Geometry on the lines of spine spaces

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Our goal

To recover the pointset of a spine space from the set of its lines using a binary relation π or ρ.

K. Petelczyc, M. Żynel
Geometry on the lines of spine spaces,
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Motivations and references

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Ann. of Math. **50** (1949), 32—67.

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Locally projective spaces which satisfy the Bundle Theorem
J. Geom. **56** (1996), 87–98

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Aequationes Math. **90** (2016), no. 3, 607–623.

Grassmann spaces

- V – a vector space of dimension n with $3 \leq n < \infty$
- $\mathrm{Sub}_k(V)$ – the set of all k-dimensional subspaces of V
- Assume that $0 < k < n$.

For $H \in \mathrm{Sub}_{k-1}(V)$, $B \in \mathrm{Sub}_{k+1}(V)$ with $H \subset B$ a k-pencil is a set of the form

$$\mathbf{p}(H, B) := [H, B]_k = \{U \in \mathrm{Sub}_k(V) \colon H \subset U \subset B\}.$$

The point-line structure

$$\mathbf{P}_k(V) = \langle \mathrm{Sub}_k(V), \mathcal{P}_k(V) \rangle,$$

where $\mathcal{P}_k(V)$ is the family of all k-pencils, is a Grassmann space.

Spine spaces

- W – a fixed subspace of V
- m – an integer such that $k - \operatorname{codim}(W) \leq m \leq k, \dim(W)$
- $\mathcal{F}_{k,m}(W) := \{U \in \mathrm{Sub}_k(V)\colon \dim(U \cap W) = m\}$
- $\mathcal{G}_{k,m}(W) := \{L \cap \mathcal{F}_{k,m}(W)\colon L \in \mathcal{P}_k(V) \text{ and } |L \cap \mathcal{F}_{k,m}(W)| \geq 2\}$

The point-line structure

$$\mathfrak{M} = \mathbf{A}_{k,m}(V, W) = \langle \mathcal{F}_{k,m}(W), \mathcal{G}_{k,m}(W) \rangle$$

is called a spine space.

- $\mathfrak{M}$ is a Gamma space
- $\mathfrak{M}$ can be
  - a projective space
  - a slit space
  - an affine space
  - the space of linear complements

Lines in spine spaces

- $\mathfrak{M}$ is a fragment of the Grassmann space
- $\mathcal{A}$ – affine lines
- $\mathcal{L}^\alpha$ and $\mathcal{L}^\omega$ – two types of projective lines
- $\mathcal{L} := \mathcal{A} \cup \mathcal{L}^\alpha \cup \mathcal{L}^\omega$

class	representative line $g = \mathbf{p}(H, B) \cap \mathcal{F}_{k,m}(W)$	g^∞
$\mathcal{A}_{k,m}(W)$	$H \in \mathcal{F}_{k-1,m}(W),\ B \in \mathcal{F}_{k+1,m+1}(W)$	$H + (B \cap W)$
$\mathcal{L}^\alpha_{k,m}(W)$	$H \in \mathcal{F}_{k-1,m}(W),\ B \in \mathcal{F}_{k+1,m}(W)$	–
$\mathcal{L}^\omega_{k,m}(W)$	$H \in \mathcal{F}_{k-1,m-1}(W),\ B \in \mathcal{F}_{k+1,m+1}(W)$	–

Table: The classification of lines in a spine space $\mathbf{A}_{k,m}(V, W)$.

Maximal strong subspaces

Grassmann space $\mathbf{P}_k(V)$: projective

stars $\mathrm{S}(H) = [H, V]_k = \{U \in \mathrm{Sub}_k(V) \colon H \subset U\}$, where $H \in \mathrm{Sub}_{k-1}(V)$

tops $\mathrm{T}(B) = [\Theta, B]_k = \{U \in \mathrm{Sub}_k(V) \colon U \subset B\}$, where $B \in \mathrm{Sub}_{k+1}(V)$

Spine space $\mathbf{A}_{k,m}(V, W)$: projective and semiaffine
(a projective space $\mathbf{P}$ with a subspace $\mathcal{D}$ removed)

class	representative subspace	$\dim(\mathbf{P})$	$\dim(\mathcal{D})$
ω-stars	$[H, H + W]_k \colon H \in \mathcal{F}_{k-1,m-1}(W)$	$\dim(W) - m$	-1
α-stars	$[H, V]_k \cap \mathcal{F}_{k,m}(W) \colon H \in \mathcal{F}_{k-1,m}(W)$	$\dim(V) - k$	$\dim(W) - m - 1$
α-tops	$[B \cap W, B]_k \colon B \in \mathcal{F}_{k+1,m}(W)$	$k - m$	-1
ω-tops	$[\Theta, B]_k \cap \mathcal{F}_{k,m}(W) \colon B \in \mathcal{F}_{k+1,m+1}(W)$	k	$k - m - 1$

Maximal strong subspaces in spine spaces

- two stars or two tops: disjoint, share a point
- a projective star and a projective top: disjoint, share a point
- (in remaining cases) a star and a top: disjoint, share a line

Fact

A line of $\mathfrak{M}$ can be in at most two maximal strong subspaces of different type: a star and a top.

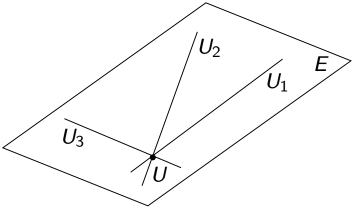
Lemma

Three pairwise coplanar and concurrent, or parallel, lines not all on a plane span a star or a top.

E – plane in $\mathfrak{M}$, $U \in \overline{E}$

$\mathbf{p}(U, E) := \{ L \in \mathcal{L} \colon U \in \overline{L} \subseteq \overline{E} \}$

- is a pencil of lines if U is proper
- is a parallel pencil otherwise



coplanarity

$L_1 \ \pi \ L_2$ iff there is a plane E such that $L_1, L_2 \subset E$

relation of being in one pencil of lines

$L_1 \ \rho \ L_2$ iff there is a pencil p such that $L_1, L_2 \in p$

- $\rho \subseteq \pi$

...and their cliques...

- X – a subspace of $\mathfrak{M}$

$$\mathrm{L}(X) = \{L \in \mathcal{L} \colon L \subset X\}$$

- flat: $\mathrm{L}(E)$, where E is a plane
- semiflat: all projective lines and "a selector" on a plane
- flats and semiflats can be: projective, punctured, affine (punctured $\cup$ affine $=$ semiaffine)

- $U \in \overline{X}$

$$\mathrm{L}_U(X) = \{L \in \mathcal{L} \colon U \in \bar{L} \text{ and } L \subseteq X\}$$

- semibundle: $\mathrm{L}_U(X)$, where X is a strong subspace of $\mathfrak{M}$
- semibundles can be: proper or improper

... maximal in particular

Lemma

1. *Flats and semibundles are π-cliques.*
2. *Semiflats and proper semibundles are ρ-cliques.*

Proposition

Every maximal π-clique is either a flat or a semibundle.

Proposition

Every maximal ρ-clique is either a semiflat or a proper semibundle.

Cliques in terms of $\delta \in \{\pi, \rho\}$

$\mathbf{\Delta}_\delta(L_1, L_2, L_3)$ iff $\neq (L_1, L_2, L_3)$ and $L_i \ \delta\ L_j$ for all $i, j = 1, 2, 3$,
and for all $M_1, M_2 \in \mathcal{L}$ if $M_1, M_2 \ \delta\ L_1, L_2, L_3$ then $M_1 \ \delta\ M_2$.

Lemma

Let $L_1, L_2, L_3 \in \mathcal{L}$.

1. $\mathbf{\Delta}_\pi(L_1, L_2, L_3)$ *iff $\overline{L_1}, \overline{L_2}, \overline{L_3}$ form a tripod or a triangle.*
2. $\mathbf{\Delta}_\rho(L_1, L_2, L_3)$ *iff L_1, L_2, L_3 form a ρ-clique, they are not in a pencil of lines, they are not on an affine plane, and in case they are on a punctured plane one of L_1, L_2, L_3 is an affine line.*

- generally speaking: 3 lines satisfying $\mathbf{\Delta}_\delta$ determine a δ-clique
- "more than a plane" assumption:

$$3 \leq n - k \quad \text{and} \quad 3 \leq k - m$$

Maximal cliques in terms of $\delta \in \{\pi, \rho\}$

The set

$$[L_1, L_2, L_3]_\delta := \{L \in \mathcal{L} \colon L \ \delta \ L_1, L_2, L_3\}$$

is the maximal δ-clique, provided that $\mathbf{\Delta}_\delta(L_1, L_2, L_3)$.

Proposition

1. *The family of maximal π-cliques is definable in $\langle \mathcal{L}, \pi \rangle$.*
2. *Maximal ρ-cliques, except affine semiflats, are definable in $\langle \mathcal{L}, \rho \rangle$.*

Lemma

A maximal ρ-clique K satisfies the following condition:

there are lines $L_1 \in K, L_2 \in \mathcal{L} \setminus K$ such that

$$(K \setminus \{L_1\}) \cup \{L_2\} \text{ is a maximal } \rho\text{-clique} \qquad (*)$$

iff K is a semiaffine semiflat.

Ternary concurrency

Let $L_1, L_2, L_3 \in \mathcal{L}$.

$\mathbf{p}_{\boldsymbol{\pi}}(L_1, L_2, L_3)$ iff $L_i \ \boldsymbol{\pi} \ L_j$ for all $i, j = 1, 2, 3$ and $\neg \boldsymbol{\Delta}_{\boldsymbol{\pi}}(L_1, L_2, L_3)$
iff L_1, L_2, L_3 form a pencil of lines or a parallel pencil

$\mathbf{p}_{\boldsymbol{\rho}}(L_1, L_2, L_3)$ iff there are $M_1, M_2, M_3 \in \mathcal{L}$ such that
$\boldsymbol{\Delta}_{\boldsymbol{\rho}}(M_1, M_2, M_3)$, $\mathbf{[}M_1, M_2, M_3\mathbf{]}_{\boldsymbol{\rho}}$ does not satisfy $(*)$,
$L_1, L_2, L_3 \in \mathbf{[}M_1, M_2, M_3\mathbf{]}_{\boldsymbol{\rho}}$, and $\neg \boldsymbol{\Delta}_{\boldsymbol{\rho}}(L_1, L_2, L_3)$
iff L_1, L_2, L_3 form a pencil of lines

Lemma

1. *The family $\mathcal{P}_{\boldsymbol{\pi}}$ of all pencils of lines and parallel pencils is definable in $\langle \mathcal{L}, \boldsymbol{\pi} \rangle$.*
2. *The family $\mathcal{P}_{\boldsymbol{\rho}}$ of all pencils of lines is definable in $\langle \mathcal{L}, \boldsymbol{\rho} \rangle$.*

Parallel pencils

Coplanar pencils in $\langle \mathcal{L}, \pi \rangle$

$p_1 \sqcap p_2$ iff for all $l_1 \in p_1, l_2 \in p_2$ we have $l_1 \ \pi \ l_2$

Parallel pencils in $\langle \mathcal{L}, \pi \rangle$

- on an affine plane: p_1 is a parallel pencil if there is another pencil p_2 such that $p_1 \sqcap p_2$ and $p_1 \cap p_2 = \emptyset$
- on a punctured plane: p is a parallel pencil if the base plane of p is not affine but every line $l \in p$ lies on some affine plane

Lemma

The family $\mathcal{P}_{||}$ of all parallel pencils is definable in $\langle \mathcal{L}, \pi \rangle$.

Pencils of lines

- $\mathcal{P}$ – the family of all pencils of lines in $\mathfrak{M}$
- for $\boldsymbol{\pi}$: $\mathcal{P} = \mathcal{P}_{\boldsymbol{\pi}} \setminus \mathcal{P}_{\|}$
- for $\boldsymbol{\rho}$: $\mathcal{P} = \mathcal{P}_{\boldsymbol{\rho}}$

Proposition

If $\mathfrak{M}$ satisfies "more than a plane" assumption, then $\langle \mathcal{L}, \mathcal{P} \rangle$ is definable in $\langle \mathcal{L}, \boldsymbol{\delta} \rangle$.

Geometry induced by pencils of lines, proper semibundles

- Proper maximal δ-cliques:
  - projective flats
  - punctured semiflats
  - proper semibundles
- "more than a 3-space" assumption: stars or tops are at least 4-dimensional projective or semiaffine spaces
- $\mathcal{P}_0$ – the family of all pencils of lines definable in $\langle \mathcal{L}, \delta \rangle$
- $\mathcal{K}^0_\delta := \{ K \in \mathcal{K}_\delta \colon \text{there is } q \in \mathcal{P}_0 \text{ such that } q \subset K \}$
- $\mathcal{B} := \{ K \in \mathcal{K}^0_\delta \colon \dim(K) \geq 3 \}$

Lemma

The family $\mathcal{B}$ coincides with the family of all proper top semibundles, the family of all proper star semibundles or the union of these two families depending on whether tops, stars or all of them as projective or semiaffine spaces are at least 4-dimensional.

Let $K_i := \mathrm{L}_{U_i}(X_i) \in \mathcal{B}$, $i = 1, 2$.

$$\Upsilon(K_1, K_2) \quad \text{iff} \quad (\exists\ L_1, L_2 \in K_1)(\exists\ M_1, M_2 \in K_2) \\ \left[L_1 \neq L_2 \wedge L_1\ \boldsymbol{\delta}\ M_1 \wedge L_2\ \boldsymbol{\delta}\ M_2\right]$$

Lemma

If $\Upsilon(K_1, K_2)$ and $K_1 \cap K_2 = \emptyset$, then X_1, X_2 are both stars or tops and $U_1 = U_2$.

Why punctured spaces are bad?

- U – a fixed point
- X_1 – a semiaffine, but not affine, star containing U; an α-star
- X_2 – a projective star containing U; an ω-star
- $K_i = \mathrm{L}_U(X_i)$, $i = 1, 2$ – proper semibundles
- L – a projective line in K_1; an α-line
- suppose that there is a line M in K_2 coplanar with L
- the plane E spanned by L, M is contained in some α-top
- there is no line in $E \cap X_2$, a contradiction
- there is no line in K_2 coplanar with L
- if L is an affine line, then E is contained in an ω-top
- to have $\Upsilon(K_1, K_2)$ we need at least two affine lines in K_1

Grouping proper semibundles into bundles cont.

Assume that: stars or tops are at least 4-dimensional projective or semiaffine but not punctured projective spaces:

$$4 \leq n - k \quad \text{and} \quad \dim(W) \neq m + 1 \qquad \text{or}$$
$$4 \leq k - m \quad \text{and} \quad k \neq m + 1$$

Lemma

If X_1, X_2 are both stars or tops and $U_1 = U_2$, then $\Upsilon(K_1, K_2)$ and $K_1 \cap K_2 = \emptyset$.

Bundles of lines

$$\Upsilon_{\emptyset}(K_1, K_2) \quad \text{iff} \quad \Upsilon(K_1, K_2), \Upsilon(K_2, K_1), \text{ and}$$
$$\text{either } K_1 \cap K_2 = \emptyset \text{ or } K_1 = K_2$$

Lemma

$\Upsilon_{\emptyset}(K_1, K_2)$ iff X_1, X_2 are both stars or tops and $U_1 = U_2$.

For $K \in \mathcal{B}$ we write

$$\Lambda_{\Upsilon_{\emptyset}}(K) := \bigcup \{K' \in \mathcal{B} \colon \Upsilon_{\emptyset}(K, K')\}.$$

Lemma

If X is a maximal strong subspace containing a point U, then
$$\Lambda_{\Upsilon_{\emptyset}}(\mathrm{L}_U(X)) = \{L \in \mathcal{L} \colon U \in L\}.$$

Theorem

Let $\mathfrak{M}$ be a spine space and let $\mathcal{L}$ be its lineset. If stars or tops in $\mathfrak{M}$ are at least 4-dimensional projective or semiaffine but not punctured projective spaces, then

$$\text{the spine space } \mathfrak{M} \text{ and the structure } \langle \mathcal{L}, \boldsymbol{\delta} \rangle \text{ are definitionally equivalent.} \qquad (**)$$

Excluded cases where $(**)$ holds true

$\dim(W) = n$

$\mathfrak{M}$ is a Grassmann space

$\dim(W) = m = k$

$\mathfrak{M}$ is a single point

$\dim(W) = m = k - 1$

$\mathfrak{M}$ is a star in $\mathbf{P}_k(V)$, i.e. it is a projective space

$\dim(W) = k + 1, m = k$

$\mathfrak{M}$ is a top in $\mathbf{P}_k(V)$, i.e. it is a projective space

Excluded cases where $(**)$ does not hold true

$\dim(W) = k, m = k - 1$

$\mathfrak{M}$ is the neighbourhood of a point W in $\mathbf{P}_k(V)$, i.e. the set of all points that are collinear with W

- X – a star in $\mathfrak{M}$
- φ – a homology $\neq \mathrm{id}$ on $\overline{X}$ with the center W
- $f\colon \mathrm{Sub}_k(V) \longrightarrow \mathrm{Sub}_k(V), \qquad f(U) = \begin{cases} \varphi(U), & U \in X \\ U, & U \notin X \end{cases}$
- $\mathcal{L}_{\mathcal{S}}$ – the set of all lines contained in all stars
- $F_{\mathcal{S}}\colon \mathcal{L}_{\mathcal{S}} \longrightarrow \mathcal{L}_{\mathcal{S}}$ such that $F_{\mathcal{S}}(L) = f(L)$
- $\mathcal{L}_{\mathcal{T}}$ – the set of all lines contained in all tops
- $F_{\mathcal{T}}\colon \mathcal{L}_{\mathcal{T}} \longrightarrow \mathcal{L}_{\mathcal{T}}$ such that $F_{\mathcal{T}}(L) = L$
- $F := F_{\mathcal{S}} \cup F_{\mathcal{T}}$ is an automorphism of $\langle \mathcal{L}, \boldsymbol{\delta} \rangle$ which does not preserve bundles of lines

Thank you for your attention