Spin$^{\mathbb{C}}$ **structures**

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Clifford algebra

Definition

By a Clifford algebra over the real numbers we shall understand an associative algebra with unity, generated by elements

$$\{e_1, e_2, \ldots, e_n\}$$

and with relations

$$\forall i, e_i^2 = -1,$$

$$\forall i, j, e_i e_j = -e_j e_i,$$

where $1 \leqslant i, j \leqslant n$. *We define* $C_0 = \mathbb{R}$.

It is easy to see that $C_1 = \mathbb{C}$ and $C_2 = \mathbb{H}$, where $\mathbb{H}$ is the four-dimensional quaternion algebra. Moreover, $\mathbb{R}^n \subset C_n$ and $\dim_{\mathbb{R}} C_n = 2^n$, where $\mathbb{R}^n$ is n-dimensional $\mathbb{R}$-vector space with the basis $e_1, e_2, \ldots, e_n$.

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We have the following homomorphisms (involutions) on C_n :

(i) $^* : e_{i_1} e_{i_2} \dots e_{i_k} \mapsto e_{i_k} e_{i_{k-1}} \dots e_{i_2} e_{i_1}$,

(ii) $' : e_i \mapsto -e_i$,

(iii) $^- : a \mapsto (a')^*, a \in C_n$.

Suppose $C_n^0 = \{x \in C_n \mid x' = x\}$. It is easy to observe that

$$\forall a, b \in C_n, (ab)^* = b^* a^*.$$

Definition

We define subgroups of C_n,

$$Pin(n) = \{x_1 x_2 \dots x_k \mid x_i \in S^{n-1} \subset \mathbb{R}^n \subset C_n, i = 1, 2, \dots k\},$$

$$Spin(n) = Pin(n) \cap C_n^0.$$

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A closed oriented manifold M^n has a Spin*-structure if and only if the second Stiefel-Whitney class $w_2(M^n) = 0$.*

Remark

A Spin*-structure on the manifold M^n is a lift of δ to* $\mathrm{BSpin}(n)$*, giving a commutative diagram*:

$$\begin{array}{ccc} & & \mathrm{BSpin}(n) \\ & \nearrow & \downarrow{\scriptstyle B(\lambda_n)} \\ M^n & \xrightarrow{\delta} & \mathrm{BSO}(n). \end{array}$$

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The group $\mathrm{Spin}^{\mathbb{C}}(n)$ is given by

$$\mathrm{Spin}^{\mathbb{C}}(n) = (\mathrm{Spin}(n) \times S^1)/\{1, -1\}$$

where $\mathrm{Spin}(n) \cap S^1 = \{1, -1\}$. Moreover, there is a homomorphism of groups

$$\bar{\lambda}_n : \mathrm{Spin}^{\mathbb{C}}(n) \to \mathrm{SO}(n)$$

given by

$$\bar{\lambda}_n[g, z] = \lambda_n(g),$$

where $g \in \mathrm{Spin}(n), z \in S^1$ and $\lambda_n : \mathrm{Spin}(n) \to \mathrm{SO}(n)$ is the universal covering.

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Spin$^{\mathbb{C}}$-structure

The oriented manifold M^n has a Spin$^{\mathbb{C}}$-structure if and only if there exists $\tilde{w}_2 \in H^2(M^n, \mathbb{Z})$ such that $red(\tilde{w}_2) = w_2$, where $w_2 \in H^2(M^n, \mathbb{Z}_2)$ and $red : H^2(M^n, \mathbb{Z}) \to H^2(M^n, \mathbb{Z}_2)$ is a homomorphism induced by the natural homomorphism $\mathbb{Z} \to \mathbb{Z}_2$.

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Theorem

The set of lifts of δ is in bijection correspondence with $[M^n, BU(1)]$.

Proof: We have the bundle

$$BU(1) \to \mathrm{BSpin}^{\mathbb{C}}(n) \to BSO(n).$$

Let h_p denote the homeomorphism from $BU(1)$ to the fiber of $\mathrm{BSpin}^{\mathbb{C}}(n)$ over the point $p \in BSO(n)$. Given a map $\lambda \in [M^n, BU(1)]$, define the lift δ_λ by $\delta_\lambda(x) = h_{\delta(x)} \circ \lambda(x)$. This is an injective map from $[M^n, BU(1)]$ into the set of lifts; it is also surjective, since two different lifts will have to disagree on at least one fiber.

Comments

1. Any Spin manifold has a $\mathrm{Spin}^{\mathbb{C}}(n)$ structure,

2. Any three manifold has a Spin structure,

3. Any four manifold has a $\mathrm{Spin}^{\mathbb{C}}$ structure (Very old result W. T. Wu (1950), F. Hirzebruch-H. Hopf (1958)).

Dimension Three I

We have the following isomorphisms of Lie groups:

$$SO(3) = SU(2)/\{\pm 1\} = U(2)/U(1),$$

$U(1) = S^1$ lies in $U(2)$ as diagonal subgroup.

$$Spin(3) = SU(2) = S^3$$

$$Spin^{\mathbb{C}}(3) = (U(1) \times Spin(3))/\{\pm 1\} = (U(1) \times SU(2))/\{\pm 1\} = U(2).$$

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Simple remark

The isomorphism classes of principal circle bundles over a CW space X are numerated by the elements of $[X, BU(1)] = [X, \mathbb{CP}^\infty] = [X, K(\mathbb{Z}, 2)] = H^2(X, \mathbb{Z})$.

As an example let us consider the circle bundle

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$\mathrm{Spin}^{\mathbb{C}}$ in dimension 3

Let M^3 be a closed, oriented Riemannian 3-manifold. Let $f_M : Fr \to M^3$ be the associated principal $SO(3)$-bundle of oriented orthonormal frames.

A $\mathrm{Spin}^{\mathbb{C}}$-structure on M^3 is a lift of f_M to a principal $U(2)$-bundle. In other words it is a pair: a principal $U(2)$-bundle $F \to M^3$ and an isomorphism α of principal $SO(3)$-bundles

$$F/U(1) \to M \quad \text{with} \quad Fr \xrightarrow{f_M} M^3.$$

We can consider the above isomorphism as the circle bundle

$$F \xrightarrow{proj} F/U(1) \xrightarrow{\alpha} Fr,$$

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Spin$^{\mathbb{C}}$ in dimension 3

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Vector Fields I

Let M^3 be an oriented closed 3-manifold. Since M^3 has trivial Euler characteristic, it admits (Th. Euler-Hopf) nowhere vanishing vector fields.

Definition

Let v_1 and v_2 be two nowhere vanishing vector fields. We say that v_1 is homologous to v_2 if there is a boll $B \subset M^3$ with the property that

$$v_1|_{M \setminus B} \simeq v_2|_{M \setminus B}.$$

(Here $\simeq$ = "is homotopic".)
This gives an equivalence relation, and we define the space of vect(M) structures over M^3 as nowhere vanishing vector fields modulo this relation. vect(M) is also sometimes called the set of Euler structures on M^3.

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Vector Fields II

An equivalent definition of Euler structures on M^3 can be given in terms of the spherical fiber bundle of unit tangent vectors $SM \to M^3$.

Definition

An Euler structure on M^3 is an element of $H^2(SM, \mathbb{Z})$ whose reduction to every fiber $S_x M, x \in M^3$ is the generator of

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Equivalence

Let u be a nowhere vanishing vector field of M^3. The mapping

$$x \mapsto u(x)/ \mid u(x) \mid : M^3 \to SM$$

defines an 3-cycle in $SM = M^3 \times S^2$.

It is an element of $H_3(SM, \mathbb{Z}) \simeq H^2(SM, \mathbb{Z})$. When we orient SM^3 properly the above element of $H^2(SM, \mathbb{Z})$ represented by this cycle is an Euler structure on M^3 in the sense of the second definition.

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Main Observation

Theorem

Let M^3 be a closed oriented Riemannian 3-manifold. There is a canonical $H_1(M^3, \mathbb{Z})$-equivariant bijection $vect(M) = \mathcal{S}(M)$.

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Sketch of the Proof

As before, consider the principal $SO(3)$-bundle $f_M : Fr \longrightarrow M^3$ and the spherical bundle $SM \longrightarrow M^3$. Denote by p the bundle morphism $Fr \longrightarrow SM$ which is given at a point of M^3 by a formula

$$p(e_1, e_2, e_3) = e_1.$$

Hence a homomorphism $p^* : H^2(SM, \mathbb{Z}) \to H^2(Fr, \mathbb{Z})$ sends $vect(M)$ to $\mathcal{S}(M)$. Moreover $vect(M) = \mathcal{S}(M) = H^2(M^3, \mathbb{Z})$.

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Final Remarks

1. Action $H_1(M^3,\mathbb{Z}) \times \mathcal{S}(M) \to \mathcal{S}(M) = H^2(M^3,\mathbb{Z})$: Let $x \in H_1(M^3,\mathbb{Z}), y \in \mathcal{S}(M), (x,y) \mapsto f_M^*(x) + y \in \mathcal{S}(M) \subset H^2(Fr,\mathbb{Z})$.

2. Action $H_1(M^3,\mathbb{Z}) \times vect(M) \to vect(M) \subset H^2(SM,\mathbb{Z})$: Let $x \in H_1(M^3,\mathbb{Z}), y \in vect(M), (x,y) \mapsto f_y^*(x) + y \in vect(M) \subset H^2(SM,\mathbb{Z})$ where $f_y : SM \to M^3$ is a spherical fiber bundle of unit tangent vectors.
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