Loop Groups with Infinite Dimensional Targets and their Unitary Representations

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1.1. From compact groups to Hilbert–Lie groups

**Definition (Hilbert–Lie algebra)**

A Hilbert–Lie algebra is a Lie algebra \( \mathfrak{k} \) which is a real Hilbert space whose scalar product is invariant: \( ([x, y], z) = (x, [y, z]) \).

A Lie group \( K \) is a **Hilbert–Lie group** if \( L(K) = \mathfrak{k} \) is a Hilbert–Lie algebra.

Finite dimensional Hilbert–Lie algebras are the **compact** Lie algebras.

**Theorem (Schue, 1960/61; Structure of Hilbert–Lie algebras)**

\( \mathfrak{k} \) is an orthogonal direct sum \( \mathfrak{k} = \mathfrak{z}(\mathfrak{k}) \oplus \bigoplus_{j \in J} \mathfrak{k}_j \), where \( \mathfrak{k}_j \) is simple.

If \( \mathfrak{k} \) is inf. dim. simple, then \( \mathfrak{k} \cong \mathfrak{u}_2(\mathcal{H}) \) (skew-herm. Hilbert–Schmidt ops) for a Hilbert space \( \mathcal{H} \) over \( \mathbb{R}, \mathbb{C} \) or \( \mathbb{H} \) with \( (x, y) = \text{tr}_{\mathbb{R}}(xy^*) = -\text{tr}_{\mathbb{R}}(xy) \).

**Example**

\( U_2(\mathcal{H}) = \{ g \in U(\mathcal{H}) : \| 1 - g \|_2 < \infty \} \) is a Hilbert–Lie group with Lie algebra \( L(U_2(\mathcal{H})) = \mathfrak{u}_2(\mathcal{H}) \).

Here \( \| X \|_2 = \sqrt{\text{tr}(X^*X)} \).
1.2. Root data of simple Hilbert–Lie algebras

\( \mathfrak{t} \) simple Hilbert–Lie algebra
\( \mathfrak{t} \subseteq \mathfrak{t} \) maximal abelian (Cartan subalgebra), \( \mathfrak{t} \cong \ell^2(J, \mathbb{R}) \)

\[ \mathfrak{t}_\mathbb{C} = \mathfrak{t}_\mathbb{C} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{t}_\mathbb{C}^\alpha \]  
(root decomposition), orthogonal direct sum

\( \Delta = \Delta(\mathfrak{t}, \mathfrak{t}) \) is a locally finite root system.

**Theorem (Stumme ’99, Classif. of infinite locally finite root systems)**

\[ A_J = \{ \varepsilon_i - \varepsilon_j : i \neq j \in J \}, \quad B_J = \{ \pm \varepsilon_i, \pm \varepsilon_i \pm \varepsilon_j : i \neq j \in J \} \]
\[ C_J = \{ \pm 2\varepsilon_i, \pm \varepsilon_i \pm \varepsilon_j : i \neq j \in J \}, \quad D_J = \{ \pm \varepsilon_i \pm \varepsilon_j : i \neq j \in J \}. \]

\( \Rightarrow 4 \) iso-classes of pairs \( (\mathfrak{t}, \mathfrak{t}) \) (for each cardinality \( |J| \)):

- **A\(_J\)**: \( \mathbb{K} = \mathbb{C}, \quad \mathfrak{t} = \mathfrak{u}_2(\mathcal{H}) \)
- **B\(_J\), D\(_J\)**: \( \mathbb{K} = \mathbb{R}, \quad \mathfrak{t} = \mathfrak{u}_2(\mathcal{H}) := \mathfrak{o}_2(\mathcal{H}), \dim(\ker(\mathfrak{t})) \in \{1, 0\} \)
- **C\(_J\)**: \( \mathbb{K} = \mathbb{H}, \quad \mathfrak{t} = \mathfrak{u}_2(\mathcal{H}) := \mathfrak{sp}_2(\mathcal{H}) \).

\( \mathfrak{t} = \mathfrak{o}_2(\mathcal{H}) \) has **two conjugacy classes** of Cartan subalgebras under \( \text{Aut}(\mathfrak{t}) \).
2.1. Loop groups and twisted loop groups

Definition (Twisted loop groups)

For a Hilbert–Lie group \( K \),

\[
\mathcal{L}(K) := \{ f \in C^\infty(\mathbb{R}, K) : (\forall t \in \mathbb{R}) f(t + 2\pi) = f(t) \}
\]

is called the corresponding loop group. For an automorphism \( \varphi \in \text{Aut}(K) \),

\[
\mathcal{L}_\varphi(K) := \{ f \in C^\infty(\mathbb{R}, K) : (\forall t \in \mathbb{R}) f(t + 2\pi) = \varphi^{-1}(f(t)) \}
\]

is called the corresponding twisted loop group.

\( \mathcal{L}_\varphi(K) \) is a Fréchet–Lie group with Lie algebra

\[
\mathcal{L}_\varphi(\mathfrak{k}) := \{ \xi \in C^\infty(\mathbb{R}, \mathfrak{k}) : (\forall t \in \mathbb{R}) \xi(t + 2\pi) = L(\varphi)^{-1}\xi(t) \}.
\]

Note: \( \mathcal{L}_\varphi(K) \) is the group of smooth sections of a \( K \)-Lie group bundle \( \mathcal{K} = (\mathbb{R} \times K)/\sim \) over \( S^1 \cong \mathbb{R}/2\pi\mathbb{Z} \), where \( (t + 2\pi, k) \sim (t, \varphi(k)) \).
2.2. Loop groups with Hilbert targets

*K*-Lie group bundles over \( S^1 \) correspond to \( \text{Aut}(K) \) principal bundles, hence are classified by

\[ \pi_0(\text{Aut}(K))/\text{conj}. \]

**Note:** \( \text{Aut}(K) \simeq \text{Aut}(\mathfrak{k}) \) if \( K \) is 1-connected.

**Theorem**

The automorphism groups of the infinite dimensional simple Hilbert–Lie algebras are given by the connected groups

\[
\text{Aut}(\mathfrak{o}_2(\mathcal{H})) \cong O(\mathcal{H})/\{\pm \mathbf{1}\}, \quad \text{Aut}(\mathfrak{sp}_2(\mathcal{H})) \cong \text{Sp}(\mathcal{H})/\{\pm \mathbf{1}\}
\]

(real and quaternionic case) and the 2-component group (complex case)

\[
\text{Aut}(\mathfrak{u}_2(\mathcal{H})) = \text{PU}(\mathcal{H}) \rtimes \{1, \sigma\}, \quad \sigma : \mathcal{H} \to \mathcal{H} \text{ antilin. isom. involution.}
\]

We thus obtain 4 iso-classes of twisted loop algebras

\[
\mathcal{L}(\mathfrak{o}_2(\mathcal{H})), \quad \mathcal{L}(\mathfrak{u}_2(\mathcal{H})), \quad \mathcal{L}(\mathfrak{sp}_2(\mathcal{H})) \quad \text{and} \quad \mathcal{L}_\sigma(\mathfrak{u}_2(\mathcal{H})).
\]
2.3. Double extensions

Definition (Double extensions)

For a Lie algebra \( \mathfrak{g} \) with invariant symmetric bilinear form \( \kappa \) (quadratic Lie algebra) and a \( \kappa \)-skew-symmetric derivation \( D \) on \( \mathfrak{g} \), the corresponding double extension is the quadratic Lie algebra \((\widehat{\mathfrak{g}}, \widehat{\kappa})\), where

\[
\widehat{\mathfrak{g}} := \mathbb{R} \times \mathfrak{g} \times \mathbb{R}
\]

\[
[(z_1, x_1, t_1), (z_2, x_2, t_2)] := (\kappa(Dx_1, x_2), t_1 Dx_2 - t_2 Dx_1 + [x_1, x_2], 0)
\]

\[
\widehat{\kappa}((z_1, x_1, t_1), (z_2, x_2, t_2)) := z_1 t_2 + z_2 t_1 + \kappa(x_1, x_2).
\]

Note: \( \widetilde{\mathfrak{g}} := \mathbb{R} \times \mathfrak{g} \) is a central ext. with cocycle \( \omega_D(x_1, x_2) := \kappa(Dx_1, x_2) \).

\( \widehat{\mathfrak{g}} \cong \widetilde{\mathfrak{g}} \times \widetilde{D} \mathbb{R} \) for \( \widetilde{D}(z, x) := (0, Dx) \).

Ex: \( \mathfrak{g} = u_2(\mathcal{H}), \quad Dx = [T, x], \quad T \in u(\mathcal{H}), \quad \kappa(Dx, y) = -\text{tr}(T[x, y]) \)

Rem: \( \mathfrak{k} \) Hilbert–Lie algebra \( \Rightarrow \) Any 2-cocycle \( \omega \) can be written as \( \omega(x, y) = (DX, y) \) with \( D \in \text{der}(\mathfrak{k}) \) \( \Rightarrow \) double extension \( \widehat{\mathfrak{k}}_D \).
2.4. Affine Kac–Moody groups

If \( k \) is a Hilbert–Lie algebra and \( \varphi \in \text{Aut}(k) \), then the loop algebra \( \mathcal{L}_\varphi(k) \) carries the scalar product \( (\xi, \eta) := \int_0^{2\pi} (\xi(t), \eta(t)) \, dt \) and the derivation \( D\xi = \xi' \) is skew-symmetric. This leads to the double extension

\[
\hat{\mathcal{L}}_\varphi(k) = \mathbb{R} \oplus \mathcal{L}_\varphi(k) \oplus \mathbb{R}
\]

\[
[(z_1, \xi_1, t_1), (z_2, \xi_2, t_2)] := ((\xi'_1, \xi'_2), t_1 \xi'_2 - t_2 \xi'_1 + [\xi_1, \xi_2], 0)
\]

**Theorem (N., ’02, N./Wockel ’09; Integrability Theorem)**

*If \( k \) is simple, then there exists a simply connected Fréchet–Lie group \( \hat{\mathcal{L}}_\varphi(K) \) with Lie algebra \( \hat{\mathcal{L}}_\varphi(k) \) and center \( \mathbb{T} \).*

**Definition**

We call \( \hat{\mathcal{L}}_\varphi(K) \) the corresponding (affine) Kac–Moody group.

**Goal:** Understand unitary rep’s of \( \hat{\mathcal{L}}_\varphi(K) \) ⇒ We need root data.
2.5. Root systems for Kac–Moody groups

Here are the candidates for root systems of $\hat{\mathcal{L}}_\varphi(\mathfrak{k})$:

**Theorem (Y. Yoshii, 2006)**

The irreducible reduced locally affine root systems of infinite rank are the following $X_J^{(1)} := X_J \times \mathbb{Z}$ for $X_J \in \{A_J, B_J, C_J, D_J\}$, $J$ an infinite set, and $B_J^{(2)} := (B_J \times 2\mathbb{Z}) \cup ((B_J)_{sh} \times (2\mathbb{Z} + 1))$, where $(B_J)_{sh} = \{\pm \varepsilon_j : j \in J\}$.

$C_J^{(2)} := (C_J \times 2\mathbb{Z}) \cup (D_J \times (2\mathbb{Z} + 1))$

$BC_J^{(2)} := (B_J \times 2\mathbb{Z}) \cup (BC_J \times (2\mathbb{Z} + 1))$, $BC_J := B_J \cup C_J$.

These root systems contain no root bases $\Rightarrow$ No Dynkin diagrams.
To obtain root decompositions of $\hat{\mathcal{L}}_\varphi(\mathfrak{k})$, we assume:

$\mathfrak{k}$ is simple, $\varphi \in \text{Aut}(\mathfrak{k})$ involution,

$\mathfrak{t} \subseteq \mathfrak{k}^\varphi := \{x \in \mathfrak{k} : \varphi(x) = x\}$ is maximal abelian.
Then \( \hat{\mathfrak{t}} := \mathbb{R} \times \mathfrak{t} \times \mathbb{R} \subseteq \hat{\mathcal{L}}_\varphi(\mathfrak{t}) \) is maximal abelian.

For \( \alpha : \mathfrak{t} \to i\mathbb{R} \) and \( n \in \mathbb{Z} \) we define \( (\alpha, n) : \hat{\mathfrak{t}} \to i\mathbb{R} \) by

\[
(\alpha, n)(z, x, t) := \alpha(x) + \text{int}.
\]

Then the (anisotropic) root system \( \hat{\Delta} := \Delta(\hat{\mathcal{L}}_\varphi(\mathfrak{t}), \hat{\mathfrak{t}}) \) is

\[
\hat{\Delta} = (\Delta_+ \times 2\mathbb{Z}) \cup (\Delta_- \times (2\mathbb{Z} + 1)) \quad \text{with} \quad \Delta_\pm := \Delta(\mathfrak{t}^\pm, \mathfrak{t}) \quad \mathfrak{t}-(\text{weights}).
\]

**Theorem (Realization of the 7 locally affine root systems)**

For \( \Delta(\mathfrak{k}, \mathfrak{t}) = X_J \) we obtain \( \hat{\Delta} = \Delta(\hat{\mathcal{L}}(\mathfrak{k}), \hat{\mathfrak{t}}) = X_J^{(1)} \),
and \( \hat{\Delta} = \Delta(\hat{\mathcal{L}}_\varphi(\mathfrak{t}), \hat{\mathfrak{t}}) = X_J^{(2)} \) is obtained for \( \varphi(x) = \sigma x \sigma^{-1} \) as follows:

- \( B_J^{(2)} \) for \( \mathfrak{k} = \mathfrak{o}_2(\mathcal{H}) \), \( \sigma \) orth. reflection in a hyperplane, \( \mathfrak{t} \) of type \( B_J \).
- \( C_J^{(2)} \) for \( \mathfrak{k} = \mathfrak{u}_2(\mathcal{H}) \), \( \sigma \) antilinear with \( \sigma^2 = -1 \).
- \( BC_J^{(2)} \) for \( \mathfrak{k} = \mathfrak{u}_2(\mathcal{H}) \), \( \sigma \) antilinear with \( \sigma^2 = 1 \) and \( \ker(\mathfrak{t}) \neq \{0\} \).

Three root systems for \( \hat{\mathcal{L}}(\mathfrak{o}_2(\mathcal{H})) \cong \hat{\mathcal{L}}_\varphi(\mathfrak{o}_2(\mathcal{H})) \): \( B_J^{(1)}, D_J^{(1)} \) and \( B_J^{(2)} \).
Two root systems for \( \hat{\mathcal{L}}_\sigma(\mathfrak{u}_2(\mathcal{H})) \) (\( \sigma \) antilinear): \( C_J^{(2)} \) and \( BC_J^{(2)} \).
3.1. Bounded and semibounded representations

Definition
A unitary representation $\pi : G \to U(\mathcal{H})$ is called \textbf{smooth} if the space $\mathcal{H}_\infty := \{ v \in \mathcal{H} : G \to \mathcal{H}, g \mapsto \pi(g)v \text{ smooth} \}$ of smooth vectors is dense.

The derived representation: $d\pi(x)v = \frac{d}{dt}|_{t=0}\pi(\exp tx)v, \quad v \in \mathcal{H}_\infty, x \in \mathfrak{g}$.

The support function: $s_\pi : \mathfrak{g} \to \mathbb{R} \cup \{\infty\}, s_\pi(x) := \sup \text{Spec}(id\pi(x))$

Cone of semiboundedness: $W_\pi := \{ x \in \mathfrak{g} : s_\pi \text{ bounded in a nbhd of } x \}$.

Definition
A smooth representation is called \textbf{semibounded} if $W_\pi \neq \emptyset$.
It is called \textbf{bounded} if $W_\pi = \mathfrak{g}$.

Theorem (N. ’08)
$\pi$ bounded iff $d\pi : \mathfrak{g} \to u(\mathcal{H})$ continuous iff $\pi : G \to U(\mathcal{H})$ norm-cont.
3.2. Automatic boundedness

Example

If $K$ is compact, then every continuous unitary representation is a direct sum of irreducible ones and irreducible reps are bounded.

Remark

(a) If $\pi$ is semibounded, then $W_\pi$ is an open $\text{Ad}(G)$-invariant convex cone in $g$.
(b) If all open invariant cones in $g/\mathfrak{z}(g)$ are trivial, then every semibounded irreducible representation of $G$ is bounded.
(c) All open invariant cones in $g/\mathfrak{z}(g)$ are trivial iff all open invariant cones in $g$ intersect $\mathfrak{z}(g)$ (=$\text{fixed points of Ad}(G)$).

Examples

Lie algebras $g$ for which open invariant cones in $g/\mathfrak{z}(g)$ are trivial:
(a) $g$ semisimple Hilbert–Lie algebra (Bruhat–Tits Fixed Point Thm)
(b) $\mathfrak{u}(\mathcal{H})$, $\mathcal{H}$ Hilbert space over $\mathbb{R}, \mathbb{C}, \mathbb{H}$. 
4. Bounded representations of Hilbert–Lie groups

$t$ simple Hilbert–Lie algebra, $t \subseteq \mathfrak{k}$ maximal abelian, $\Delta$ corresp. roots

Coroots: $\check{\alpha} \in it \cap [t^\alpha_C, t^{-\alpha}_C]$ with $\alpha(\check{\alpha}) = 2$, for $\alpha \in \Delta$

$K$ the 1-connected Lie group with Lie algebra $\mathfrak{k}$; $T := \exp(t)$

$\mathcal{P}_T := \{ \lambda \in i t' : (\forall \alpha \in \Delta) \lambda(\check{\alpha}) \in \mathbb{Z} \} \cong \text{Hom}(T, \mathbb{T}) \subseteq i t'$ ($T$-weights)

Weyl group: $\mathcal{W} = \langle r_\alpha : \alpha \in \Delta \rangle \subseteq \text{GL}(t_C)$, $r_\alpha(x) = x - \alpha(x)\check{\alpha}$.

Theorem (Classification Theorem, N. ’98, ’11)

Bounded unitary representations of $K$ are direct sums of irreducible ones. The irreducible bounded reps $\pi_\lambda$ are characterized by their $T$-weight set

$$\text{conv}(\mathcal{W}\lambda) \cap (\lambda + Q), \quad Q = \langle \Delta \rangle_{\text{grp}} \subseteq \mathcal{P}_T \ (\text{root group}).$$

Classification of bounded irreps by $\mathcal{P}_T / \mathcal{W}$ (every $\lambda \in \mathcal{P}_T$ occurs).

Remark

(a) Bounded reps of $K$ behave like reps of a compact group.
(b) The continuous representation theory of $K$ is not type I (Boyer ’80)
5. Semibounded representations of Kac–Moody groups

\[ G = \hat{\mathcal{L}}_\varphi(K) \] (1-connected) as above (7 types), \( d := (0, 0, -i) \in i\hat{\mathcal{L}}_\varphi(\mathfrak{k}) \)

- \( \pi \) irreducible semibounded rep of \( G \) \( \Rightarrow \) \( i \cdot d \in W_\pi \cup -W_\pi \)
  (positive/negative energy representations if \( \pm d\pi(d) \) bounded below).
- We use that \( \mathfrak{k}^\varphi \) is simple, hence all its open inv. cones are trivial.
- On the minimal/maximal eigenspace of \( d\pi(d) \) we find a bounded irreducible representation \( \rho_\lambda \) of \( Z_G(d) \cong \mathbb{T} \times K^\varphi \times \mathbb{R} \)

**Theorem (Classification Theorem, Part 1)**

Irreducible semibounded representations \( \pi_\lambda \) of \( \hat{\mathcal{L}}_\varphi(K) \) are extremal weight representations characterized by their \( \hat{\mathfrak{t}} \)-weight set

\[ \mathcal{P}_\lambda := \text{conv}(\hat{\mathcal{W}}\lambda) \cap (\lambda + \hat{Q}) \quad \text{with} \quad \text{Ext}(\text{conv}(\mathcal{P}_\lambda)) = \hat{\mathcal{W}}\lambda \]

(\( \hat{\mathcal{W}} \) is the Weyl group of \( \hat{\Delta} \)). The set of occurring extremal weights \( \lambda \) is

\[ \mathcal{P}^\pm := \{ \mu \in \mathcal{P}_{\hat{\varphi}} : \pm (\hat{\mathcal{W}}\mu)(d) \text{ bounded from below} \} \]
Let $\mathcal{P}^\pm_d \subseteq \mathcal{P}^\pm$ denote those elements $\mu$ for which $\mu(d)$ is minimal/maximal in $\hat{\mathcal{W}}\mu$. With $c := \mu(i, 0, 0)$ (central charge), the elements $\mu \in \mathcal{P}^+_d$ are characterized by:

$$c \geq 0, \quad |\mu(\tilde{\alpha})| \leq \frac{2c}{(\alpha, \alpha)}, \quad |\mu(\tilde{\beta})| \leq \frac{4c}{(\beta, \beta)} \quad \text{for} \quad (\alpha, 1), (\beta, 2) \in \hat{\Delta}. \quad \tag{1}$$

\textbf{Theorem (Classification Theorem, Part 2)}

\textit{Classification of semibounded irreps:} $\mathcal{P}^\pm / \hat{\mathcal{W}} \cong \mathcal{P}^\pm_d / \mathcal{W}$.

\textbf{Methods:}

- **Convex geometry** of $\hat{\mathcal{W}}$-orbits (local Coxeter theory).
- **Complex geometry**: Realization of $\pi_\lambda$ in holomorphic sections of a complex Hilbert bundle with fiber representation $\rho_\lambda$ of $Z_G(d)$ over the complex manifold $G/Z_G(d) \cong \mathcal{L}_\varphi(K)/K^\varphi$ (holomorphic induction).
- **Harmonic analysis**: Locally defined operator-valued analytic positive definite functions; automatic extension.
6. Semibounded projective reps of Hilbert–Lie groups

**Problem:** Boundedness of representations of a Hilbert–Lie group $K$ is rather restrictive. It excludes important representations like “infinite wedge representations”. These lead to projective representations, hence to central extensions and further to double extensions.

**Setup:** $\mathfrak{k}$ simple Hilbert–Lie algebra, $\mathfrak{t} \subseteq \mathfrak{k}$ maximal abelian, $\check{\Delta} \subseteq i\mathfrak{t}$ coroots

A $\mathfrak{t}$-invariant continuous cocycle $\omega(x, y)$ on $\mathfrak{k}$ can be represented by $D \in \text{der}(\mathfrak{k})$ via $\omega(x, y) = (Dx, y)$ and there exists a linear functional $\lambda: \mathfrak{t} \cap [\mathfrak{k}, \mathfrak{k}] \rightarrow i\mathbb{R}$ with

$$\omega(x, y) = i\lambda([x, y]) \quad \text{for} \quad x, y \in \mathfrak{k}.$$

We call $\lambda$ a bounded weight if $\lambda(\check{\alpha}) \in \mathbb{Z}$ for $\alpha \in \Delta$; $\mathcal{P}_b$ set of bd weights.

**Definition**

For $\lambda \in \mathcal{P}_b$ we write $\hat{\mathfrak{k}}_\lambda = \mathbb{R} \oplus \mathfrak{t} \oplus \mathbb{R}$ for the corresp. double extension. $\hat{\mathfrak{t}} := \mathbb{R} \oplus \mathfrak{t} \oplus \mathbb{R} \subseteq \hat{\mathfrak{k}}_\lambda$ is maximal abelian.

$\hat{K}_\lambda$ is the corresponding 1-connected group; $\hat{T} := \exp \hat{\mathfrak{t}} \subseteq \hat{K}_\lambda$.

$\mathcal{P}_{\hat{T}} \subseteq \text{Hom}(\hat{\mathfrak{t}}, i\mathbb{R})$ (group of $\hat{T}$-weights).
Irreducible semibounded representations $\pi_\mu$ of $\hat{K}_\lambda$ are extremal weight representations characterized by their $\hat{t}$-weight set

$$P_\mu := \text{conv}(\mathcal{W}_\mu) \cap (\mu + \mathbb{Q}) \quad \text{with} \quad \text{Ext}(\text{conv}(P_\mu)) = \mathcal{W}_\mu.$$

Put $d := (0, 0, -i) \in \hat{t}$. The set of occurring extremal weights is

$$P^\pm := \{\mu \in P_{\hat{t}} : \pm (\mathcal{W}_\mu)(d) \text{ bounded from below}\}.$$

By minimizing/maximizing, we get the $d$-extremal weights

$$P^\pm_d = \{\mu \in P_{\hat{t}} : (\forall \alpha \in \Delta) \lambda(\check{\alpha}) > 0 \Rightarrow \pm \mu(\check{\alpha}) \geq 0\}.$$

Classification: $P^\pm / \mathcal{W} \cong P^\pm_d / \mathcal{W}_\lambda$, where $\mathcal{W}_\lambda \subseteq \mathcal{W}$ is the stabilizer of $\lambda$.

Remark: (a) Representations of $\hat{K}_\lambda$ are projective representations of $K$. (b) For $K = U_2(\mathcal{H})$ we cover in particular infinite wedge representations.
Again, projective representations of \( \hat{\mathcal{L}}_{\varphi}(K) \) lead to double extensions of \( g = \hat{\mathcal{L}}_{\varphi}(\mathfrak{k}) \), hence to iterated double extensions \( \hat{\hat{\mathcal{L}}}_{\varphi}(\mathfrak{k}) \). Here the cocycle is of the form

\[
\omega((z_1, \xi_1, t_1), (z_2, \xi_2, t_2)) := \frac{1}{2\pi} \int_0^{2\pi} i\lambda([\xi_1(t), \xi_2(t)]) \, dt
\]

for some bounded weight \( \lambda \in \mathcal{P}_b \) for \((\mathfrak{k}, \mathfrak{t})\). Corresponding Lie groups \( \hat{\hat{\mathcal{L}}}_{\varphi}(K) \) exist, and for \( d = (0, 0, -i) \in i\hat{\mathcal{L}}_{\varphi}(\mathfrak{k}) \) we have

\[
Z_g(d) = \mathbb{R} \oplus (\mathbb{R} \oplus \mathfrak{t}_{\varphi} \oplus \mathbb{R}) \oplus \mathbb{R} = \mathbb{R} \oplus \hat{\mathfrak{t}}_{\lambda} \oplus \mathbb{R}.
\]

Semibounded representations of \( \hat{\hat{\mathcal{L}}}_{\varphi}(K) \) now lead to semibounded representations of the double extension \((\hat{K}_{\varphi})_{\lambda}\). These representations are classified!
Conjecture

Irreducible semibounded representations $\pi_\mu$ of $\hat{L}_\varphi(K)$ are extremal weight representations characterized by their $\hat{t}$-weight set

$$\mathcal{P}_\mu := \text{conv}(\hat{\mathcal{W}}\mu) \cap (\mu + \hat{Q}) \quad \text{with} \quad \text{Ext}(\text{conv}(\mathcal{P}_\mu)) = \hat{\mathcal{W}}\mu.$$  

The set of occurring extremal weights is

$$\mathcal{P}^\pm := \{ \mu \in \mathcal{P}_{\hat{t}} : \pm (\hat{\mathcal{W}}\mu)(d) \text{ bounded from below} \}.$$  

By minimizing/maximizing, we get the $d$-extremal weights $\mathcal{P}_d^\pm$.

Classification of semibounded irreps: $\mathcal{P}^\pm / \hat{\mathcal{W}} \cong \mathcal{P}_d^\pm / \hat{\mathcal{W}}_d$.

Problems: (a) The complex geometric Banach methods (holomorphic induction) break down because the representations of $\hat{K}_\lambda$ are unbounded. We need a weaker notion of a complex Hilbert bundle.
(b) The iterated double extension creates 2 “$d$-elements”, but semiboundedness should be controlled by the first one. This requires refined information on convexity properties of coadjoint orbits.
Positive energy vs. semiboundedness

- **Semiboundedness is stronger than the positive energy condition** $d \pi(d) \text{bd. below.}$ It is crucial that semiboundedness implies boundedness of the $K$-representation on the minimal energy space. This is automatic if $K$ is compact. In general $K$ has many irreducible unbounded representations which are harder to control, f.i., Boyer's factor representations of $U_2(H)$. We do not expect that the positive energy condition implies semiboundedness in general.

- **Semiboundedness is intrinsic**, it does not refer to the specification of an element $d \in \mathfrak{g}$, such as the positive energy condition. It also does not refer to a specific Cartan subalgebra.

- Our classification results hold for each of the 7 types of root systems of the 4 classes of Lie algebras. For different root systems, resp., conjugacy classes of Cartan subalgebras, we obtain different parameters for the same representations.
Concluding remarks

- **Non-connected loop groups**: \( \pi_0(\mathcal{L}(K)) \cong \pi_1(K) \) is non-trivial in general. Which semibounded representations extend to non-connected groups?

- We need a better understanding of the concept of a Cartan subalgebra for \( \widehat{\mathcal{L}_\varphi}(\mathfrak{k}) \). Are there finitely many conjugacy classes?

- Describe the automorphism group of \( \mathcal{L}_\varphi(\mathfrak{k}) \).

- Are there also semibounded representations for double extensions of mapping groups \( C^\infty(M, K) \), where \( \dim M > 1 \)? The corresponding derivations should correspond to divergence free vector fields on \( M \). Possibly one has to consider \( n \)-fold iterated double extensions, where \( n = \dim M \). Here \( M = \mathbb{T}^2 \) is the natural testing case.

- For \( K = U_2(\mathcal{H}) \), \( \mathcal{H} \) complex, we have \( \text{Aut}(K)_0 \cong \text{PU}(\mathcal{H}) \), so that \( K \)-group bundles with this structure group over \( X \) are classified by their Dixmier–Douady classes in

\[
[X, B \text{PU}(\mathcal{H})] = [X, K(\mathbb{Z}, 3)] \cong H^3(X, \mathbb{Z}).
\]