighborhood elements x_n exist, that means it is exactly that what we call an accumulation point. Bolzano shows that f is not continuous at c. For the existence of c he refers to a paragraph § that is not contained in his manuscript. We know that the "Functionenlehre" was planned as a part of a certain "Grössenlehre", but that text remained unfinished. As a corollary he states, that continuous functions on closed intervals are bounded. Quite analogously, Bolzano proves (§22): Let f be a continuous function on a closed interval $[a,b]$. If there are infinitely many $x_n \in [a,b]$ such that $\lim_{n\to\infty} f(x_n) = C$, then there is a point $c \in [a,b]$ such that $f(c) = C$. It is followed by the theorem that a continuous function on a closed interval attains its maximum and minimum (§24). The intermediate value theorem is stated too. Moreover, in that manuscript Bolzano described a continuous function on a closed interval that is nondifferentiable on a subset dense everywhere, about four decades before Weierstrass presented his famous example. But – as already mentioned – nobody knew of these achievements of Bolzano at that time. The basic theorems on continuous functions, except for the concept of uniform continuity, are stated by Bolzano. With respect to his precise treatment, his clear concept of arithmetization of analysis, Bolzano is a forerunner of the further developments in the 19th century.

For what does Weierstrass use the theorem of Bolzano-Weierstrass in his lectures?
In the first lecture in 1863/64, as already mentioned, to prove the uniqueness theorem for power series;

in the summer of 1868, according to the notes of Killing, to discuss the difference between polynomials and infinite power series convergent on the whole complex plane, that means the transcendental entire functions – today known as the Casorati-Weierstrass theorem;
Thus, in the first years Weierstrass stated the theorem only for the complex plane because he was concerned in his lectures exclusively with complex power series.
In his lectures later on, for instance in the summer of 1874, functions of one real variable are considered and therefore the Bolzano-Weierstrass theorem is stated for subsets of real numbers and is applied in the proofs of the basic theorems on continuous functions. A new aspect is the concept of uniform continuity. In his proof that a continuous funtion on a closed interval is uniform continuous Weierstrass uses that "*fundamental theorem of the theory of quantities*", as he called it in the lectures 1886. In one of the existing lecture notes one reads: *Weierstrass was the first who drew the attention to the fact that this property* [of uniform continuity] *has to be assumed for the concept of definite integral.*

It is through Weierstrass's lectures that the strict foundation of analysis, its exact control over basic concepts like limit, convergence, continuity and differentiability, and its systematic construction with watertight proofs, became known to mathematicians. The so-called "Weierstrassian rigour" became legendary. Today we are familiar with such deductions. No lecture on analysis without these concepts. So, hardly anyone would assume that theorems of this kind could be something to start controversial discussions. In the 1880's, the conflict between Weierstrass and his Berlin colleague Leopold Kronecker became more acute. Their friendship, which had lasted for more than twenty years, ended once and for all. This escalation seems surprising, the more so because for a long time these fundamental existence theorems (as for instance the completeness property or the intermediate value theorem) were generally regarded as self-evident before and also during the nineteenth century when efforts to provide an exact foundation of analysis started.

Lets come back again to the last third of the 19^{th} century. One has to keep in mind, that although Weierstrass, Dedekind and Cantor presented their constructions in their lectures, the mathematical community was not aware of an exact theory of real numbers or even refused such concepts. Still in 1884 Cantor declares:

Its worth to be remarked that our proof [Cantor refers to the method of nested intervals] *is attacked by some geometers* [that means mathematicians]. *Many are embarrassed, intimidated or confused.*

I would like to give some more attention to the controversy concerning the foundation of analysis.
It was Weierstrass' intention to develop analysis without any reference to geometrical imagination. A quotation from the lecture in the summer of 1874:

To establish analysis we need a purely arithmetical foundation [...] *analysis has to be cleaned of geometry.*

Was that the right way? Lively discussions started, especially around 1870. The most prominent critic was Leopold Kronecker (1823 – 1891). He, Weierstrass and Eduard Kummer (1810 – 1893) were the leading mathematicians in Berlin at that time, a distinguished group with world-wide recognition .
Both Kronecker and Weierstrass agreed that there was a need for a more exact foundation of analysis. In 1864 Felice Casorati (1835 – 1890) came as a young professor to Berlin. He made notes on his discussions with Weierstrass, Kronecker and others that make clear what a central theme the foundation of analysis was, for instance how to define the notion of continuity. Heine also reports that Weierstrass and Kronecker discussed such questions frequently.

Kronecker wrote to Schwarz a letter, dated 1870, 3rd July:

Beloved friend,

[...] *through your correspondence with Heine and Cantor you are very well informed about their work on Fourier series. I would like to tell you the same thing that I already said to Heine and Cantor personally, namely, that I did not get to like those papers* [...]. *Cantor's approach is based on Weierstrassian "method" using the "upper and lower bound"* [supremum and infimum in our terminology] *that you are also applying, which I cannot accept like the more obvious fallacies Bernard Bolzano's as well.* [...] *By the way, in my disbelief of Bolzano's kind of reasoning and the analogous, although more subtle reasoning of Weierstrass, I have Heine and Borchardt on my side.* [With respect to Heine it can only be viewed as an illusion.] *I'm even convinced that it will be possible to construct functions which are so unreasonable that they don't have an upper bound despite Weierstrass's assumptions being satisfied. All such general theorems have their hiding places where they do not hold. In order to prove such statements one has to go back to the first principles, to the explanation of "quantities" where one is confronted with the whole difficulties and controversies that arose* *whenever efforts are made to a renewal of the foundations of our science. In order to overcome them it requires years of reasoning. As for my part, having been engaged in such matters for a few months, I'm glad to get away from such things. I would like to warn you not to waste your youthful freshness and inventive talent for these relatively fruitless things. As already said, it became quite clear to me that the Weierstrassian claim that an "upper bound" exists cannot be proved and that it is, perhaps, even not true.*

A few days later, Kronecker again explains in a letter to Schwarz his criticism.

Beloved friend,

[…] *In my opinion, Cantor's attempt to extend his proof of uniqueness to* <u>*arbitrary*</u> *trigonometric series failed. For his deductions are based on the existence of a "maximum" that is at least not yet proved. The many attempts Weierstrass made to overcome my doubts were all in vain. I think, Weierstrass himself is now convinced that this maximum is* <u>*unprovable*</u>*. However, he regards it as existent. But Heine* […] *neither does believe in the rigour of the Bolzano-Weierstrass method nor in the rigour of Cantor's proof based on it.* […] <u>*I*</u> *wished that Heine and Cantor had withdrawn their publications; after the intervention of Weierstrass I decided for the present not to publish my point of view; but later I shall surely find the opportunity to do so.*

Not only Kronecker, but also the third of the Berlin triumvirate, Eduard Kummer (father-in-law of Schwarz) explains his objections at the same time. He writes to his son-in-law in 1870, 1st June:

I for myself would like to remark that I regard as very slippery the whole ground on which the investigations of the Dirichlet principle grow, namely the ground where completely indetermined functions are studied such that a solid building is impossible to establish. Yet you know my words: "In the field of <u>*transcendental*</u> *functions everything is possible." Here it can always appear that in special cases the theorems stated stop being valid. I would really like it if you don't cultivate that slippery ground with such an eagerness, but that you turn to those problems to which you are excellently gifted, namely, the more concrete ones, where the success doesn't depend on the opinions about rigour which can have that or the other mathematician and where one simply, by drive for rigour, goes into the most fruitless investigations which never reach the absolute rigour in such a generality as it is sought-after.*

A few weeks later, on July 14th, Weierstrass spoke in the Berlin Academy on his criticism of Dirichlet principle – the basis of Riemann's function theory with the result that Riemann's approach was no longer accepted. Two years later (1872, on July 18th) Weierstrass presented a function continuous for all real numbers that is differentiable nowhere. A shocking fact. For it was a common belief that continuous functions on finite intervals are always differentiable except, perhaps, at a finite number of points. There were even "proofs" of this wrong statement, for instance by such renowned mathematicians like Galois (1830) and Bertrand (1864). Such a function represented in the eyes of Charles Hermite (1822 – 1901) a "deplorable wound". Still in 1893 he declares: *I turn away with fright and horror from this lamentable evil of functions that do not have derivatives.*

By the way, it seems that Weierstrass knew for years that such functions do exist. I was surprised when I read in Schwarz's notes of the lecture 1863/64 (that lecture where the theorem of Bolzano-Weierstrass occurs for the first time):

It is not justified that such functions have derivatives; - proofs are wrong, for I shall show that there are such functions that are continuous but that they have at no point a derivative.

The criticism to Weierstrass's approach was not without influence. Schwarz became doubtful. Answering a letter of him Cantor writes in 1870, July 27th :

You ask me, if our Weierstrass admits a lack of rigour in his characteristic kind of proving. That's in no way the case. Although the professors Kummer, Kronecker and Borchardt have made the most intensive attempts to drive him to a corner, I find that his point of view has been changed not an inch. When on the one hand your father-in-law raises the objection that it is impossible to state general theorems without assumptions, on the other hand Weierstrass emphasizes that his theorem of the existence and the attainment of the upper bound for continuous functions has just its power and its applicability in the fact that it is general.

I for myself have no doubts about its truth. Moreover, I'm convinced that it will be accepted over the years.

As already said, in the 1880's the conflict between Weierstrass and Kronecker became more acute. Kronecker speaks of the "so-called analysis". He writes to Schwarz in 1884, December 28th:

If enough years and powers are left to me I shall show to the mathematical world that not only geometry but also algebra can show analysis the ways and certainly the stronger ones. If I cannot do that, then those will do it who shall come after me and, moreover, they will realize the incorrectness of all these reasonings, used today by the so-called analysis.

That's especially a slap in the face of Weierstrass. In his attacks Kronecker never mentioned the name of Weierstrass explicitly. Although a concrete criticism by Kronecker himself is not known – despite promising it on several occasions – , the position he adopted in basic questions can be describes as follows. For him, mathematical concepts and deductions are only admissible if they can be described through constructive procedures requiring only a finite number of steps. For instance, the introduction of the notion of irreducibility of a polynomial is only justified if, at the same time, a procedure is stated which allows one to decide in a finite number of steps whether a given polynomial is irreducible or not. For this reason it is also quite clear that Kronecker could not accept a theorem like that of Bolzano-Weierstrass, because it is a pure existence theorem: it provides for any given infinite bounded set of real numbers no way of determining in a finite number of steps an accumulation point.

Sofya Vasilievna Kovalevskaya
(1850 – 1891)

In the correspondence between Weierstrass and Sofya Kovalevskaya, his unforgettable Russian student, we find also traces of the controversy with Kronecker. In the autumn of 1870 she had come from Heidelberg to Berlin in order to continue her studies with Weierstrass. She remained in Berlin until the summer of 1874. A connection grew between them which is difficult to find anywhere else in the history of science. In 1874 she received the doctoral degree from the University of Göttingen. In 1884 she was appointed professor for higher

analysis at the then still young University of Stockholm. For the first time, a woman held a chair in mathematics in an unrestricted sense.
About that time, Weierstrass writes to her in 1885, March 24th:

So it happens not seldom, that I present in a lecture a theorem together with a proof – at least that is my opinion – that is viewed in another lecture as untenable and misleading. [Weierstrass had mentioned before that his colleagues Kronecker and Fuchs "work against him"] *Whereas I say that a so-called irrational number has such a real existence as anything else in the world of thinking, it is now an axiom for Kronecker that there are only equations between integers.*

Finally, the controversy between Weierstrass and Kronecker led to a permanent discord between them. For Weierstrass the strain was so great that he even had the intention, a few weeks before his 70th birthday, to leave Berlin and to settle in Switzerland. We know that from one of his letters to Sofya Kovalevskaya.
Weierstrass and his two sisters all lived together in one apartment. All three remained unmarried. The sisters were worried about their brother. Was he on the right way? Was his work of any value?
Sister Clara wrote to Sofya and asked her for an evaluation of the work of her brother. A letter arrived (undated, but presumably written in 1887):

Nothing on this world is more certain to me than that: the mathematical truths discovered by Weierstrass will be recognized as long as there are mathematicians on this earth at all. His name will be forgotten only, when the names of Gauss and Abel are forgotten too.

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