Wreath products as isometry groups of non standard metric products

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Let \((X_i, d_i), \ i = 1, \ldots, n\), be metric spaces. To define a metric on their cartesian products \(X = \prod_{i=1}^{n} X_i\) one can use, for instance, one of the following equalities
\[
d((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = d_1(x_1, y_1) + \ldots + d_n(x_n, y_n);
\]
\[
\tilde{d}((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = \sqrt{d_1^2(x_1, y_1) + \ldots + d_n^2(x_n, y_n)}.
\]
There are different generalizations of these constructions. They include $f$-products (M. Moszynska, 1992), warped products (C.-H. Chen, 1999), $\mu$-products (S. Avgustinovich, D. Fon-Der-Flaass, 2000), etc. Following A. Bernig, T. Foertsch, V. Schroeder (2003) we consider non standard metric products or $\Phi$-products of metric spaces.
Assume that $\Phi : [0, \infty)^n \rightarrow [0, \infty)$ be a function such that the following conditions hold

(A) $\Phi(p_1, p_2, \ldots, p_n) = 0$ iff $p_1 = p_2 = \ldots = p_n = 0$;

(B) for arbitrary $q_i, r_i, p_i \in [0, \infty)$ such that $q_i \leq r_i + p_i$, $1 \leq i \leq n$, the inequality

$$
\Phi(q_1, \ldots, q_n) \leq \Phi(r_1, \ldots, r_n) + \Phi(p_1, \ldots, p_n)
$$

holds.
Then the function

\[ d_\Phi((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = \Phi(d_1(x_1, y_1), \ldots, d_n(x_n, y_n)) \]

is a metric on \( X \).

**Definition**
The metric space \((X, d_\Phi)\) is called a \( \Phi \)-product or non standard metric product of metric spaces \( X_1, \ldots, X_n \).
Let $q$ be a positive real number. It is easy to see that the function

$$
\hat{\Phi}(p_1, p_2, \ldots, p_n) = \begin{cases} 
0, & \text{if } p_1 = p_2 = \ldots = p_n = 0 \\
q, & \text{in other cases}
\end{cases}
$$

meets conditions (A) and (B).

The isometry group of $(X, d_{\hat{\Phi}})$ is isomorphic as a permutation group to the symmetric group $S_{|X|}$. This is the largest possible isometry group of $\Phi$-products of $X_1, \ldots, X_n$. 

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Proposition 1 Let $X$ be a $\Phi$-product of metric spaces $X_1, \ldots, X_n$, $n \geq 2$. Then the transformation group $(Isom X, X)$ contains a subgroup isomorphic to the direct product of the transformation groups

$$(Isom X_1, X_1) \times \ldots \times (Isom X_n, X_n).$$
Let now \((X_1, d_1), \ldots, (X_n, d_n)\) be discrete spaces, i.e., for different points \(u, v \in X_i\) \(d_i(u, v) = 1, 1 \leq i \leq n\). And let \(|X_i| = k_i, 1 \leq i \leq n\). We can introduce the function \(\Phi_1 : [0, \infty)^n \to [0, \infty)\) putting

\[
\Phi_1(q_1, \ldots, q_n) = \begin{cases} 
q_1, & \text{if } q_1 \neq 0; \\
\frac{1}{2} q_2, & \text{if } q_1 = 0 \text{ and } q_2 \neq 0; \\
\vdots & \\
\frac{1}{n} q_n, & \text{if } q_1 = 0, \ldots, q_{n-1} = 0, q_n \neq 0; \\
0, & \text{if } q_1 = 0, \ldots, q_n = 0.
\end{cases}
\]
Let $T$ be a finite $n$-levels rooted tree with root $v_0$. Assume, that a rooted tree $T$ is level homogenous with level index $[k_1; k_2; \ldots, k_n]$, where $k_i$ is the number of edges joining a vertex of the $i$-th level with vertices of the $(i+1)$-st level. The metric space $\delta T$ is defined to be the set of all rooted path of $T$ equipped with a natural ultrametric

$$\rho(\gamma_1, \gamma_2) = 1/(m + 1),$$

where $m$ is the length of the maximal common part of rooted paths $\gamma_1$ and $\gamma_2$. 
The space $\delta T$ of paths in the rooted level homogeneous tree $T$ and the $\Phi_1$-product of discrete metric spaces $X_1, \ldots, X_n$ are isometric. It is well known that the isometry group of the space $\delta T$ is isomorphic as a permutation group to the wreath product of symmetric group $S_{k_i}$, $i = 1, \ldots, n$. Therefore, the isometry group of the space $(X_1 \times \ldots \times X_n, d_{\Phi_1})$ is isomorphic as a permutation group to the wreath product of isometry groups of discrete spaces $X_i$, $i = 1, \ldots, n$. 

Let now \((X_i, d_i), i = 1, \ldots, n,\) be arbitrary metric spaces. And let as before \(C_i\) be the set of values of metric \(d_i, 1 \leq i \leq n.\) Assume that there exist functions \(f_i : [0, \infty) \rightarrow [0, \infty), 1 \leq i \leq n,\) such that

\[
\Phi(q_1, \ldots, q_n) = \begin{cases}
    f_1(q_1), & \text{if } q_1 \neq 0; \\
    f_2(q_2), & \text{if } q_1 = 0 \text{ and } q_2 \neq 0; \\
    \ldots & \ldots \\
    f_n(q_n), & \text{if } q_1 = 0, \ldots, q_{n-1} = 0, q_n \neq 0; \\
    0, & \text{if } q_1 = 0, \ldots, q_n = 0
\end{cases}
\]

for arbitrary \(q_i \geq 0, 1 \leq i \leq n.\)
For each $i, 1 \leq i \leq n$, denote by $\hat{X}_i$ the space $(X_i, \hat{d}_i)$, where for arbitrary $u, v \in X_i$

$$\hat{d}_i(u, v) = \begin{cases} f_i(d_i(u, v)), & \text{if } u \neq v \\ 0, & \text{in other cases} \end{cases}.$$ 

Assume that for all $i, 1 \leq i \leq n - 1$, the inequalities

$$\inf_{q_i \in C_i, q_i \neq 0} f_i(q_i) > \sup_{q_{i+1} \in C_{i+1}} f_{i+1}(q_{i+1})$$

(2)

hold.
Theorem
Let $\Phi : [0, \infty)^n \to [0, \infty)$ be a function such that conditions (A),(B), (1) and (2) hold. Then the isometry group of the $\Phi$-product of metric spaces $X_1, X_2, \ldots, X_n$ is isomorphic as a permutation group to the wreath product of isometry groups of spaces $\hat{X}_i, i = 1, \ldots, n$,

$$(Isom(X, d_\Phi), X) \simeq \wr_{i=1}^n (Isom\hat{X}_i, X_i).$$
Corollary
Let $\Phi : [0, \infty)^n \to [0, \infty)$ be a function such that the conditions (A), (B), (1) and (2) hold. If

$$Isom(X_i, f_i(d_i)) = Isom(X_i, d_i)$$

for all $i, 1 \leq i \leq n$, then

$$(IsomX, X) \simeq \smallsum_{i=1}^{n}(IsomX_i, X_i).$$
Example
Let $X_i = \mathbb{Z}$ and $d_i$ be the Euclidean distance, $1 \leq i \leq n$. The function

$$
\Phi_5(q_1, \ldots, q_n) = \begin{cases}
  n + 1 - \frac{1}{q_1+1}, & \text{if } q_1 \neq 0; \\
  n - \frac{1}{q_2+1}, & \text{if } q_1 = 0 \text{ and } q_2 \neq 0; \\
  \ldots \ldots \ldots \ldots & \\
  2 - \frac{1}{q_n+1}, & \text{if } q_1 = 0, \ldots, q_{n-1} = 0, q_n \neq 0; \\
  0, & \text{if } q_1 = 0, \ldots, q_n = 0.
\end{cases}
$$

meets conditions (A),(B), (1) and (2). Therefore, one can consider the $\Phi_5$-product $(\mathbb{Z} \times \ldots \times \mathbb{Z}, d_{\Phi_5})$ of $X_i$, $1 \leq i \leq n$. The isometry group of $(\mathbb{Z} \times \ldots \times \mathbb{Z}, d_{\Phi_5})$ is isomorphic as a permutation group to the wreath product of $n$ infinite dihedral groups $D_\infty$.

$$(Isom(\mathbb{Z} \times \ldots \times \mathbb{Z}, d_{\Phi_5}), \mathbb{Z} \times \ldots \times \mathbb{Z}) \simeq \wr_{i=1}^{n}(D_\infty, \mathbb{Z}).$$
Example
Let \((X_1, d_1)\) and \((X_2, d_2)\) be metric spaces of finite diameters \(D_1, D_2\). Assume that there exists a positive number \(r\) such that for arbitrary points \(x_1, x_2 \in X_1, x_1 \neq x_2\), the inequality \(d_1(x_1, x_2) \geq r\) holds. Let \(\Phi_3(q_1, q_2) = \max(q_1, q_2)\).
If the inequality
\[
    r > D_2
\]
holds then
\[
    \text{Isom}(X_1 \times X_2, d_{\Phi_3}) \simeq \text{Isom}X_1 \wr \text{Isom}X_2.
\]
Now we consider Φ-products \((X_1 \times X_2, d_\Phi)\) of two metric spaces \((X_1, d_1), (X_2, d_2)\). For each \(a_1 \in X_1, a_2 \in X_2\) let

\[
X_{a_1}^2 = \{(a_1, x_2) \mid x_2 \in X_2\}, \quad X_{a_2}^1 = \{(x_1, a_2) \mid x_1 \in X_1\}
\]

be subspaces of \((X_1 \times X_2, d_\Phi)\). The points of spaces \(X_{a_1}^2\), \(a_1 \in X_1\) are in natural one-to-one correspondence with the points of the space \(X_2\), while the points of spaces \(X_{a_2}^1\), \(a_2 \in X_2\) are in natural one-to-one correspondence with the points of \(X_1\). Hence we can assume that the group \(Isom X_{a_1}^2\) acts on the set \(X_2\) and the group \(Isom X_{a_2}^1\) acts on the set \(X_1\).
Denote by $C_i$ the set of values of the metric $d_i$, $i = 1, 2$. Assume that inequalities

$$
\inf_{q_1 \in C_1, q_1 \neq 0} \Phi(q_1, 0) > \sup_{q_2 \in C_2} \Phi(0, q_2),
$$

$$
\inf_{q_2 \in C_2, q_2 \neq 0} \Phi(0, q_2) > \frac{1}{2} \sup_{q_1 \in C_1} \Phi(q_1, 0). \quad (3)
$$

hold.
**Theorem**

Let $\Phi : [0, \infty)^2 \to [0, \infty)$ be a function such that conditions (A),(B) and inequalities (3) hold. Assume that

$$\Phi(q_1, q_2) = \Phi(q_1, 0) + \Phi(0, q_2).$$  \hspace{1cm} (4)

Then

$$(IsomX, X) \simeq (IsomX_{a_2}^1, X_1) \times (IsomX_{a_1}^2, X_2)$$

for any $(a_1, a_2) \in X_1 \times X_2$. 

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Corollary
Let $\Phi : [0, \infty)^2 \to [0, \infty)$ be a function such that conditions (A),(B) and inequalities (3) hold. Assume that

$$\Phi(q_1, q_2) = \Phi(q_1, 0) + \Phi(0, q_2).$$

If $IsomX_{a_2}^1 = IsomX_1$, $IsomX_{a_1}^2 = IsomX_2$ for some $(a_1, a_2) \in X_1 \times X_2$, then

$$(IsomX, X) \cong (IsomX_1, X_1) \times (IsomX_2, X_2).$$
Example
Let \((X_1, d_1)\) and \((X_2, d_2)\) be uniformly discrete metric spaces of finite diameters \(D_1, D_2\) correspondingly. And let \(r_1, r_2\) be positive numbers, such that for arbitrary points \(x_1, x_2 \in X_i, x_1 \neq x_2\), the inequalities \(d_i(x_1, x_2) \geq r_i\) hold, \(i = 1, 2\). Denote \(\Phi_2(q_1, q_2) = q_1 + q_2\). Then the function \(\Phi_2(q_1, q_2)\) meets conditions (A) and (B). If the inequalities

\[
r_1 > D_2 \geq r_2 > \frac{1}{2}D_1 \text{ or } r_2 > D_1 \geq r_1 > \frac{1}{2}D_2
\]

hold then the inequalities (3) hold as well. Therefore

\[
Isom(X_1 \times X_2, d_{\Phi_2}) \simeq IsomX_1 \times IsomX_2.
\]
Example
Let \((X_1, d_1)\) and \((X_2, d_2)\) be metric spaces. Let

\[
\Phi_1(q_1, q_2) = \begin{cases} 
0, & \text{if } p_1 = p_2 = 0 \\
4, & \text{if } p_1 \neq 0, \ p_2 = 0 \\
3, & \text{if } p_1 = 0, \ p_2 \neq 0 \\
q_1 + q_2, & \text{in other cases}
\end{cases}
\]

Then

\[
Isom(X_1 \times X_2, d_{\Phi_1}) \cong S_{|X_1|} \times S_{|X_2|}.
\]
Let \((G_1, X_1), \ldots, (G_n, X_n)\) be a sequence of transformation groups. Following Kaloujnine L.A., Beleckij P.M., Feinberg V.T the transformation group \((G, \prod_{i=1}^n X_i)\) is called the \textit{wreath products of groups} \((G_1, X_1), \ldots, (G_n, X_n)\) if for all elements \(u \in G\) the following conditions hold:

1) if \((x_1, \ldots, x_n)^u = (y_1, \ldots, y_n)\), then for all \(i, 1 \leq i \leq n\), the value of \(y_i\) depends only on \(x_1, \ldots, x_i\);

2) for fixed \(x_1, \ldots, x_{i-1}\) the mapping \(g_i(x_1, \ldots, x_{i-1})\) defined by the equality

\[
g_i(x_1, \ldots, x_{i-1})(x_i) = y_i, \quad x_i \in X_i
\]

is a permutation on the set \(X_i\) which belongs to \(G_i\). Denote the wreath products of groups \((G_1, X_1), \ldots, (G_n, X_n)\) by \(\wr_{i=1}^n(G_i, X_i)\).