Khovanov homologies for knots

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To understand Khovanov homologies for links we need first to recall some basic facts about the Jones polynomial. We start with the Kauffman version of the Jones polynomial. This is a polynomial f_L, assigned to any oriented link L, $f_L \in \mathbb{Z}[A, A^{-1}]$ It was originally defined in the following way. First, we define Kauffman bracket. This is a polynomial $\langle D \rangle$ assigned to an unoriented diagram D by the following rules:

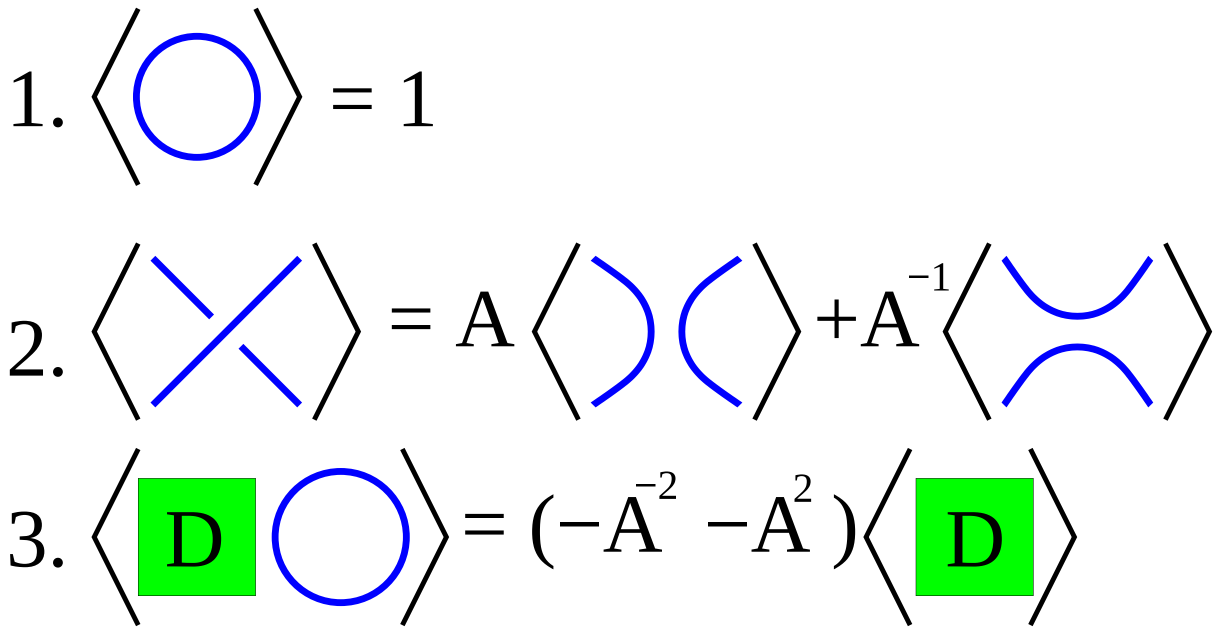

figure KH1

The bracket, as defined, is invariant under Reidemeister moves 2 and 3. In order to make it invariant under all three moves, we define the polynomial f_L by the following formula.

$$f_D = (-A)^{-3tw(D)} \langle D \rangle.$$

Of course, by the bracket of an oriented diagram we mean the bracket of the same diagram with orientation forgotten.
We classify crossings in the (oriented) diagram as positive or negative as shown in figure KH2.
We need to stress that in the absence of orientation we still can classify a smoothing of a crossing in the diagram as positive or negative as shown in figure KH2.

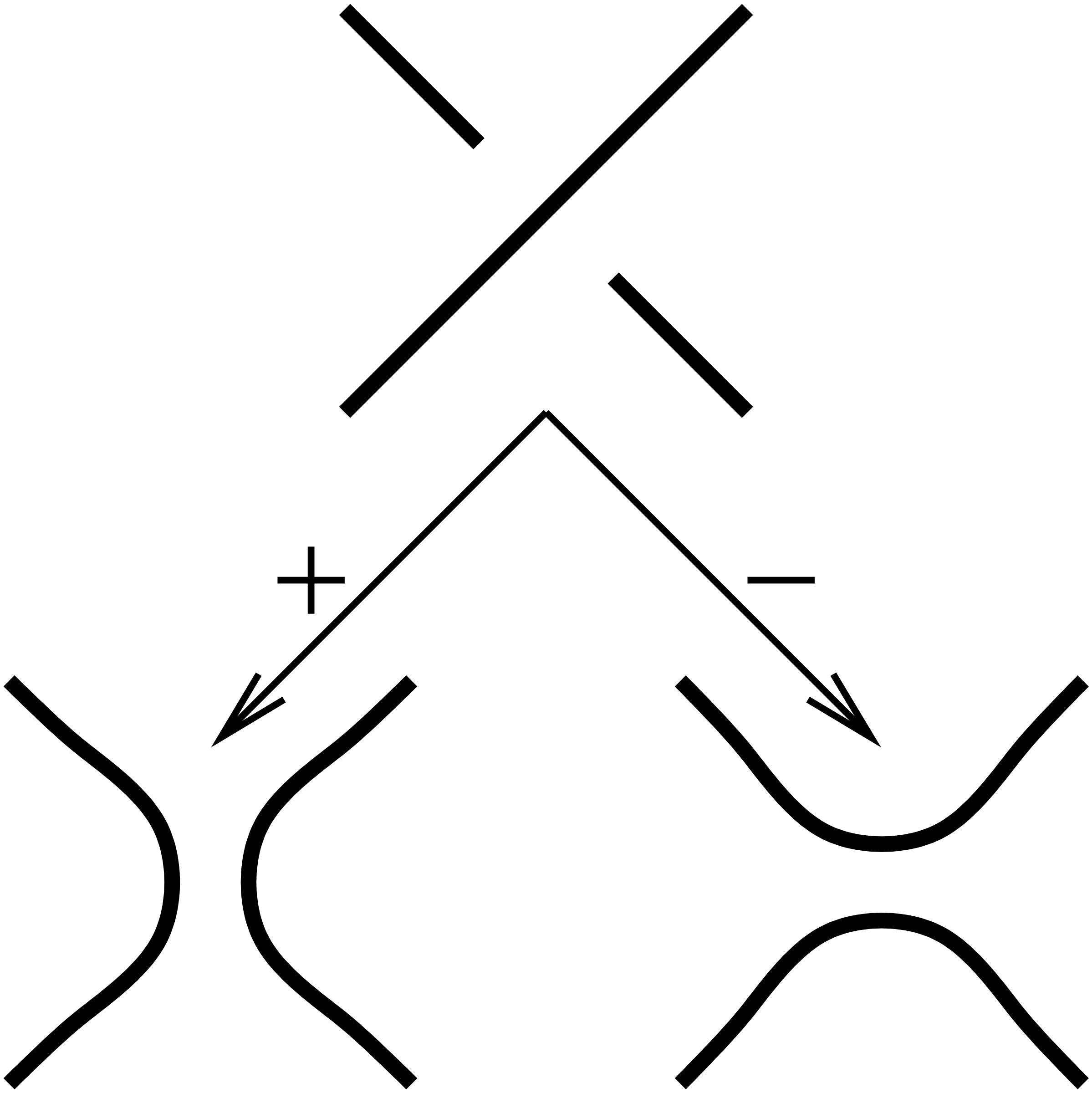

figure KH2

The bracket polynomial may be calculated by applyig the given rules systematically crossing after crossing. The configuration obtained after all crossings are smoothed is called a **state** of the original diagram. Figure KH3 shows the eight states of the standard trefoil diagram arranged in a systematic manner. The way the states are arranged is the following: in the $i{-}th$ row (counting from 0) we have the states obtained by splitting i crossings negatively and the remaining ones positively.

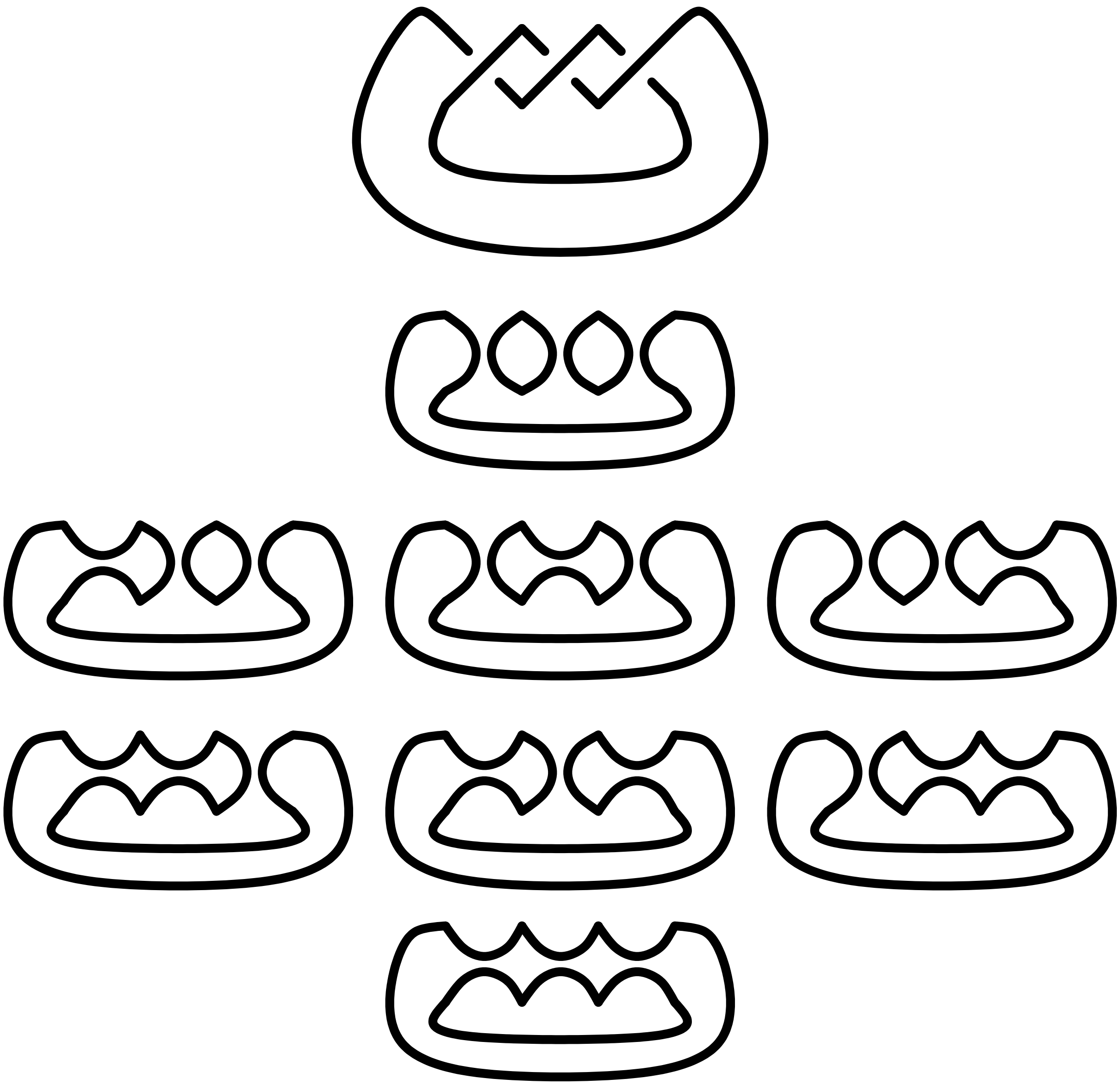

figure KH3

As it happens, the states in one row have the same number of components. This is very unusual but it helps to calculate the bracket polynomial of this diagram. It is easy to see that the result is (we write the terms from the bottom row to the top row)

$$A^{-3}\cdot(-A^{-2}-A^{2})+3A^{-1}\cdot 1+3A\cdot(-A^{-2}-A^{2})$$
$$+A^{3}(-A^{-2}-A^{2})^{2}$$

When going from the bracket to the f polynomial we still need to multiply by $-A^{9}$. The final result for the f polynomial is

$$A^{4}+0+A^{12}-A^{16},$$

while the bracket is equal to

$$-A^{-5}-0\cdot A^{-1}-A^{3}+A^{7}.$$

This was not a particularly clever method of calculating the Jones polynomial of the trefoil. Nevertheless it illustrates well the significance of the states in the calculation.

In fact, while it might seem natural to assume that the bracket polynomial of the single trivial circle is the constant 1, it is useful to assume that we consider the empty link also, and that it is for the empty link that the constant 1 is assigned. Then the bracket of the single circle diagram is $-A^{-2} - A^2$.

While it is possible to define the Khovanov homologies using this convention we prefer to change the conventions slightly more. We define a version of the Kauffman bracket (to be denoted as $\langle D \rangle_K \in \mathbb{Z}[q, q^{-1}]$ as described in figure KH4.

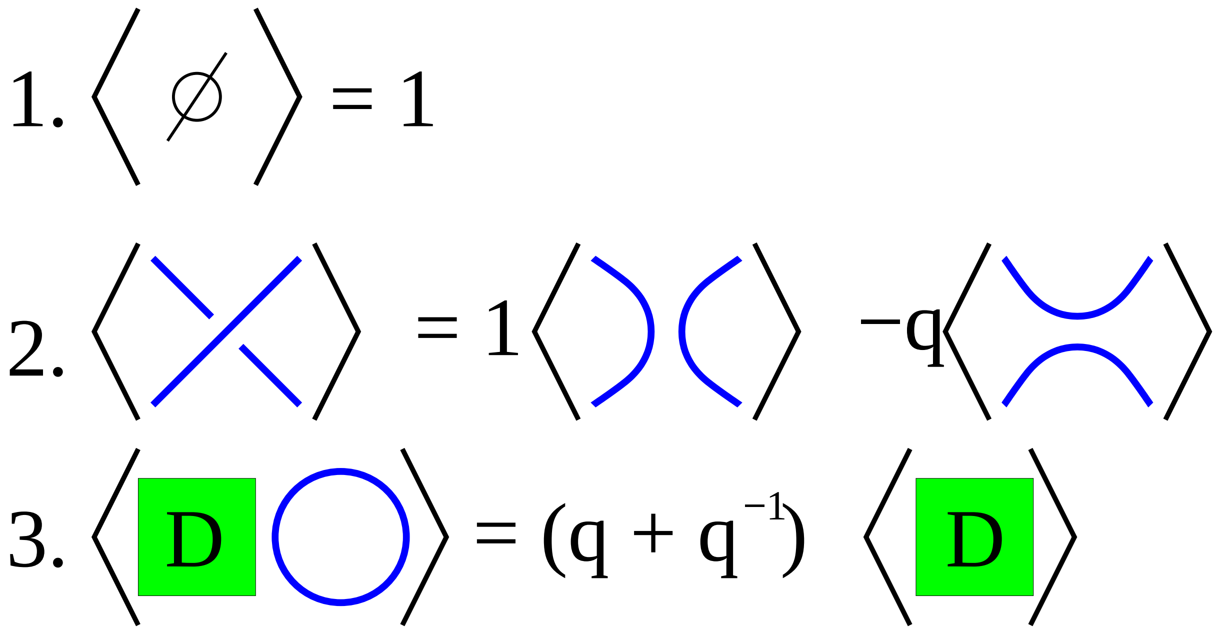

figure KH4

Figure KH5 shows how the new bracket changes under Reidemeister 1. Obviously, the behaviour of the new bracket in this aspect is less regular than that of the original Kauffman bracket. We omit calculations but we show in figure KH6 how the new bracket changes under Reidemeister 2 (expanding Reidemeister 2 needs to be balanced by multiplying by $-q$). Then, it is easily checked that it is invariant under Reidemeister 3.

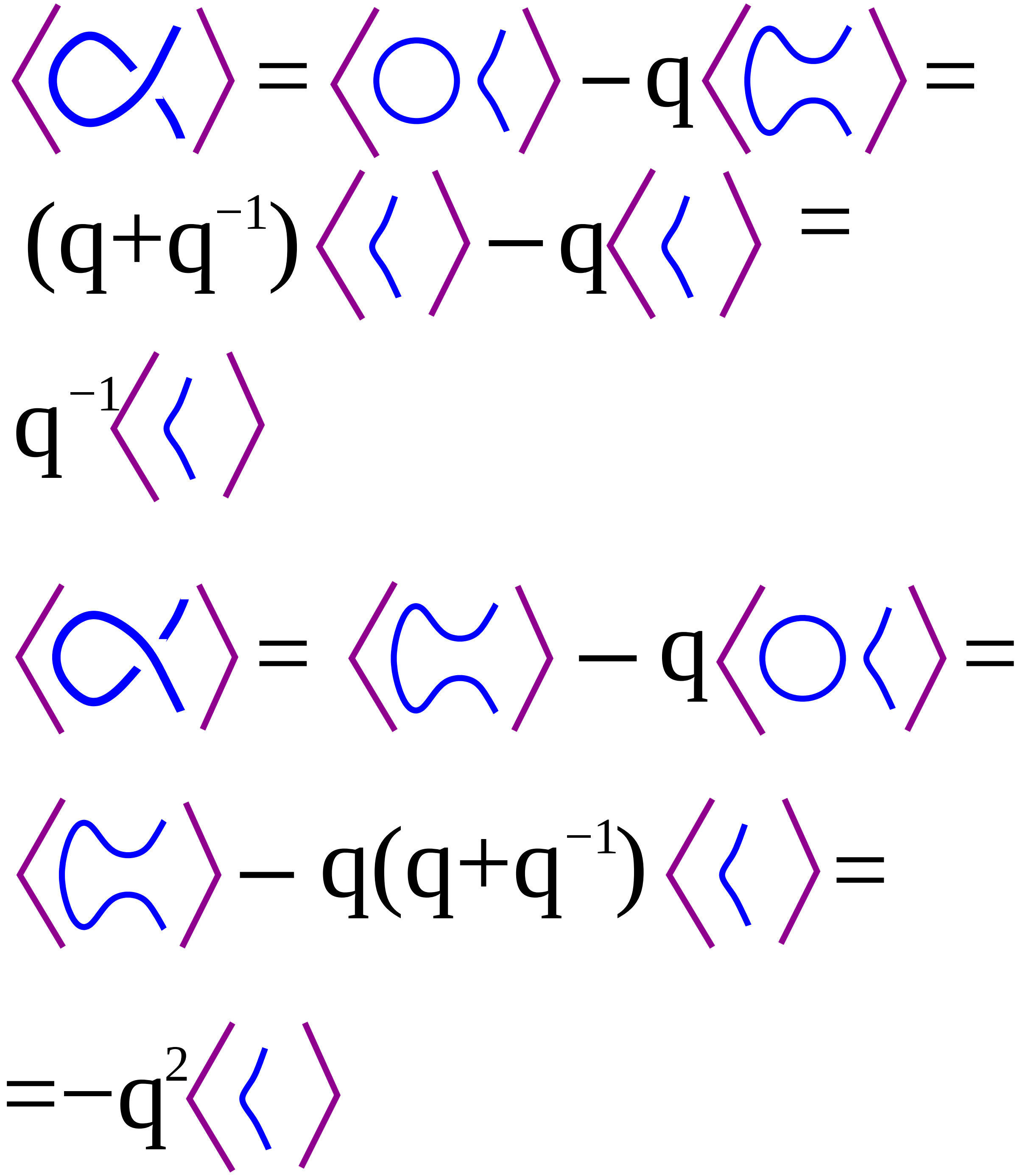

figure KH5

For a given diagram D of an oriented link we denote by n_+ (n_-) the number of positive (negative) crossings of D. It is easily checked by direct calculation that the following polynomial is an invariant of oriented links:

$$\tilde{J}(D) = (-1)^{n_-} q^{n_+ - 2n_-} \langle D \rangle_K.$$

For a given state S of a diagram D we denote by β_S the number of crossings that were smoothed negatively.

We can write the following *state expansion formula* for the $\tilde{J}$ polynomial.

$$\tilde{J}(D) = (-1)^{n_-} q^{n_+ - 2n_-} \Sigma_S (-q)^{\beta_S} (q + q^{-1})^{|S|}.$$

We will also use the β-state expansion, a version of the formula given above with the right hand side rewritten as

$$(-1)^{n_-} q^{n_+ - 2n_-} \Sigma_\beta (-q)^\beta \Sigma_{S:\beta_S = \beta} (q + q^{-1})^{|S|}$$

One can easily check that the substitution $A \mapsto \sqrt{q^{-1}}$ transforms $(-A^{-2} - A^2) f$ into $\tilde{J}$ (just write the analogous state expansion for the f polynomial and compare the two).

Khovanov chain complex

We will define a bigraded chain complex C. The vertical grading will be referred to as the β-grading. The horizontal grading will be referrd to as the q-grading. The differential will be of degree $(0,1)$. The homologies (Khovanov homologies) will be invariant under Reidemeister moves. The coefficients will be in $\mathbb{Z}_2$. For a fixed q grading i the sum

$$\chi_i = \Sigma(-1)^j dim H_{i,j}$$

will be the i-th coefficient of the $\tilde{J}$ polynomial. Before we do it let us consider one more example. We will calculate the $\tilde{J}$ polynomial for the trefoil knot, from the diagram shown in figure KH5. We will do it according to the state expansion formula.

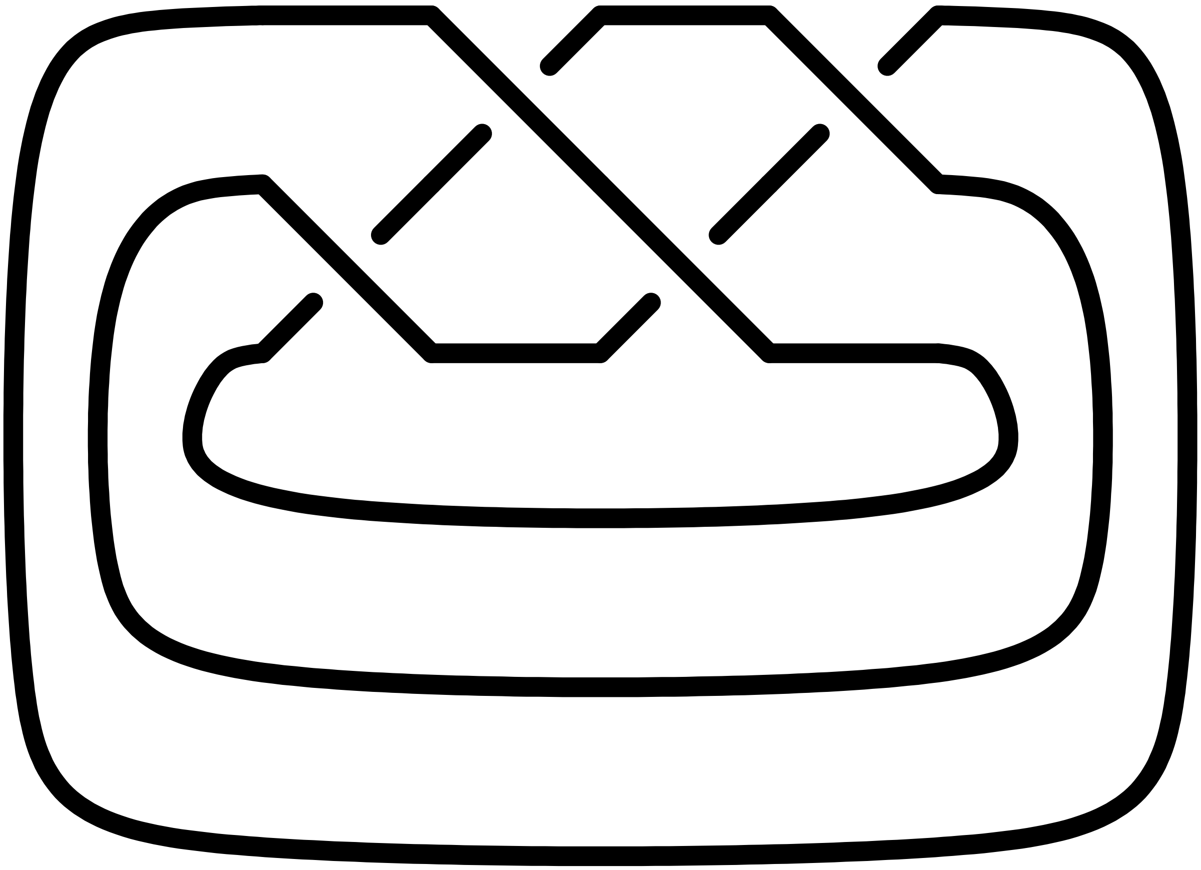

figure KH6

We will arrange the state sum for $\langle\cdot\rangle_{kh}$ in five rows, according to the increasing order of the number of crossings smoothed negatively, counting from the bottom to the top row (the β parameter). So, in the top row we will put the contribution of the single state obtained by smoothing all four crossings negatively (which for the diagram shown in figure KH5 means: vertically), then one line below there will be the sum of the four states obtained by smoothing three crossing negatively and one crossings positively, and so on —– alltogether there will be five rows, each containing a polynomial. The result is showed in table KH7. The comments on the next page explain how the table should be read.

4	0	0	0	0	0	0	1	0	1
3	0	0	0	0	−4	0	−8	0	−4
2	0	0	2	0	10	0	10	0	2
1	0	0	−4	0	−8	0	−4	0	0
0	1	0	3	0	3	0	1	0	0
	−3		−1		1		3		5

table KH7

Columns of the table are labled with exponents of variable q. Rows are labled with values of the parameter β. For example:
The first row from the top (with β-label 4) shows the contribution to the state sum of the state obtained by smoothing all crossings negatively (vertically in this case). The result is a single circle, with the bracket equal to $q^{-1}+q$, but we need to multiply by $(-q)^4$ so the contribution of this state to the bracket is

$$q^3+q^5.$$

The third row from the top (with β-label 2) describes the joint contribution of the six states obtained by smoothing two crossings positively and two crossings negatively. The contribution is

$$2q^{-1}+10q+10q^3+2q^5.$$

Let me explain this in more detail. Obviously, we have six possible choices of two vertices to be smoothed negatively (out of four vertices altogether). These are shown in figure KH8. You will see that there two classes — two states with three components in the top row and four states with one component in two rows below. Then again, we need to multiply by a suitable monomial — this time it is q^2. it follows that the contribution of these six states to the bracket is

$$q^2 \cdot (4(q^{-1} + q) + 2 \cdot (q^{-3} + 3q^{-1} + 3q + q^3)).$$

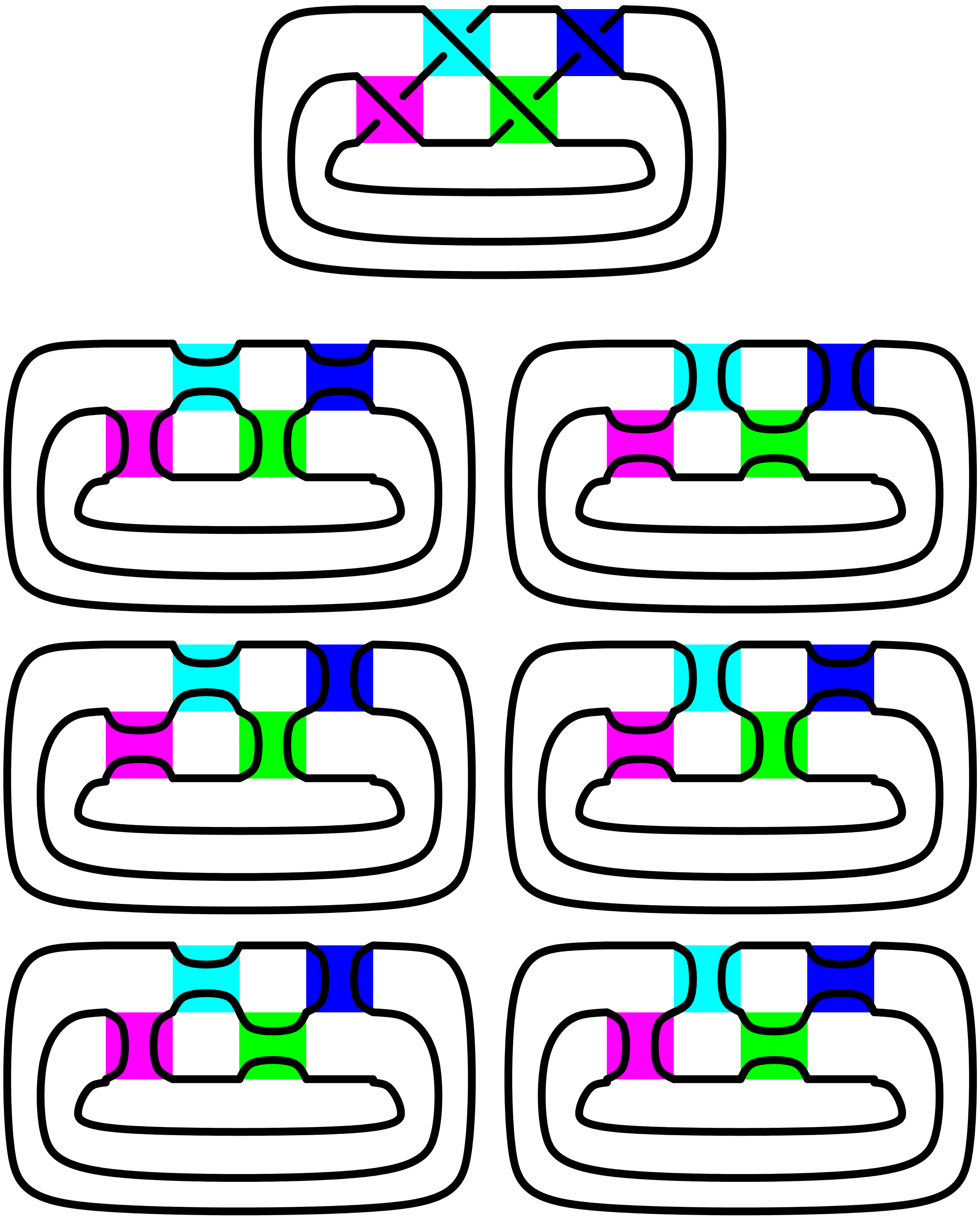

figure KH8

There is one characteristic regularity visible in table KH7. In the rows marked by odd values of the parameter β all non-zero entries are negative and in the rows marked by even values of β all non-zero entries are positive. For all practical purposes we could equally well specify just the absolute values of the considered entries.

This is no accident and it is obvious that this must be so for all tables constructed in this manner, for all diagrams of links.

Suppose that we find some natural way of constructing a bigraded chain complex, so that the dimension at any given position be equal to the absolute value of the number in our state sum table. Then the Euler characteristic condition will be automatically satisfied by an elementary theorem of homological algebra.

Description of the bigraded complex C

We will now describe the Khovanov complex C as a bigraded space. Differentials will be defined later. Let V be a graded two dimensional space over $\mathbb{Z}_2$ generated by two vectors denoted by e_-, e_+ of grading -1 and 1 respectively. There is the obvious grading induced on any tensor power $V^{\otimes n}$. It is clear that the dimension of the subspace generated by vectors of grading k is the same as the coefficient of q^k in the polynomial $(q^{-1}+q)^n$ — in both cases it is either zero or $\binom{n}{\frac{n+k}{2}}$, depending on parity of n and k.

Description of the total space of C

Now, let us consider a linear space over $\mathbb{Z}_2$ defined as

(1) $$\bigoplus_{\beta}\Big(\bigoplus_{S:\beta_S=\beta} V^{\otimes|S|}\Big)$$

If we introduce the natural (q,β) grading on this space, then the table of dimensions is almost identical with the one we considered for the β state sum formula for the $\tilde{J}$ polynomial. The two difference are:

1. All non-zero entries are now positive.
2. The q grading is shifted by $\beta+n_+-2n_-$ to the left.

The first difference is expected and there is no particular harm in it. As for the second, we correct it by shifting the horizontal grading by $\beta+n_+-2n_-$ to the right in the β height.

Now, we know that whatever differentials of degree $(0,1)$ we define on C, the q-graded Euler characteristic will be the sequence of coefficients of the $\tilde{J}$ polynomial — or this multiplied by -1. The second possibility is the consequence of the fact that so far we did nothing to take into account the $(-1)^{n_-}$ term in the formula for $\tilde{J}$. Now, we do it in the simplest possible way: we modify the vertical grading by shifting it down vertically by n_- (so row number 0 in the original table will now become row number $-n_-$.

So the total space of chain complex C is

$$\bigoplus_{\beta}\Big(\bigoplus_{S:\beta_S=\beta} V^{\otimes|S|}\Big).$$

While this is usually considered sufficiently precise, it will be more obvious how this is to be understood if the term $V^{\otimes|S|}$ is rewritten as

$$\bigotimes_{C\subseteq S} V_C,$$

where C is a component of the state S and V_C is a copy of V. We need this to stress that the generators are corresponding with the individual circles of the states.

Definition of the differential

We need to define the differential ∂ which is supposed to increase the vertical (β) gradig by 1 and to preserve the horizontal (q) grading. Working over $\mathbb{Z}_2$ we just need to describe the images of generators by specifying which other generatores appear in the image with coefficient 1. The condition on the horizontal grading means that we have already decided that non-trivial differentials can only go from a standard generator of the total space of the complex to generators corresponding to states with the number of negative splittings bigger by precisely one: if we have a generator corresponding to a state S obtained by smoothing k crossings negatively, then non-trivial differentials can only go to generators corresponding to states obtained by smoothing $k+1$ crossings negatively. In fact we will use another quite natural condition:

Assume that S and T are two states that differ by the character of splitting at two points or more. Then there is no non-trivial differential from any generator corresponding to S to any generator corresponding to T. This arrangement is conveniently visualized by an example in figure KH11. The figure shows the standardly positioned cube in $\mathbb{R}^3$, whose set of vertices is $\{0,1\}^3$. Vertices of the cube correspond to states of the diagram. The zero coordinate at position i means that the i-th crossing of the diagram was smoothed positively. It follows that the number of coefficients equal to 1 corredsponds to the number of crossings smoothed negatively — the unadjusted β grading. Vertices marked with the same colour correspond to states with the same vertical grading in the complex. The arrows show the possible non-trivial differentials, going from a point with a given number of coefficients equal to 1 to the points with the number bigger by exactly one.

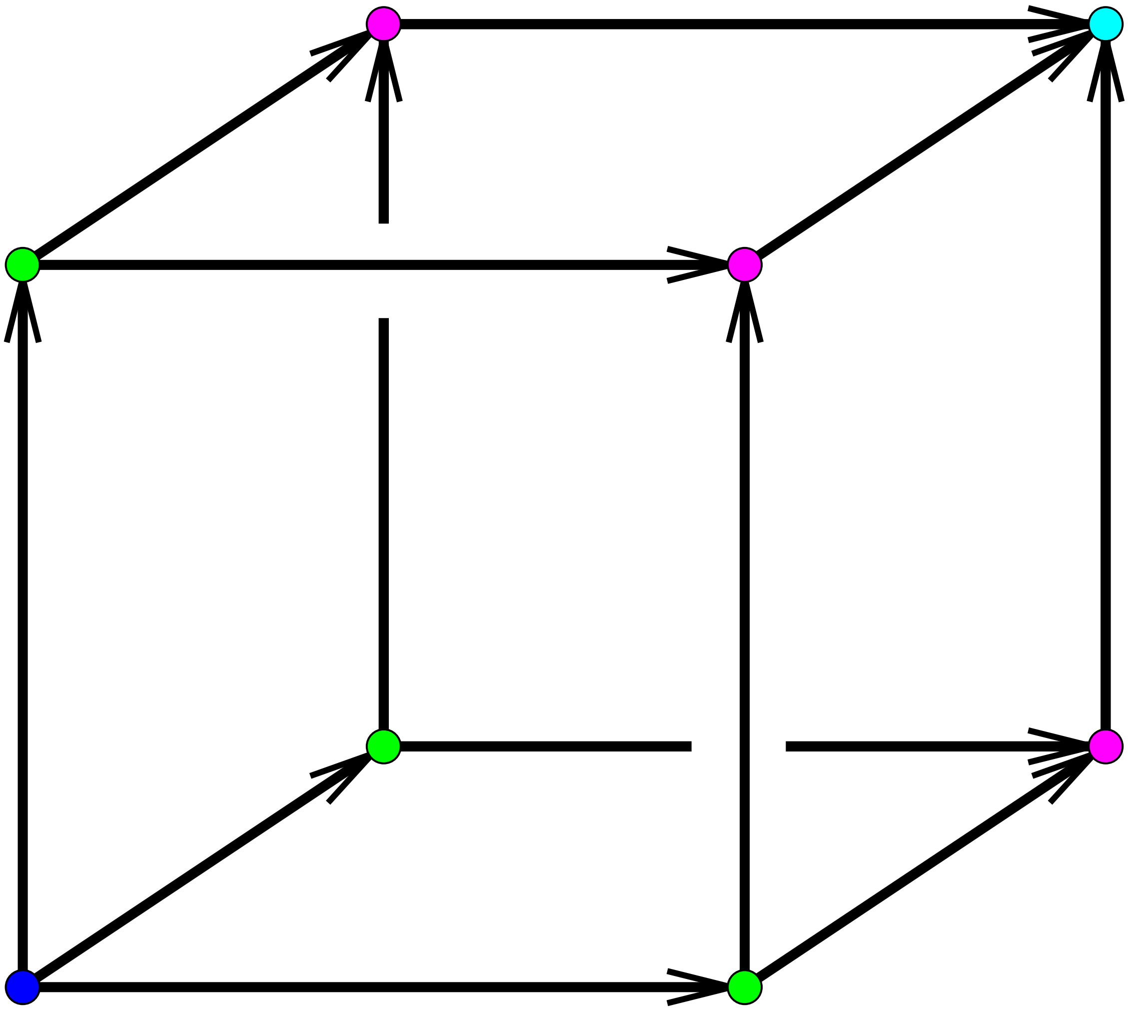

figure KH11

In principle we need to define the differential for simple tensors made up of vectors e_- and e_+. For example, we might need to explain the image of $e_- \otimes e_- \otimes e_- \otimes e_+ \otimes e_+$. Now, this simple tensor is one of the 32 simple tensors related to a state with five components. When we change the splitting at one crossing it could possibly involve the two that are now specially marked: $[e_-] \otimes e_- \otimes e_- \otimes [e_+] \otimes e_+$. This means that all the remaining circles of the considered state survive virtually unchanged. We assume that the differential acst as identity on the corresponding vectors. Therefore we need to describe *only* how the differential behaves on the two specially marked circles. It may also happen that the change of splitting involves only one circle. Then we need only to specify what happens to the corresponding vector. To avoid unnecessry complications of notation we will show only these vecors on which the differential acts non-trivially, putting them one next to another (if there are two). Here are the formulae describing the differential in this setting.

First we give the definition for a switch of splitting that decreases the number of circles in the state.

$e_- \otimes e_- \mapsto 0$

$e_- \otimes e_+ \mapsto e_-$

$e_+ \otimes e_- \mapsto e_-$

$e_+ \otimes e_+ \mapsto e_+$

Now, for a switch of splitting that increases the number of circles in the state.

$e_- \mapsto e_- \otimes e_-$

$e_+ \mapsto e_- \otimes e_+ + e_+ \otimes e_-$

It is easy to check directly that the differential defined in this way has the property $\partial\partial = 0$. Let us check one case directly. Suppose that we change the type of splitting at two positions an the result is that at first one of the circles is split into two and then one of these is split into another two. Then the same must happen when we change the splittings in reversed order. Let us do the calculation for the first case, for e_+:
We have

$$e_+ \mapsto e_- \otimes e_+ + e_+ \otimes e_-.$$

But the second case (changing the splitting of the second crossing first) gives the result that looks formally the same. Now, let us assume than in the second instance we need to apply the rule to the first term of the tensor product. We obtain:

$e_- \otimes e_+ + e_+ \otimes e_- \mapsto$
$e_- \otimes e_- \otimes e_+ + (e_- \otimes e_+ + e_+ \otimes e_-) \otimes e_-$
We need to check what happens when we do the change of splitting at the second crossing first. There is no natural order of the two circles obtained from the original one so we can as well assume that we need to apply the rule to the first vector again. Obviously, we will obtain the same formal expression. But did we not decide too easily to choose the order of components as was convenient? No: whatever the order of the final three components, the expression is symmetric with respect to the order: a sum of all possible simple tensors of length three with e_- appearing twice and e_+ appearing once.
As a result we have the two identical terms cancelling each other over $\mathbb{Z}_2$.
If you object to this style of discussion of the $\partial\partial$ composition, it might be helpful to use cobordism pictures as shown in figure KH12. With such a figure we adopt the convention of writing base vectors from left to right, as the circles appear in the picture.

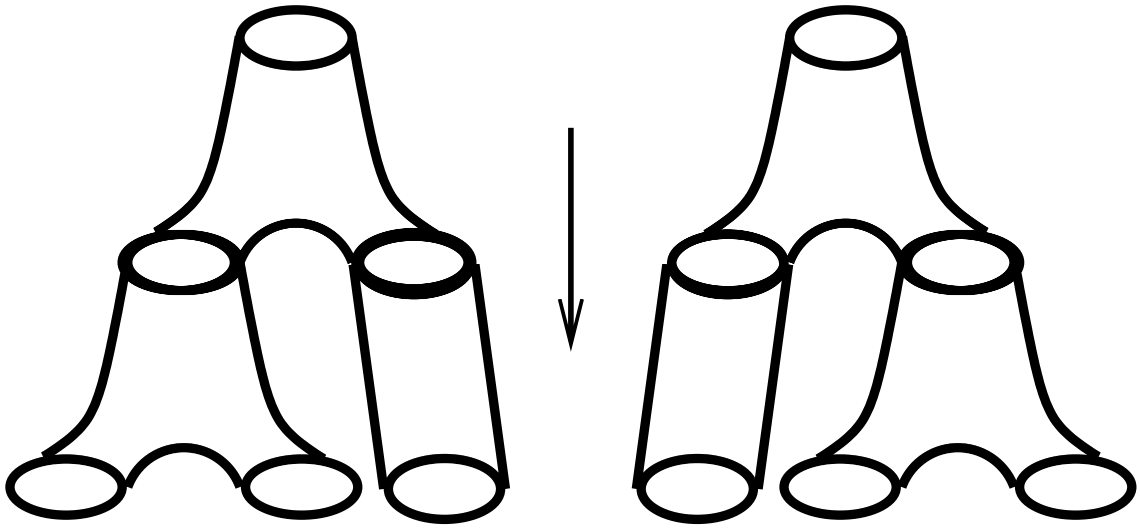

figure KH12

Here is the calculation for $\partial\partial(e_+)$ for this case again. We omit $\otimes$ signs. For the left:

$$e_+ \mapsto e_+e_- + e_-e_+ \mapsto (e_+e_- + e_-e_+)e_- + e_-e_-e_+$$

And for the right:

$$e_+ \mapsto e_+e_- + e_-e_+ \mapsto e_+e_-e_- + e_-((e_+e_- + e_-e_+))$$

This way we new exactly which term in the formula correspond to which circle in the figure. Obviously, we obtain the same result.
Let me show just one more calculation for the situation shown in figure KH13, for the tensor $e_+e_+e_-$. For the left:

$$e_+e_+e_- \mapsto e_+e_- \mapsto e_-$$

For the right:

$$e_+e_+e_- \mapsto e_+e_- \mapsto e_-$$

Again, we obtain the same results both way, so they cancel because the coefficients are in $\mathbb{Z}_2$.

If you check all possibilities you will see that the differential has the required property. The homologies of Khovanov complex are invariant under Reidemeister moves. in fact, the complex itself is invariant up to chain homoptopy equvalence.