ARE SCALES FRÉCHET?

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Abstract. We continue the study of Dow spaces of a $\mathfrak{b}$-scale, originally introduced by Dow in [14]. We prove that it is consistent that all such spaces are Fréchet, but it is also consistent that none of them is. We use these spaces to exhibit (consistently) a $\triangle^1_2$ ideal that does not satisfy the Category Dichotomy. Finally, we prove that the Category Dichotomy holds for all co-analytic ideals.

1. Introduction

During the 2012 thematic program on Forcing and its Applications at the Fields Institute, Juhász asked the following question:

Problem 1 (Juhász). *Is there a countable, Fréchet*[1] *space with uncountable π-weight?*

Although the consistency of existence of such spaces is easy to establish, no ZFC result was known. The problem was motivated by the following result of Hrušák and Ramos:

Theorem 2 (H., Ramos [33]). *It is consistent that every countable, Fréchet topological group is second countable.*

Since for topological groups the notions of weight and π-weight coincide, Juhász's question asks to what extent the algebraic structure is needed for the consistent result above. The question is particularly interesting, as many of the results in [33] do not use the algebraic structure at all. The problem remained open until Dow provided a solution:

Theorem 3 (Dow [14]). *There exists a countable, Fréchet, zero dimensional space with π-weight at least $\mathfrak{b}$.*

Not only is this result impressive, but its proof is highly ingenious and illuminating. Given a $\mathfrak{b}$-scale $\mathcal{B}$, Dow defined a topological space $\mathbb{D}(\mathcal{B}) = (\omega^{<\omega}, \tau_{\mathcal{B}})$ (which we call the *Dow space of $\mathcal{B}$*) that is zero dimensional and

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[1] Undefined concepts will be reviewed in the following sections.

whose π-weight is exactly the bounding number $\mathfrak{b}$. Dow then proceeds to take a sequential modification to obtain a topology $\sigma_{\mathcal{B}}$ extending $\tau_{\mathcal{B}}$ such that $\mathbb{D}_1(\mathcal{B}) = (\omega^{<\omega}, \sigma_{\mathcal{B}})$ is Fréchet, zero dimensional and its π-weight is at least $\mathfrak{b}$ (the exact π-weight is currently unknown). Evidently, the advantage of $\mathbb{D}_1(\mathcal{B})$ over $\mathbb{D}(\mathcal{B})$ is that it is Fréchet. Nevertheless, the original space $\mathbb{D}(\mathcal{B})$ has certain advantages over $\mathbb{D}_1(\mathcal{B})$. Mainly, the open sets of $\mathbb{D}(\mathcal{B})$ admit a very nice combinatorial description and are intuitive to work with, but the same is no longer true for $\mathbb{D}_1(\mathcal{B})$. It is then natural to ask the following question:

Can the Dow space of a scale be a Fréchet space?

In other words, we wanted to know if taking the sequential modification is really needed. We will say that a $\mathfrak{b}$-scale is *Fréchet* if its Dow space is Fréchet. In this article, we will prove the following result:

Theorem 4. *Both of the following statements are consistent (but not at the same time) with ZFC:*

1. *Every $\mathfrak{b}$-scale is Fréchet.*
2. *No $\mathfrak{b}$-scale is Fréchet.*

We finish the paper with an application to the Katětov order on ideals and prove that the Category Dichotomy of [30] is true for all co-analytic ideals.

The structure of the paper is as follows. Section 2 covers the necessary notation and definitions that will be used throughout the paper. The next five sections provide the required background on cardinal invariants of the continuum, filters and ideals, topology, and forcing that will be needed in the article. The reader may skip these preliminary sections and return to them as needed. The definition of the Dow space of a $\mathfrak{b}$-scale from [14] is reviewed in Section 8. In Section 9 we prove that $\mathfrak{p} = \mathfrak{b}$ implies that all $\mathfrak{b}$-scales are Fréchet. In Section 10 we produce a model where there is a non-Fréchet $\mathfrak{b}$-scale and in Section 11 we build a model where no Fréchet $\mathfrak{b}$-scales exist. In Section 12 we use results from the previous sections to find a consistent example of a $\triangle^1_2$ ideal that does not satisfy the Category Dichotomy and prove that this is the least possible complexity. Section 13 contains open questions that we do not know how to answer.

It is worth pointing out that countable, Fréchet spaces with uncountable π-weight have received a lot of attention recently, as can be seen from the recently published papers [17] and [15].

2. NOTATION

Our notation is basically standard and follows [43] and [38]. Given $s, t \in \omega^{<\omega}$ by $s^\frown t$ we denote the *concatenation* of s and t. We write $s^\frown n$ instead of $s^\frown (n)$. If $F \subseteq \omega^{<\omega}$, define $s^\frown F = \{s^\frown z \mid z \in F\}$. We say that $T \subseteq \omega^{<\omega}$ is a *tree* if it is closed under taking initial segments. For $s \in T$, define

$\mathsf{suc}_T(s) = \{n \mid s^\frown n \in T\}$. We say s *is the stem of* T (denoted by $s = \mathsf{st}(T)$) if every node of T is comparable with s and s is maximal with this property. The domain of a function f is denoted by $\mathsf{dom}(f)$ and its image by $\mathsf{im}(f)$. By $f\ ; X \longrightarrow Y$ we mean that f is a partial function from X to Y (i.e., $\mathsf{dom}(f) \subseteq X$). The expression "for almost all" means "for all except finitely many".

If κ is a cardinal, define $\mathsf{H}(\kappa)$ as the set consisting of all sets whose transitive closure has size less than κ. We will work with elementary submodels of these sets. For an introduction to this very important technique, we refer the reader to the survey [13].

We will need the concepts of F_σ, G_δ, Borel, analytic, co-analytic and projective subsets of a Polish space, which can be consulted in [40] or [56].

3. PRELIMINARIES ON CARDINAL INVARIANTS OF THE CONTINUUM

Cardinal invariants of the continuum play an important role in this article. We now review some basic definitions that will be needed. To learn more, see [7] and [58]. For $f, g \in \omega^\omega$, define $f \leq^* g$ if $f(n) \leq g(n)$ holds for almost all $n \in \omega$. We say a family $\mathcal{B} \subseteq \omega^\omega$ is *unbounded* if $\mathcal{B}$ is unbounded with respect to $\leq^*$. A family $\mathcal{D} \subseteq \omega^\omega$ is a *dominating family* if for every $f \in \omega^\omega$, there is $g \in \mathcal{D}$ such that $f \leq^* g$. The *bounding number* $\mathfrak{b}$ is the size of the smallest unbounded family and the *dominating number* $\mathfrak{d}$ is the smallest size of a dominating family. We say that $\mathcal{B} = \{f_\alpha \mid \alpha \in \mathfrak{b}\} \subseteq \omega^\omega$ is a $\mathfrak{b}$*-scale* if every f_α is an increasing function, $\mathcal{B}$ is unbounded and $f_\alpha <^* f_\beta$ whenever $\alpha < \beta$. A *scale* is a dominating $\mathfrak{b}$-scale. It is not hard to see that the existence of a scale is equivalent to $\mathfrak{b} = \mathfrak{d}$.

The order $\leq^*$ can be extended to partial functions. Given $f, g\ ; \omega \longrightarrow \omega$ define $f \leq^* g$ if $f(n) \leq g(n)$ holds for almost all n in their common domain. It is well-known that $\mathfrak{b}$-scales are not only unbounded with respect to total functions, but also with respect to infinite partial functions. See [58, Fact 3.4] for the proof of the following lemma:

Lemma 5. *Let $\mathcal{B} \subseteq \omega^\omega$ be a $\mathfrak{b}$-scale and $g; \omega \longrightarrow \omega$ an infinite partial function. There is $f \in \mathcal{B}$ such that $f \not\leq^* g$.*

Let A and B be two subsets of ω. Define $A \subseteq^* B$ (A *is an almost subset of* B) if $A \setminus B$ is finite. For $\mathcal{H} \subseteq [\omega]^\omega$ and $A, B \subseteq \omega$, we say that A *is a pseudo-intersection of* $\mathcal{H}$ if it is almost contained in every element of $\mathcal{H}$. A family $\mathcal{P} \subseteq [\omega]^\omega$ is *centered* if the intersection of finitely many of its elements is infinite. The pseudo-intersection number $\mathfrak{p}$ is the least size of a centered family without an infinite pseudo-intersection. We say $\mathcal{T} = \{A_\alpha \mid \alpha < \kappa\}$ is a *pre-tower* if it is $\subseteq^*$-decreasing and it is a *tower* it has no infinite pseudo-intersection. The *tower number* $\mathfrak{t}$ is the least length of a tower. By $\mathfrak{c}$ we denote the size of the real numbers. It is easy to see that $\mathfrak{p} \leq \mathfrak{t} \leq \mathfrak{b} \leq \mathfrak{d} \leq \mathfrak{c}$. Moreover, an impressive theorem of Malliaris and Shelah establishes that $\mathfrak{p} = \mathfrak{t}$ (see [48]). The following is essentially [4, Theorem 4.2].

Proposition 6 (Baumgartner, Dordal [4])**.** *If there are no towers of length* $\mathfrak{b}$, *then every* $\mathfrak{b}$*-scale is a scale (so* $\mathfrak{b} = \mathfrak{d}$*).*

Proof. Assume that there is a $\mathfrak{b}$-scale $\mathcal{B} = \{f_\alpha \mid \alpha \in \mathfrak{b}\}$ that is not a dominating family. Let $g \in \omega^\omega$ be not dominated by any function in $\mathcal{B}$. It follows that the set $A_\alpha = \{n \mid f_\alpha(n) < g(n)\}$ is infinite for every $\alpha < \mathfrak{b}$ and $\mathcal{T} = \{A_\alpha \mid \alpha \in \mathfrak{b}\}$ is a pretower. Since there are no towers of length $\mathfrak{b}$, there is $X \in [\omega]^\omega$ which is a pseudointersection of $\mathcal{T}$. We get $f_\alpha \restriction X \leq^* g \restriction X$ for every $\alpha < \mathfrak{b}$, which contradicts Lemma 5. □

We will need the following result in Section 9.

Lemma 7. *Let* $\mathcal{B} = \{f_\alpha \mid \alpha \in \mathfrak{b}\}$ *be a* $\mathfrak{b}$*-scale and* M *an elementary submodel of* $\mathsf{H}(\kappa)$ *(for some large enough regular cardinal* κ*) such that* $\mathcal{B} \in M$*, the size of* M *is less than* $\mathfrak{b}$ *and* $\delta = M \cap \mathfrak{b} \in \mathfrak{b}$*. If* $g; \omega \longrightarrow \omega$ *is an infinite partial function in* M*, then* $f_\delta \not\leq^* g$*.*

Proof. By Lemma 5, there is $\alpha < \mathfrak{b}$ such that $f_\alpha \not\leq^* g$. Moreover, by elementarity, we may assume that $\alpha < \delta$. Since $f_\alpha \leq^* f_\delta$, the result follows. □

Let M be a model of ZFC and $f \in \omega^\omega$. We say that f *is dominating (unbounded) over* M if for every $g \in M \cap \omega^\omega$, it is the case that $g \leq^* f$ ($f \not\leq^* g$).

4. Preliminaries on filters and ideals

We denote the power set of a set X by $\mathcal{P}(X)$. $\mathcal{I} \subseteq \mathcal{P}(X)$ is an *ideal on* X if $\emptyset \in \mathcal{I}$ and $X \notin \mathcal{I}$, for every $A, B \subseteq X$, if $A \in \mathcal{I}$ and $B \subseteq A$ then $B \in \mathcal{I}$ and if $A, B \in \mathcal{I}$ then $A \cup B \in \mathcal{I}$. A family $\mathcal{F} \subseteq \wp(X)$ is a called a *filter on* X if $X \in \mathcal{F}$ and $\emptyset \notin \mathcal{F}$, for every $A, B \subseteq X$, if $A \in \mathcal{F}$ and $A \subseteq B$ then $B \in \mathcal{F}$ and if $A, B \in \mathcal{F}$ then $A \cap B \in \mathcal{F}$. Given a family $\mathcal{B}$ of subsets of X, we define $\mathcal{B}^* = \{X \setminus B \mid B \in \mathcal{B}\}$. Note that if $\mathcal{F}$ is a filter, then $\mathcal{F}^*$ is an ideal (called the *dual ideal of* $\mathcal{F}$) and if $\mathcal{I}$ is an ideal, then $\mathcal{I}^*$ is a filter (called the dual filter of $\mathcal{I}$). Let $\mathcal{I}$ be an ideal on X. The collection of $\mathcal{I}$*-positive sets* is $\mathcal{I}^+ = \wp(X) \smallsetminus \mathcal{I}$. If $\mathcal{F}$ is a filter, we define $\mathcal{F}^+ = (\mathcal{F}^*)^+$. It is easy to see that $\mathcal{F}^+$ is the family of all sets that intersects every member of $\mathcal{F}$. If $A \in \mathcal{I}^+$ then *the restriction of* $\mathcal{I}$ to A, defined as $\mathcal{I} \restriction A = \wp(A) \cap \mathcal{I}$, is an ideal on A. By $\mathcal{I}^\perp$ we denote the set of all sets that have finite intersection with every member of $\mathcal{I}$. It is easy to see that $\mathcal{I}^\perp$ is an ideal. We review some properties of ideals that will be needed in this article.

Definition 8. *Let* $\mathcal{I}$ *be an ideal on* ω *(or any countable set).*

1. $\mathcal{I}$ *is* tall *if for every* $X \in [\omega]^\omega$ *there is* $Y \in \mathcal{I}$ *such that* $Y \cap X$ *is infinite.*
2. $\mathcal{I}$ *is a* Fréchet ideal *if for every* $A \in \mathcal{I}^+$*, there is* $B \in [A]^\omega \cap \mathcal{I}^\perp$*.*
3. $\mathcal{I}$ *is* ω-hitting *if for every* $\{X_n \mid n \in \omega\} \subseteq [\omega]^\omega$ *there is* $Y \in \mathcal{I}$ *such that* $Y \cap X_n$ *is infinite for every* $n \in \omega$*.*

(4) *$\mathcal{I}$ is* weakly selective *if for every $X \in \mathcal{I}^+$ and $\mathcal{P}$ a partition of X either $\mathcal{P} \nsubseteq \mathcal{I}$ or $\mathcal{P}$ has a (partial) selector in $\mathcal{I}^+$ (that is, a set that intersects each element of $\mathcal{P}$ in no more than one point).*

Let $\mathcal{F}$ be a filter on a set W. The filter $\mathcal{F}^{<\omega}$ is the filter on $[W]^{<\omega} \setminus \{\emptyset\}$ generated by $\left\{[A]^{<\omega} \setminus \{\emptyset\} \mid A \in \mathcal{F}\right\}$. Note that a set X is in $(\mathcal{F}^{<\omega})^+$ if and only if every $A \in \mathcal{F}$ contains an element of X.

Definition 9. *Let $\mathcal{F}$ be a filter. We say that $\mathcal{F}$ is a* FUF filter *if for every $X \in (\mathcal{F}^{<\omega})^+$ there is $Y \in [X]^\omega$ such that every $F \in \mathcal{F}$ contains almost all elements of Y.*

In this way, $\mathcal{F}$ is a FUF filter if and only if $(\mathcal{F}^{<\omega})^*$ is a Fréchet ideal. It is not hard to see that every countably generated filter is a FUF filter. Gruenhage and Szeptycki asked if there was a ZFC example of a FUF filter on ω that is not countably generated. In [8] it was proved that it is consistent that every FUF filter generated by less than $\mathfrak{c}$ many elements is countably generated and the main theorem of [33] implies that all FUF filters are countably generated.

We now review the Katětov order, which will be needed in Section 12.

Definition 10. *Let X, Y be two sets, $\mathcal{I}$ an ideal on X, $\mathcal{J}$ an ideal on Y and $f : X \longrightarrow Y$.*

(1) *f is a* Katětov function *from $\mathcal{I}$ to $\mathcal{J}$ if for every $A \subseteq Y$, we have that if $A \in \mathcal{J}$, then $f^{-1}(A) \in \mathcal{I}$.*
(2) *$\mathcal{J} \leq_{\mathsf{K}} \mathcal{I}$ means that there is a Katětov function from $\mathcal{I}$ to $\mathcal{J}$.*

The *eventually different* ideal $\mathcal{ED}$ is the ideal on ω^2 generated by the set of columns $\{\{n\} \times \omega \mid n \in \omega\}$ and the graphs of functions from ω to ω. Fix X a topological space and $N \subseteq X$. We say that N is *nowhere dense* if for every non-empty open set $U \subseteq X$, there is another open set $\emptyset \neq V \subseteq U$ such that $V \cap N = \emptyset$. By $\mathsf{nwd}(X)$ we denote the ideal of nowhere dense subsets of X. By nwd we mean the ideal of nowhere dense subsets of the rational numbers.

Readers interested in learning more about the Katětov order are encouraged to see [30], [31], [28], [26], [10], [2], [16], [49], [53], [24], [1], [41], [44] or [19] among others.

5. Preliminaries on Topology

We now review some basic topological concepts that will be used in the paper. Let X be a topological space[2] and $b \in X$. Recall that (X, τ) is *zero-dimensional* if it has a basis of clopen sets. For $A \subseteq X$ a countable set, we say that A *converges to* b (denoted by $A \longrightarrow b$) if every open subset of b almost contains A. X is a *Fréchet space* if for every $a \in X$ and $Y \subseteq X$ such that $a \in \overline{Y} \setminus Y$, there is $A \in [Y]^\omega$ that converges to a. Let $\mathcal{B}$ be a

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[2] All spaces under discussion are Hausdorff.

collection of non-empty open subsets of X. We say that $\mathcal{B}$ is a *π-base* if every non-empty open subset of X contains an element of $\mathcal{B}$. The *π-weight of X* is the smallest size of a π-base of X. Moreover, $\mathcal{B}$ is a *local π-base at b* if every neighborhood of b contains an element of $\mathcal{B}$. The *π-character of b* is the smallest size of a *local π-base at b*. We say that X *has uncountable π-character everywhere* if every point of X has uncountable π-character. Let φ be a topological property. By $X \models \varphi$ we mean that "X has property φ". We mainly use this notation when we are working with several topological spaces with the same underlying set.

Let X be a topological space and $a \in X$. Denote by $\mathcal{N}_X(a)$ the neighborhood filter of a and $\mathcal{I}_X(a)$ its dual ideal. Note that $\mathcal{I}_X(a) = \left\{A \subseteq X \mid a \notin \overline{A}\right\}$. If there is no risk of confusion, we simply write $\mathcal{N}(a)$ and $\mathcal{I}(a)$. Many topological properties at the point a can be expressed as properties of its neighborhood filter and its dual ideal, as shown in Table 1.

Topological property	**Combinatorial translation**
$a \in \overline{B}$	$B \in \mathcal{N}(a)^+$
$A \longrightarrow a$	A is a pseudointersection of $\mathcal{N}(a)$ equivalently, $A \in \mathcal{I}(a)^\perp$
a is a Fréchet point	$\mathcal{I}(a)$ is a Fréchet ideal
$a \in \overline{Y}$ but no sequence from Y converges to a	$Y \in \mathcal{I}(a)^+$ and $\mathcal{I}(a) \upharpoonright Y$ is a tall ideal

Table 1. Topological properties and their translations

We will need the following result in Section 11, which is Proposition 44 of [16].

Proposition 11. *Let X be a countable Fréchet space with no isolated points. The ideal $\mathsf{nwd}(X)$ is weakly selective.*

6. Preliminaries on Forcing

We review some preliminaries on forcing that will be needed. We assume the reader is already familiar with the method of forcing as presented in [42]. Let $\mathcal{F}$ be a filter on ω. The *Laver forcing of $\mathcal{F}$* (denoted by $\mathbb{L}(\mathcal{F})$) consists of all trees $p \subseteq \omega^{<\omega}$ that have a stem and if $t \in T$ extends the stem, then $\mathsf{suc}_T(s) \in \mathcal{F}$. Given $p, q \in \mathbb{L}(\mathcal{F})$, denote $p \leq q$ if $p \subseteq q$. It is easy to see that $\mathbb{L}(\mathcal{F})$ is a $\mathsf{c.c.c.}$ partial order. If $p \in \mathbb{L}(\mathcal{F})$ and $s \in p$, define $p_s = \{t \in p \mid t \subseteq s \vee s \subseteq t\}$. It is clear that $p_s \in \mathbb{L}(\mathcal{F})$, it extends p and in case $\mathsf{st}(p) \subseteq s$, we have that $\mathsf{st}(p_s) = s$. The *Laver generic real* will be denoted by $l_{gen} \in \omega^\omega$ and is the only element that is a branch of every tree in the generic filter. To learn more about this forcings, see [32].

By $\mathbb{D}$ we denote *Hechler forcing*, which consists of all pairs (s, f) where $s \in \omega^{<\omega}$ and $f \in \omega^{\omega}$. Define $(s, f) \leq (t, g)$ if $t \subseteq s$, $g(n) \leq f(n)$ for every $n \geq |t|$ and $s(i) \geq g(i)$ for every $i \in \mathsf{dom}(s) \setminus \mathsf{dom}(t)$. Hechler forcing is the standard c.c.c. forcing for adding a dominating real. To learn about Hechler forcing, see [3], [9] or [51].

A family $\mathcal{W} \subseteq [\omega]^{\omega}$ is *ω-hitting* if for every $\{X_n \mid n \in \omega\} \subseteq [\omega]^{\omega}$, there is $W \in \mathcal{W}$ that intersects every X_n. We say $\mathbb{P}$ *preserves ω-hitting families* if every ω-hitting family remain ω-hitting after forcing with $\mathbb{P}$. It is not hard to see that a forcing preserving ω-hitting families can not fill towers. A more refined version of the previous notion is the following:

Definition 12. *Let $\mathbb{P}$ be a partial order. We say that $\mathbb{P}$* strongly preserves ω-hitting families *if for every $\mathbb{P}$-name $\dot{B}$ for an infinite subset of ω, there is $\{B_n \mid n \in \omega\} \subseteq [\omega]^{\omega}$ such that for every $X \in [\omega]^{\omega}$, if $|X \cap B_n| = \omega$ for every $n \in \omega$, then $\mathbb{P}$ forces that $X \cap \dot{B}$ is infinite.*

In [8], the reader can find a characterization of the filters whose Laver forcing strongly preserves ω-hitting families. Moreover, it is also proved that $\mathbb{L}(\mathcal{F})$ preserves ω-hitting families if and only if $\mathbb{L}(\mathcal{F})$ strongly preserves ω-hitting families. It is also known that Hechler forcing strongly preserves ω-hitting families. The following iteration theorem can be found in [8].

Theorem 13 (Brendle, H. [8]). *The finite support iteration of c.c.c. forcings that strongly preserve ω-hitting families, strongly preserves ω-hitting families.*

Let $\mathbb{P}$ be a partial order. We say that $C \subseteq \mathbb{P}$ is *centered* if any finitely many elements of C have a lower bound in $\mathbb{P}$. Recall that $\mathbb{P}$ is *σ-centered* if it is the union of countably many of its centered sets. On the other hand, we say that $\mathbb{P}$ is *σ-filtered* if it is the union of countably many filters. Evidently, every σ-filtered forcing is σ-centered. Juhász and Kunen proved that the converse is not true (see [39]). Nevertheless, we have the following:

Proposition 14. *Let $\mathbb{P}$ be a partial order and $\mathbb{B}$ its Boolean completion. The following are equivalent:*

1. *$\mathbb{B}$ is σ-centered.*
2. *$\mathbb{B}$ is σ-filtered.*
3. *$\mathbb{P}$ is σ-centered.*

We will need the following well-known preservation result (see [57] and [21]).

Proposition 15. *Let $\gamma < \mathfrak{c}^{+}$ and $\langle \mathbb{P}_{\alpha}, \dot{\mathbb{Q}}_{\alpha} \mid \alpha < \gamma \rangle$ be a finite support iteration of σ-centered forcings. $\mathbb{P}_{\gamma}$ is σ-centered.*

A remarkable theorem of Bell is the following:

Theorem 16 (Bell [5]). *Let κ be a cardinal. The following are equivalent:*

1. $\kappa < \mathfrak{p}$.

(2) *For every σ-centered forcing $\mathbb{P}$ and $\{D_\alpha \mid \alpha < \kappa\}$ a collection of dense subsets of $\mathbb{P}$, there is a filter $G \subseteq \mathbb{P}$ such that $G \cap D_\alpha \neq \emptyset$ for every $\alpha < \kappa$.*

7. Preliminaries on Forcing and topology

In this section we review the method for destroying the Fréchet property at a point, as developed in [33] (which was based on the techniques from [8]). Let X be a topological space and $a \in X$. According to Table 1, to destroy the Fréchet property at a by a forcing $\mathbb{P}$, we must add a set $\dot{A}$ such that $\mathcal{I}(a) \restriction \dot{A}$ is a tall ideal. But this is not enough, since the tallness of an ideal may not be preserved under forcing iterations. The solution is to ensure that $\mathcal{I}(a) \restriction \dot{A}$ is not only a tall ideal but an ω-hitting one, for which we can prove preservation theorems under forcing.

Definition 17. *Let $\mathcal{I}$ be an ideal on a countable set, $\mathbb{P}$ a forcing notion, and $\dot{A}$ a $\mathbb{P}$-name. We say that $\mathbb{P}$* seals $\mathcal{I}$ with $\dot{A}$ *if $\mathbb{P}$ forces that $\dot{A} \in \mathcal{I}^+$ and $\mathcal{I} \restriction \dot{A}$ is ω-hitting.*

Let X be a countable space. It is easy to see that $\mathbb{L}(\mathsf{nwd}(X)^*)$ forces $\dot{A}_{\mathsf{gen}}$ to be a dense subset of X. The following is Proposition 5.2 of [33]:

Proposition 18 (H., Ramos [33])**.** *Let X be a countable space with no isolated points and $a \in X$:*

(1) *If a has uncountable π-weight, then $\mathbb{L}(\mathsf{nwd}(X)^*)$ seals $\mathcal{I}_X(a)$ via $\dot{A}_{\mathsf{gen}}$.*
(2) *If X is Fréchet, then $\mathbb{L}(\mathsf{nwd}(X)^*)$ strongly preserves ω-hitting families.*

In this way, if X and a are as in the above proposition, then $\mathbb{L}(\mathsf{nwd}(X)^*)$ forces $\dot{A}_{\mathsf{gen}}$ to be a dense subset of X, yet it does not contain sequences converging to a. This property will be preserved under any further forcing extension that preserves ω-hitting families.

We would like to point out that the method of [33] has been greatly refined and expanded by Shibakov and the third author (see [35], [36] and [34]).

8. The Dow space of a $\mathfrak{b}$-scale

We now review the construction of the Dow space from [14]. We expect that the study of the topological properties of Dow spaces will be useful for investigating the combinatorial properties of $\mathfrak{b}$-scales, much as the study of the Mrówka-Isbell spaces resides in understanding the combinatorics of almost disjoint families (see [58], [29] and [27]). Instead of working with $\mathfrak{b}$-scales on ω, we find it more convenient to work on $\mathfrak{b}$-scales consisting of functions from $\omega^{<\omega}$ to ω. To this end, we first adapt the relevant definitions to our setting.

For convenience, given $m \in \omega$, we will denote $\triangle_m = m^{\leq m}$. Let $f, g : \omega^{<\omega} \longrightarrow \omega$. We say that f *is increasing* if for every $s, t \in \omega^{<\omega}$, if s is a proper initial segment of t, then $f(s) < f(t)$ and for every $n, m \in \omega$, if $n < m$, then $f(s^\frown n) < f(s^\frown m)$. As expected, define $f <^* g$ if $f(s) < g(s)$ holds for almost all $s \in \omega^{<\omega}$. Moreover, define $f <_m g$ if $f(s) < g(s)$ holds for all $s \notin \triangle_m$. It follows that $f <^* g$ if and only if there is $m \in \omega$ such that $f <_m g$.

Definition 19. *Let $\mathcal{B}$ be a family of functions from $\omega^{<\omega}$ to ω.*

1. *$\mathcal{B}$ is a* weak $\mathfrak{b}$-scale *if the following conditions hold:*
   - (a) *$\mathcal{B}$ consists of increasing functions.*
   - (b) *$\mathcal{B}$ is unbounded (there is no function g such that $f \leq^* g$ for every $f \in \mathcal{B}$).*
   - (c) *$\mathcal{B}$ is well-ordered by $\leq^*$.*
   - (d) *For every $n \in \omega$, the set $\{f \in \mathcal{B} \mid n < f(\emptyset)\}$ is cofinal in $\mathcal{B}$.*
2. *$\mathcal{B}$ is a* $\mathfrak{b}$-scale *if its order type (with respect to $\leq^*$) is $\mathfrak{b}$.*

Although our main interest is $\mathfrak{b}$-scales, it is convenient to also consider weak $\mathfrak{b}$-scales. We will now define the sets that will be part of a subbase in a Dow space.

Definition 20. *Let $f : \omega^{<\omega} \longrightarrow \omega$. Define the tree $U(f) \subseteq \omega^{<\omega}$ recursively as follows:*

1. *$\emptyset \in U(f)$.*
2. *If $s \in U(f)$, then $\mathsf{suc}_{U(f)}(s) = \omega \setminus \{f(s)\}$.*

In this way, $U(f)$ is a very wide tree, since every node branches into all elements of ω except one. The following is easy, but worth pointing out.

Lemma 21. *Let $f : \omega^{<\omega} \longrightarrow \omega$ and $s \in \omega^{<\omega}$. If $s \notin U(f)$, then there is $i < |s|$ such that $f(s \restriction i) = s(i)$.*

We now have the following easy lemma:

Lemma 22. *Let $\mathcal{B}$ be a weak $\mathfrak{b}$-scale, $s \in \omega^{<\omega}$ and $f \in \mathcal{B}$.*

1. *If $s \in U(f)$, then $s^\frown \triangle_{f(s)} \subseteq U(f)$.*
2. *The set $\{g \in \mathcal{B} \mid s \in U(g)\}$ is cofinal in $\mathcal{B}$.*

Proof. The first point follows since f is an increasing function. For the second point, choose $n \in \omega$ such that $s \in \triangle_n$. Since $\{f \in \mathcal{B} \mid n < f(\emptyset)\}$ is cofinal in $\mathcal{B}$, we get the desired conclusion. $\square$

Let $s \in \omega^{<\omega}$. In this paper, we denote $\langle s \rangle = \{t \in \omega^{<\omega} \mid s \subseteq t\}$. This set should not be confused with $\{f \in \omega^\omega \mid s \subseteq f\}$, which is also denoted by $\langle s \rangle$ in the literature. Sets of the form $\langle s \rangle$ will be often referred as *cones*, while a *cocone* is a set of the form $\langle s \rangle^{\mathsf{c}} = \omega^{<\omega} \setminus \langle s \rangle$. A *non-trivial cocone* is simply a non-empty cocone. Define[3] $s^\uparrow = \{t \in \omega^{<\omega} \mid t \subseteq s\}$ and for a set $A \subseteq \omega^{<\omega}$, denote $A^\uparrow = \bigcup \{t^\uparrow \mid t \in A\}$. We can now define the Dow space of a weak $\mathfrak{b}$-scale.

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[3] In [14] our $s^\uparrow$ is denoted by $s^\downarrow$. We reverse this notation, as we picture our trees as growing downward.

Definition 23. *Let $\mathcal{B}$ be a weak $\mathfrak{b}$-scale. The* Dow space of $\mathcal{B}$ *is the space* $\mathbb{D}(\mathcal{B}) = (\omega^{<\omega}, \tau_{\mathcal{B}})$ *where $\tau_{\mathcal{B}}$ is the topology generated by the subbase consisting of the following sets:*

1. $\langle s \rangle, \langle s \rangle^c$ *for* $s \in \omega^{<\omega}$.
2. $U(f)$ *for* $f \in \mathcal{B}$.

In [14] the topology is not explicitly defined from the $\mathfrak{b}$-scale (as in our presentation), but it is instead constructed recursively. Nevertheless, our presentation falls under the scope of [14] because the family $\{U(f) \mid f \in \mathcal{B}\}$ satisfies Lemma 3.1 of that paper. The following notion was introduced in [14]:

Definition 24. *We say a topological space* $(\omega^{<\omega}, \tau)$ *is* $\uparrow$-sequential *if for every* $s \in \omega^{<\omega}$, *the following conditions hold:*

1. $\langle s \rangle$ *is an open set.*
2. $\langle s^\frown n \rangle_{n\in\omega}$ *converges to* s.
3. *If* $A \subseteq \omega^{<\omega}$ *converges to* s, *then* $A^{\uparrow}$ *also converges to* s.

Denote by $A^{(1)}$ the set of all convergence points of sequences contained in A. In [14] Dow proved the following:

Theorem 25 (Dow)**.** *Let $\mathcal{B}$ be a $\mathfrak{b}$-scale.*

1. $\mathbb{D}(\mathcal{B})$ *is zero dimensional.*
2. $\mathbb{D}(\mathcal{B})$ *is* $\uparrow$*-sequential.*
3. *The π-character of every point in $\mathbb{D}(\mathcal{B})$ is $\mathfrak{b}$.*
4. *If* $A \subseteq \omega^{<\omega}$, *then* $\left(A^{(1)}\right)^{(1)} = A^{(1)}$.
5. $\mathbb{D}(\mathcal{B})$ *has no isolated points.*
6. *Every* $\uparrow$*-sequential topology extending* $\tau_{\mathcal{B}}$ *has π-character at least $\mathfrak{b}$.*

Let $U \subseteq \omega^{<\omega}$ and $n \in \omega$. Define $U_{>n} = \{s \in U \mid s = \emptyset \vee s(0) > n\}$. We have the following:

Lemma 26. *Let $\mathcal{B}$ be a weak $\mathfrak{b}$-scale. The family:*

$$\left\{(U(f_1) \cap ... \cap U(f_n))_{>m} \mid f_1, ..., f_n \in \mathcal{B} \wedge m \in \omega\right\}$$

is a local base of $\emptyset$.

More constructions of countable Fréchet spaces with uncountable π-weight can be found in [17].

9. All $\mathfrak{b}$-scales may be Fréchet

In this section we will prove that it is consistent that every $\mathfrak{b}$-scale is Fréchet. In fact, we will prove that the equality $\mathfrak{p} = \mathfrak{b}$ implies that the Dow space of every $\mathfrak{b}$-scale satisfies a strong form of the Fréchet property. We recall the following definition:

Definition 27. *Let X be a topological space. We say that X is* Fréchet-Urysohn for finite sets *if for every $a \in X$, its neighborhood filter $\mathcal{N}(a)$ is a* FUF *filter.*

This class of spaces has been studied in [18], [52], [22], [23], [8] and [20] among many others. It is not hard to see that every space that is Fréchet-Urysohn for finite sets is also Fréchet.

Theorem 28 ($\mathfrak{p} = \mathfrak{b}$)**.** *The Dow space of every* $\mathfrak{b}$*-scale is Fréchet-Urysohn for finite sets (and in particular, it is a Fréchet space).*

Proof. Let $\mathcal{B} = \{f_\alpha \mid \alpha < \mathfrak{b}\}$ be a $\mathfrak{b}$-scale. For simplicity, we will prove that the neighborhood filter of $\emptyset$ is a FUF filter. The argument for an arbitrary $s \in \omega^{<\omega}$ is essentially the same, only requiring more notation. For ease of notation, let $\mathcal{F} = \mathcal{N}(\emptyset)$ and $U_\alpha = U(f_\alpha)$ for $\alpha < \mathfrak{b}$. Let $X \in (\mathcal{F}^{<\omega})^{+}$. We first find M an elementary submodel of $\mathsf{H}(\kappa)$ (for some large enough regular cardinal κ) such that $X, \mathcal{B} \in M$, the size of M is less than $\mathfrak{b}$ and $\delta = M \cap \mathfrak{b} \in \mathfrak{b}$. Denote by $\mathcal{U}$ the set of all $\bigcap\limits_{\alpha \in F} U_\alpha$ for $F \in [\delta]^{<\omega}$.

Since every non-trivial co-cone is a neighborhood of $\emptyset$, if follows that for every $U \in \mathcal{N}(\emptyset)$ and $n \in \omega$, there is $a \in X$ such that $a \subseteq U$ and $n < s(0)$ for every $s \in a \setminus \{\emptyset\}$. In this way, for every $U \in \mathcal{U}$, we can define the function $g_U \in \omega^\omega$ such that for every $n \in \omega$, it is the case that $g_U(n)$ is the least natural number for which there is $a \in X$ with the following properties:

1. $a \subseteq U$.
2. $n < s(0)$ for every $s \in a \setminus \{\emptyset\}$.
3. If $s \in a$, then $s \in \triangle_{g_U(n)}$.

Note that $g_U \in M$ for every $U \in \mathcal{U}$. Lemma 7 implies that for every $V \in \mathcal{U}$, there are infinitely many $n \in \omega$ such that $g_U(n) < f_\delta((n))$ (of course, $f_\delta((n))$ is the result of applying f_δ to $(n) \in \omega^{<\omega}$). We now define $\mathbb{P}$ as the set of all $p = (F, h, H)$ with the following properties:

1. $F \in [\omega]^{<\omega}$, $h : F \longrightarrow X$ and $H \in [\delta]^{<\omega}$.
2. For every $n \in F$ and $s \in h(n) \setminus \{\emptyset\}$, the following conditions hold:
   - (a) $n < s(0)$.
   - (b) $s \in \triangle_{f_\delta((n))}$.

Let $p, q \in \mathbb{P}$. Define $p \leq q$ in case the following conditions are met:

1. $F_q \subseteq F_p$, $h_q \subseteq h_p$ and $H_q \subseteq H_p$.
2. For every $n \in F_p \setminus F_q$ and $\alpha \in H_q$, we have that $h_p(n) \subseteq U_\alpha$.

It is easy to see that $\mathbb{P}$ is a σ-centered forcing (any finite set of conditions that share the same first two coordinates are compatible). The following claim is also not hard to prove:

1. For every $n \in \omega$, the set $D_n = \{p \in \mathbb{P} \mid F_p \nsubseteq n\}$ is dense.
2. For every $\alpha < \delta$, the set $E_\alpha = \{p \in \mathbb{P} \mid \alpha \in H_p\}$ is dense.

Since $\delta < \mathfrak{b} = \mathfrak{p}$, by Theorem 16, we can find a filter $G \subseteq \mathbb{P}$ that intersects all of the dense sets described above. Define $F_G = \bigcup\limits_{p \in G} F_p$, $h_G = \bigcup\limits_{p \in G} h_p$ (note that $h_G : F_G \longrightarrow \omega$) and $Y \subseteq X$ the image of h_G. We claim that Y is as desired. We need to prove that every $U \in \mathcal{N}(\emptyset)$ contains almost every element of Y. Note that we may assume that U is a sub-basic set. Moreover, if U is a non-trivial co-cone, the conclusion is straightforward, since the first coordinate of every element of $h_G(n) \setminus \{\emptyset\}$ is larger than n.

It remains to prove that if $\alpha < \mathfrak{b}$, then $a \subseteq U_\alpha$ for almost all $a \in Y$. This is clearly the case if $\alpha < \delta$ (since $G \cap E_\alpha \neq \emptyset$ and G is a filter), so we may assume that $\delta \leq \alpha$. Let $n \in F_G$ such that $f_\alpha(\emptyset) < n$ and $f_\delta((m)) \leq f_\alpha((m))$ for every $m \geq n$ (almost every element of F_G satisfies this conditions). We need to prove that $h_G(n) \subseteq U_\alpha$. Write $h_G(n) \setminus \{\emptyset\} = \{s_0, ..., s_l\}$. We know that $n < s_0(0), ..., s_l(0)$ and $s_0, ..., s_l \in \triangle_{f_\delta((n))}$. Moreover, $(s_0(0)), ..., (s_l(0)) \in U_\alpha$, since these values are above $f_\alpha(\emptyset)$. For each $i \leq l$, we have that $s_i \in \triangle_{f_\delta((n))} \subseteq \triangle_{f_\alpha((n))}$. It follows by Lemma 22 that each s_i is in U_α. ☐

In [17] Dow and Pecoraro proved that there is a countable, zero dimensional space that is not H-separable and has π-weight $\mathfrak{b}$. In [50] Nyikos proved that $\mathfrak{p} = \mathfrak{b}$ implies that there is an uncountably generated FUF filter. Our result provides another proof of this result.

In response to Theorem 3, Moore posed the following problem:

Problem 29 (Moore). *Is there a countable, Fréchet, zero dimensional space with π-weight exactly $\mathfrak{b}$?*

The Dow space of a Fréchet $\mathfrak{b}$-scale provides such an example. However, while Dow spaces have π-weight $\mathfrak{b}$, the π-weight of their sequential modification remains unknown. Of course, the above problem would have a positive solution if there existed a Fréchet $\mathfrak{b}$-scale, but we will see in a later section that this is consistently false. Nevertheless, we have the following result, which was independently proved by Dow and Pecoraro in [17].

Proposition 30. *If $\mathfrak{c} \leq \omega_2$, then there is a countable, Fréchet, zero-dimensional space of π-weight exactly $\mathfrak{b}$.*

Proof. If $\mathfrak{p} = \mathfrak{b}$, then every $\mathfrak{b}$-scale is Fréchet by Theorem 28. Otherwise, we $\mathfrak{p} < \mathfrak{b} = \mathfrak{c}$, so there is such space by Theorem 3. ☐

In [15] Dow investigated the possible π-weights of countable, regular, Fréchet spaces. He proved that in the Miller model every such space has π-weight at most ω_1. On the other hand, after adding κ many random reals, for any infinite cardinal $\lambda \leq \kappa$, there exists a countable, regular, Fréchet space with π-weight exactly λ. An open question remains whether there are (consistently) uncountable cardinals $\lambda < \mu < \kappa$ such that both λ and κ are realized as the π-weight of a countable, regular, Fréchet space, while μ is not.

10. THERE MAY BE A NON-FRÉCHET $\mathfrak{b}$-SCALE

In this section, we prove that it is consistent that there is a $\mathfrak{b}$-scale whose Dow space is not Fréchet. This result will be strengthened in the next section, where we show that it is consistent that no $\mathfrak{b}$-scale is Fréchet. Although the proof in this section motivates some of the ideas used later, the argument here is not a special case of the one in the next section. More importantly, we conjecture that there will be Fréchet $\mathfrak{b}$-scales in the model constructed in this section, which would establish the consistency of the simultaneous existence of both a Fréchet and a non-Fréchet $\mathfrak{b}$-scale, which is unknown at the moment. We recommend that the reader consult Section 7 as this section and the next one, use definitions and results from there.

For ease of writing, if $\mathcal{D}$ is a weak $\mathfrak{b}$-scale, we will write $\mathbb{L}(\mathcal{D})$ instead of $\mathbb{L}(\mathsf{nwd}(\mathbb{D}(\mathcal{D}))^*)$. By Theorems 25, 28 and Proposition 18, we get the following:

Lemma 31. (1) *Let $\mathcal{D}$ be a weak $\mathfrak{b}$-scale such that $\mathbb{D}(\mathcal{D})$ is Fréchet. $\mathbb{L}(\mathcal{D})$ strongly preserves ω-hitting families and forces that $\dot{A}_{\mathsf{gen}}$ is a dense set, yet it does not contain sequences convergent to $\emptyset$.*

(2) *(CH) If $\mathcal{B}$ is a $\mathfrak{b}$-scale, then $\mathbb{L}(\mathcal{B})$ strongly preserves ω-hitting families and forces that $\dot{A}_{\mathsf{gen}}$ is a dense set, yet it does not contain sequences converging to $\emptyset$.*

We can now prove the main result of this section.

Theorem 32. *It is consistent that there is a $\mathfrak{b}$-scale whose Dow space is not Fréchet.*

Proof. We start with a model of CH. Let $\mathcal{D} = \{f_\alpha \mid \alpha \in \omega_1\}$ be a $\mathfrak{b}$-scale. Define $\mathbb{P} = \mathbb{L}(\mathcal{D}) * \dot{\mathbb{D}}_{\omega_2}$, where $\mathbb{D}_{\omega_2}$ denotes the finite-support iteration of Hechler forcing of length ω_2. Let $G \subseteq \mathbb{P}$ be a generic filter. We will prove that in $V[G]$ there exists a non-Fréchet $\mathfrak{b}$-scale.

We go to $V[G]$. Let $A = A_{\mathsf{gen}} \setminus \{\emptyset\}$, where A_{gen} is the range of the generic real of $\mathbb{L}(\mathcal{D})$. By Lemma 31 and the strong preservation of ω-hitting of Hechler forcing, we know that in $\mathbb{D}(\mathcal{D})$ the set A is dense and does not contain converging sequences to $\emptyset$. Of course, $\mathcal{D}$ is no longer a $\mathfrak{b}$-scale (it is bounded). Let $\mathcal{B} = \{f_\alpha \mid \alpha \in \omega_2\}$ be a $\mathfrak{b}$-scale extending $\mathcal{D}$ such that f_{ω_1} is dominating over $V[A]$. We will prove that $\mathcal{B}$ is not Fréchet.

Claim 33. (1) $\mathbb{D}(\mathcal{B}) \models \emptyset \in \overline{A}$.

(2) $\mathbb{D}(\mathcal{B}) \models$ *A does not contain a convergent sequence to $\emptyset$.*

The second point is easy. Since A does not contain a convergent sequence to $\emptyset$ in $\mathbb{D}(\mathcal{D})$, then it cannot contain one in $\mathbb{D}(\mathcal{B})$ (every open set in $\mathbb{D}(\mathcal{D})$ is open in $\mathbb{D}(\mathcal{B})$). We now prove the first point. The argument is similar to the proof of Theorem 28.

For convenience, denote $U_\alpha = U(f_\alpha)$ for $\alpha < \omega_2$. Let $\mathcal{U}$ be the set of all $\bigcap_{\alpha \in F} U_\alpha$ for $F \in [\omega_1]^{<\omega}$. Evidently, every element of $\mathcal{U}$ is an open set in $\mathbb{D}(\mathcal{D})$.

Subclaim. Let $W \in \mathcal{U}$. The set $\langle (n) \rangle \cap W \cap A$ is not empty for almost all $n \in \omega$.

Indeed, if n is such that $\langle (n) \rangle \cap W \neq \emptyset$ (which are almost all $n \in \omega$), since $\mathbb{D}(\mathcal{D}) \models A$ is dense, it follows that $\langle (n) \rangle \cap W \cap A \neq \emptyset$. This finishes the proof of the subclaim.

Given $W \in \mathcal{U}$, we can define $g_W \in \omega^\omega$ such that for every $n \in \omega$, the following holds:

1. $g_W(n) = 0$ if $\langle (n) \rangle \cap W \cap A = \emptyset$.
2. If $\langle (n) \rangle \cap W \cap A \neq \emptyset$, let $g_W(n)$ be the least natural number such that $(n)^\frown \triangle_{g_W(n)} \cap W \cap A \neq \emptyset$.

Note that if $W \in \mathcal{U}$, then $g_W \in V[A]$. It follows that $g_W(n) < f_{\omega_1}((n))$ for almost all $n \in \omega$. We are in position to prove that $\mathbb{D}(\mathcal{B}) \models \emptyset \in \overline{A}$ now. It is enough to prove that A intersects every set of the form $(W \cap U_{\alpha_1} \cap \ldots \cap U_{\alpha_m})_{>k}$ where $W \in \mathcal{U}$, $\omega_1 \leq \alpha_1, \ldots, \alpha_m < \omega_2$ and $k \in \omega$. Find $n \in \omega$ with the following properties:

1. $g_W(n) \neq 0$.
2. $g_W(n) < f_{\omega_1}((n)) \leq f_{\alpha_1}((n)), \ldots, f_{\alpha_m}((n))$
3. $n \neq f_{\alpha_1}(\emptyset), \ldots, f_{\alpha_m}(\emptyset)$.
4. $k < n$.

Since $g_W(n) \neq 0$, there is $s \in (n)^\frown \triangle_{g_W(n)} \cap W \cap A$. It follows by Lemma 22 that $s \in U_{\alpha_i}$ for all $i \leq m$. Since $s(0) = n > k$, we have that s is in $A \cap (W \cap U_{\alpha_1} \cap \ldots \cap U_{\alpha_m})_{>k}$. □

Let us review the proof of the previous theorem. We started with a $\mathfrak{b}$-scale $\mathcal{D}$ and forced with $\mathbb{L}(\mathcal{D})$. The proof actually shows that if we complete $\mathcal{D}$ to a scale $\mathcal{B}$ in any ω-hitting preserving extension such that the first element of $\mathcal{B} \setminus \mathcal{D}$ is dominating over $V[A_{\mathsf{gen}}]$, then $\mathcal{B}$ will not be Fréchet. Of course, this does not need to be the case for every $\mathfrak{b}$-scale, so a more careful approach is required in order to construct a model in which no $\mathfrak{b}$-scale is Fréchet.

We do not know whether there are Fréchet scales in the model of Theorem 32. In fact, we conjecture that the scale induced by the Hechler reals is Fréchet, but we do not know how to prove it.

11. THERE MAY BE NO FRÉCHET $\mathfrak{b}$-SCALE

In this section, we will prove that it is consistent that no $\mathfrak{b}$-scale is Fréchet. We follow the approach used in the model constructed in [33]. We will perform a finite support iteration of forcings of the type $\mathbb{L}(\mathcal{D})$ for $\mathcal{D}$ a weak $\mathfrak{b}$-scale. As in [33], we will need a diamond sequence in order to guess an initial segment of a $\mathfrak{b}$-scale "at the right time". By Lemma 31, its Laver forcing will add A_{gen}, a dense set containing no sequences that converge to $\emptyset$. The main difficulty lies in proving that A_{gen} still accumulate to $\emptyset$, even after extending the $\mathfrak{b}$-scale. This is where our approach diverges from the one taken in [33], where the algebraic structure is used, which is not available in our setting. A completely different argument is required.

Definition 34. *Let $\mathcal{F}$ be a filter on the countable set X, $\dot{\mathbb{P}}$ an $\mathbb{L}(\mathcal{F})$-name for a partial order, and $\langle \dot{F}_n \mid n \in \omega \rangle$ a sequence of $\mathbb{L}(\mathcal{F})$-names of filters of $\mathbb{P}$ such that $\mathbb{L}(\mathcal{F}) \Vdash$ "$\mathbb{P} = \bigcup_{n \in \omega} \dot{F}_n$". Let $(p, \dot{u}) \in \mathbb{L}(\mathcal{F}) * \dot{\mathbb{P}}$.*

1. *We say that* $(p, \dot{u})$ *is* suitable *if there is* $n \in \omega$ *such that* $(p, \dot{u}) \Vdash$ "$\dot{u} \in \dot{F}_n$".
2. *A* type *is a pair of the form* (a, m) *where* $a \in X^{<\omega}$ *and* $m \in \omega$.
3. *We say that* $(p, \dot{u})$ *is of type* (a, m) *if* $\mathsf{st}(p) = a$ *and* $p \Vdash$ "$\dot{u} \in \dot{F}_n$".
4. *Let* φ *be a formula. We say that* (a, m) prefers "φ" *if there is no* $(q, \dot{v})$ *of type* (a, m) *that forces the negation of* φ.

it follows that a condition is suitable if it has a type. Note that the set of suitable conditions is dense. The following lemma is easy to verify.

Lemma 35. *Let* $\mathcal{F}$ *be a filter on a countable set,* $\dot{\mathbb{P}}$ *an* $\mathbb{L}(\mathcal{F})$*-name for a* σ*-filtered forcing,* (a, m) *a type and* φ *a formula.*

1. *If* $(p_1, \dot{u}_1), ..., (p_n, \dot{u}_n)$ *are of type* (a, m)*, then there is* $(q, \dot{v})$ *of type* (a, m) *such that* $(q, \dot{v}) \leq (p_1, \dot{u}_1), ..., (p_n, \dot{u}_n)$.
2. *If* (a, m) *prefers* "φ" *and* $(p, \dot{u})$ *is of type* (a, m)*, then there is* $(q, \dot{v}) \leq (p, \dot{u})$ *such that* $(q, \dot{v}) \Vdash$ "φ".
3. *If* $(p, \dot{u})$ *is of type* (a, m) *and* (a, m) *does not prefer* "φ"*, then there is* $(q, \dot{v}) \leq (p, \dot{u})$ *of type* (a, m) *that forces the negation of* φ.

Note that in the second point above, $(q, \dot{v})$ may not be of type (a, m). Another very important property is the following:

Lemma 36 (Pure Preference Property)**.** *Let* $\mathcal{F}$ *be a filter on a countable set,* $\dot{\mathbb{P}}$ *an* $\mathbb{L}(\mathcal{F})$*-name for a* σ*-filtered forcing,* X *a finite set and* $\dot{y}$ *an* $\mathbb{L}(\mathcal{F}) * \dot{\mathbb{P}}$*-name for an element of* X*. For every type* (a, m)*, there is* $z \in X$ *such that* (a, m) *prefers* "$\dot{y} = z$".

Proof. Assume this is not the case, so for every $z \in X$ there is $(p_z, \dot{u}_z)$ of type (a, m) such that $(p_z, \dot{u}_z) \Vdash$"$\dot{y} \neq z$". Find $(q, \dot{v})$ that extends $(p_z, \dot{u}_z)$ for every $z \in X$. It follows that $(q, \dot{v}) \Vdash$"$\dot{y} \notin X$", which is a contradiction. $\square$

We will be working with forcings of the type $\mathbb{L}(\mathcal{B}) * \dot{\mathbb{P}}$, where $\mathcal{B}$ is a weak $\mathfrak{b}$-scale. The elements of $\mathbb{L}(\mathcal{B})$ are subtrees of $(\omega^{<\omega})^{<\omega}$ that branch into subsets of $\omega^{<\omega}$. For the convenience of the reader, we adopt the following notational conventions:

1. Elements of $\omega^{<\omega}$ will be denoted by s, t and z.
2. Elements of $(\omega^{<\omega})^{<\omega}$ will be denoted by a, b and c.
3. Elements of $\mathbb{L}(\mathcal{B})$ will be denoted by p, q and r.
4. Names for elements of $\dot{\mathbb{P}}$ will be denoted by $\dot{u}, \dot{v}$ and $\dot{w}$.

Accordingly, if $p \in \mathbb{L}(\mathcal{B})$, then a typical element of p will be denoted by a,b or c and a typical element of $\mathsf{suc}_p(a)$ will be denoted by s, t or z.

Proposition 37. *Let* $\mathcal{B}$ *a weak* $\mathfrak{b}$*-scale that is dominating and Fréchet,* $\dot{\mathbb{P}}$ *an* $\mathbb{L}(\mathcal{B})$*-name for a* σ*-filtered forcing and* $\dot{g}_1, ..., \dot{g}_m$ *be* $\mathbb{L}(\mathcal{B}) * \dot{\mathbb{P}}$*-names for functions from* $\omega^{<\omega}$ *to* ω *that are dominating. For every* $U \in \mathcal{N}_{\mathbb{D}(\mathcal{B})}(\emptyset)$*, we have that:*

$$\mathbb{L}(\mathcal{B}) * \dot{\mathbb{P}} \Vdash \text{“}U \cap U(\dot{g}_1) \cap ... \cap U(\dot{g}_m) \cap \dot{A}_{\mathsf{gen}} \setminus \{\emptyset\} \neq \emptyset\text{”}.$$

Proof. We proceed by contradiction. Assume there is a condition $(\overline{p},\overline{u})$ forcing that the intersection is empty. For convenience, we may assume that if $a \in \overline{p}$ extends the stem, then $\emptyset \notin \mathsf{suc}_{\overline{p}}(a)$ and the range of a and $\mathsf{suc}_{\overline{p}}(a)$ are disjoint. Let $\dot{g}$ be the name of the function from $\omega^{<\omega}$ to ω such that $\dot{g}(s) = \mathsf{min}\{\dot{g}_1(s), ..., \dot{g}_m(s)\}$. It is easy to see that $\dot{g}$ is forced to be dominating over V. We may assume there is $\overline{n} \in \omega$ such that:

1. $(\overline{p},\overline{u}) \Vdash$"$\dot{g}_1(\emptyset), ..., \dot{g}_m(\emptyset) < \overline{n}$".
2. If $k \geq \overline{n}$, then $(k) \in U$.

Let $a \in \overline{p}$ and $n \in \omega$. We define the following:

1. $l(a,n) = \mathsf{min}\{k > \overline{n} \mid a \subseteq \triangle_k\} + n$.
2. $M(a,n)$ is the set of all $s \in \omega^{<\omega}$ for which there is $k \in \omega$ such that (a,n) prefers "$\dot{g}(s) < k$".
3. We say (a,n) *is ugly* if $M(a,n) \setminus \triangle_{l(a,n)}$.
4. Let $(p,\dot{u}) \leq (\overline{p},\overline{u})$. We say that (a,n) *can be realized below* $(p,\dot{u})$ if there is $(q,\dot{v}) \leq (p,\dot{u})$ of type (a,n).

Intuitively, $M(a,n)$ is the collection of all $s \in \omega^{<\omega}$ such that (a,n) can bound $\dot{g}(s)$ (in term of preference) and a type is ugly if it can bound something that is "very far away". We now have the following:

Claim 38. *There is $(p,\dot{u}) \leq (\overline{p},\overline{u})$ such that no ugly type is realized below $(p,\dot{u})$.*

For every $s \in \omega^{<\omega}$, define $X(s)$ as the set of all types (a,n) such that $s \in M(a,n) \setminus \triangle_{l(a,n)}$. In other words, $X(s)$ is the set of all types (a,n) such that s testifies that (a,n) is ugly. Note that $X(s)$ is a finite set. This is simply because s is in $\triangle_{l(a,n)}$ for almost all types (a,n). We can then define $h : \omega^{<\omega} \longrightarrow \omega$ such that if $s \in \omega^{<\omega}$ and $(a,n) \in X(s)$, then (a,n) prefers "$\dot{g}(s) < h(s)$" (in case $X(s) = \emptyset$, we can simply take $h(s) = 0$).

Since $\dot{g}$ is forced to be a dominating real, there are $k \in \omega$ and $(p,\dot{u}) \leq (\overline{p},\overline{u})$ such that $(p,\dot{u}) \Vdash$"$h \leq_k \dot{g}$" and $\mathsf{st}(p) \nsubseteq \triangle_k$. We claim that $(p,\dot{u})$ is as desired. If this was not true, then there is an ugly type (a,n) that can be realized below $(p,\dot{u})$. Let $(q,\dot{v}) \leq (p,\dot{u})$ that is of type (a,n). Since $\mathsf{st}(q) \nsubseteq \triangle_k$ (because $\mathsf{st}(p) \subseteq \mathsf{st}(q)$), it follows that $l(a,n) > k$. Since (a,n) is an ugly type, we know that there is $s \in M(a,n) \setminus \triangle_{l(a,n)}$. Since $(a,n) \in X(s)$, it follows that (a,n) prefers "$\dot{g}(s) < h(s)$". In this way, we can find $(r,\dot{w}) \leq (q,\dot{v})$ such that $(r,\dot{w}) \Vdash$"$\dot{g}(s) < h(s)$". But this is a contradiction since $s \notin \triangle_k$ (recall that $l(a,n) > k$ and s is not even in $\triangle_{l(a,n)}$) and $(p,\dot{u}) \Vdash$"$h \leq_k \dot{g}$". This finishes the proof of the claim.

Fix $(p,\dot{u}) \leq (\overline{p},\overline{u})$ such that no ugly type is realized below it. For simplicity, we will assume that $\mathsf{st}(p) = \emptyset$. The argument for the general case is essentially the same, only requiring much more notation. Let T be the set of all types that can be realized below $(p,\dot{u})$. For every type $(a,n) \in T$, fix a condition $(p(a,n), \dot{u}(a,n)) \leq (p,\dot{u})$ of type (a,n). Note that the stem of $p(a,n)$ is a.

We record some properties of T. For every $(a,n) \in T$, the following holds:

1. $a \in p$.
2. If $b \in p_{(a,n)}$ and $a \subseteq b$, then $(b,n) \in T$. In particular, if $z \in \mathsf{suc}_{p(a,n)}(a)$, then $(a^\frown z, n) \in T$.
3. If $q \leq p(a,n)$, then $(\mathsf{st}(q), n) \in T$.

We now have the following:

Claim 39. *For every $(a,n) \in T$ and $z \in \mathsf{suc}_{p(a,n)}(a) \cap U_{>l(a,n)}$ there is $i(z,a,n)$ such that:*

1. *$i(z,a,n) < |z|$.*
2. *There is $j \leq m$ such that $(a^\frown z, n)$ prefers "$\dot{g}_j(z \upharpoonright i(z,a,n)) = z(i(z,a,n))$". In particular, $(a^\frown z, n)$ prefers "$\dot{g}(z \upharpoonright i(z,a,n)) \leq z(i(z,a,n))$".*

Let $q = p(a,n)_{a^\frown(z)}$ (in other words, q is the tree obtained by adding z to the stem of $p(a,n)$). Note that $q \Vdash$"$z \in \dot{A}_{\mathsf{gen}} \cap U$". Since $U \cap U(\dot{g}_1) \cap ... \cap U(\dot{g}_m) \cap \dot{A}_{\mathsf{gen}} \setminus \{\emptyset\}$ is forced to be empty, Lemma 21 implies that:

$$(q, \dot{u}(a,n)) \Vdash \text{“}\exists j, k(\dot{g}_j(z \upharpoonright k) = z(k))\text{”}.$$

We can now use the Pure Preference Property to find the exact j and k. This finishes the proof of the claim.

Let $(a,n) \in T$. Define the following items:

1. $W(a,n) = \{z \upharpoonright i(z,a,n) \mid z \in \mathsf{suc}_{p(a,n)}(a) \cap U_{>l(a,n)}\}$.
2. For every $s \in W(a,n)$, denote $Ext_s(a,n) = \{z \in \mathsf{suc}_{p(a,n)}(a) \cap U_{>l(a,n)} \mid s = z \upharpoonright i(z,a,n)\}$.

We will now prove the following claim:

Claim 40. *Let $(a,n) \in T$ and $s \in W(a,n)$. The following holds:*

1. *$s \neq \emptyset$ and $s(0) > l(a,n)$.*
2. *$W(a,n)$ is infinite.*
3. *$\mathbb{D}(\mathcal{B}) \models Ext_s(a,n)$ is nowhere dense.*

We prove the first point. Pick any $z \in Ext_s(a,n)$. It follows by definition that $z(0) > l(a,n)$. In order to prove that $s \neq \emptyset$ and $s(0) > l(a,n)$, it is enough to show that $i(z,a,n) \neq 0$. If it was the case that $i(z,a,n) = 0$, then we would have that $(a^\frown z, n)$ prefers "$\dot{g}_j(\emptyset) = z(0)$" for some $j \leq m$. It follows that $(a^\frown z, n)$ prefers "$\dot{g}_j(\emptyset) > \overline{n}$", which is impossible (see the properties of $\overline{n}$ at the beginning of the proof).

We now prove the second point. Choose any $k > l(a,n)$. Since $\mathsf{suc}_p(a)$ is dense, we can find $z \in \mathsf{suc}_p(a) \cap U_{>l(a,n)}$ with $z(0) = k$. The conclusion follows by the first point of the claim.

It is time to prove the third point. Let $W \subseteq \omega^{<\omega}$ be a non-empty open set. We need to find a non-empty open set $U \subseteq W$ that is disjoint with $Ext_s(a,n)$. If $W \cap \langle s \rangle^{\mathsf{c}} \neq \emptyset$, then this is a non-empty set disjoint with $Ext_s(a,n)$, since this set is contained in $\langle s \rangle$. Moreover, we may assume there is $t \in \omega^{<\omega}$ such that $s \subsetneq t$ and $W \subseteq \langle t \rangle$ (since $W \subseteq \langle s \rangle$, there is $t \supsetneq s$ such that $t \in W$, we then change W for $W \cap \langle t \rangle$).

By the first point of the claim, we know that $s \notin \triangle_{l(a,n)}$. Since (a,n) is not ugly, it follows that (a,n) does not prefer "$\dot{g}(s) \leq t(|s|)$". By Lemma 35, we can find $(q,\dot{v}) \leq (p(a,n), \dot{u}(a,n))$ of type (a,n) such that $(q,\dot{v}) \Vdash$"$\dot{g}(s) > t(|s|)$". We may assume that $\mathsf{suc}_q(a)$ is an open dense subset of $\mathbb{D}(\mathcal{B})$. We claim that $U = W \cap \mathsf{suc}_q(a)$ (which is non-empty) is disjoint with $Ext_s(a,n)$. Assume there is $z \in U \cap Ext_s(a,n)$. We have the following:

1. $s \subsetneq t \subseteq z$ (recall that $W \subseteq \langle t \rangle$).
2. $z \restriction i(z,a,n) = s$ ($z \in Ext_s(a,n)$). In this way, there is $j \leq m$ such that $(a^\frown z, n)$ prefers "$\dot{g}_j(s) = z(|s|) = t(|s|)$". In particular, $(a^\frown z, n)$ prefers "$\dot{g}(s) \leq t(|s|)$".

Let $(r,\dot{x}) \leq (q,\dot{v})$ such that $(r,\dot{x}) \Vdash$"$\dot{g}(s) \leq t(|s|)$". But this is impossible since $(q,\dot{v}) \Vdash$"$\dot{g}(s) > t(|s|)$". This finishes the proof of the claim.

We need another claim:

Claim 41. *Let $(a,n) \in T$. There is $h_{(a,n)} : W(a,n) \longrightarrow \omega^{<\omega}$ with the following properties:*

1. $h_{(a,n)}(s) \in Ext_s(a,n)$ *(so $s \subsetneq h_{(a,n)}(s)$).*
2. $\mathit{im}(h_{(a,n)}) \in \mathit{nwd}(\mathbb{D}(\mathcal{B}))^+$.

Take an enumeration $W_{(a,n)} = \{s_i \mid i \in \omega\}$ (recall that this set is infinite). Define $N_0 = Ext_{s_0}(a,n)$ and $N_{i+1} = Ext_{s_i}(a,n) \setminus N_0 \cup ... \cup N_i$. It follows that $\{N_i \mid i \in \omega\}$ is a partition of $\mathsf{suc}_{p(a,n)}(a) \cap U_{>l(a,n)}$ into nowhere dense sets. Since $\mathbb{D}(\mathcal{B})$ is Fréchet, Proposition 11 implies that there is $Z \subseteq \mathsf{suc}_{p(a,n)}(a) \cap U_{>l(a,n)}$ that is not nowhere dense such that $|Z \cap N_k| \leq 1$ for every $k \in \omega$ (we could make sure that $|Z \cap N_k| = 1$ whenever $N_k \neq \emptyset$ if we wanted). We can now define $h_{(a,n)} : W(a,n) \longrightarrow \omega^{<\omega}$ as follows: For $s_i \in W(a,n)$, if $Z \cap N_i \neq \emptyset$, let $h_{(a,n)}(s_i)$ be the only point in $Z \cap N_i$. In case $Z \cap N_i = \emptyset$, let $h_{(a,n)}(s_i)$ be any element of $Ext_s(a,n)$. It follows that $\mathsf{im}(h_{(a,n)})$ contains Z, so it is not nowhere dense. This finishes the proof of the claim.

Let $(a,n) \in T$. We now define the function $\overline{h}_{(a,n)} : W(a,n) \longrightarrow \omega^{<\omega}$ given by $\overline{h}_{(a,n)}(s) = h_{(a,n)}(s)(|s|)$ (this is possible because $s \subsetneq h_{(a,n)}(s)$). Note that if $s \in W(a,n)$, then we have the following:

1. $h_{(a,n)}(s) \in \mathsf{suc}_{p(a,n)}(a) \cap U_{>l(a,n)}$ and $s \subsetneq h_{(a,n)}(s)$.
2. $\left(a^\frown h_{(a,n)}(s), n\right)$ prefers "$\dot{g}(s) \leq \overline{h}_{(a,n)}(s)$"
   (this is because $h_{(a,n)}(s) \restriction |s| = s$).

Since $\{\overline{h}_{(a,n)} \mid (a,n) \in T\}$ is a countable set and $\mathcal{B}$ is a dominating family, there is $f \in \mathcal{B}$ such that $\overline{h}_{(a,n)} \leq^* f$ for every $(a,n) \in T$. Recall that $\dot{g}$ is forced to be a dominating real. We can find a suitable $(q,\dot{v}) \leq (p,\dot{u})$ and $k \in \omega$ such that $(q,\dot{v}) \Vdash$"$f <_k \dot{g}$". Let (a,n) be the type of $(q,\dot{v})$. We may assume that $k < l(a,n)$.

Claim 42. *There is $s \in W(a,n)$ with the following properties:*

1. $h_{(a,n)}(s) \in \mathit{suc}_q(a)$.

(2) $(q, \dot{v}) \Vdash$ "$f(s) < \dot{g}(s)$".
(3) $\overline{h}_{(a,n)}(s) < f(s)$.

Since $\mathsf{im}(h_{(a,n)}) \in \mathsf{nwd}(\mathbb{D}(\mathcal{B}))^+$, it follows that $\mathsf{im}(h_{(a,n)}) \cap \mathsf{suc}_q(a)$ is infinite. Moreover, we know that $\overline{h}_{(a,n)} \leq^* f$, so we can find $s \in W(a,n)$ such that $h_{(a,n)}(s) \in \mathsf{suc}_q(a)$ and $\overline{h}_{(a,n)}(s) < f(s)$. Since $s(0) > l(a,n)$, we conclude that $s \notin \triangle_k$ (recall that $k < l(a,n)$). It follows that $(q, \dot{v}) \Vdash$"$f(s) < \dot{g}(s)$". This finishes the proof of the claim.

We can now finish the proof. Let $s \in W(a,n)$ as above and $z = h_{(a,n)}$. Since $(a^\frown z, n)$ prefers "$\dot{g}(s) \leq \overline{h}_{(a,n)}(s)$", there is $(r, \dot{w}) < (q_z, \dot{v})$ such that $(r, \dot{w}) \Vdash$"$\dot{g}(s) \leq \overline{h}_{(a,n)}(s)$". It follows that $(r, \dot{w}) \Vdash$"$\dot{g}(s) < f(s)$". But this is a contradiction because $(q, \dot{v}) \Vdash$"$f <_k \dot{g}$". □

Let $S_{\omega_1}(\omega_2) = \{\alpha < \omega_2 \mid \mathsf{cof}(\alpha) = \omega_1\}$. Recall the following principle:

$\Diamond(S_{\omega_1}(\omega_2))$ There is $\{D_\alpha \mid \alpha \in S_{\omega_1}(\omega_2)\}$ such that $D_\alpha \subseteq \alpha$ for all $\alpha \in S_{\omega_1}(\omega_2)$ with the property that for every $X \in \omega_2$, the set $\{\alpha \mid X \cap \alpha = D_\alpha\}$ is stationary.

A sequence as above is called a $\Diamond(S_{\omega_1}(\omega_2))$-*sequence.* It is well known that $\Diamond(S_{\omega_1}(\omega_2))$ holds in the constructible universe of Gödel (see [38] or [11]). We can now prove the main theorem of the section, which was inspired in the proof of the main theorem of [33].

Theorem 43. *It is consistent that no* $\mathfrak{b}$*-scale is Fréchet.*

Proof. We start with a model of $\mathsf{CH} + \Diamond(S_{\omega_1}(\omega_2))$. Let $\{D_\alpha \mid \alpha \in S_{\omega_1}(\omega_2)\}$ be a $\Diamond(S_{\omega_1}(\omega_2))$-sequence. Let us construct a finite support iteration $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha < \omega_2 \rangle$ such that for every $\alpha < \omega_2$, the following holds:

(1) If $\alpha \in S_{\omega_1}(\omega_2)$ and D_α codes a $\mathbb{P}_\alpha$-name for a $\mathfrak{b}$-scale $\mathcal{B}_\alpha$ that is dominating and Fréchet, we let $\dot{\mathbb{Q}}_\alpha$ be a $\mathbb{P}_\alpha$-name for $\mathbb{L}(\mathcal{B}_\alpha)$.
(2) If α is not as above, let $\dot{\mathbb{Q}}_\alpha$ be a $\mathbb{P}_\alpha$-name for Hechler forcing.

It follows that $\mathbb{P}_{\omega_2}$ is a $\mathsf{c.c.c.}$ forcing that preserves ω-hitting families. Since at every step we add a dominating real, it follows that $\mathbb{P}_{\omega_2} \Vdash$"$\mathfrak{b} = \mathfrak{d} = \mathfrak{c} = \omega_2$" and by the preservation of ω-hitting families, $\mathbb{P}_{\omega_2}$ forces that there are no towers of length ω_2 (see [4] for more details). Proposition 6 implies that $\mathbb{P}_{\omega_2}$ forces that every $\mathfrak{b}$-scale is a dominating family.

Let $G \subseteq \mathbb{P}_{\omega_2}$ be a generic filter. We shall prove that there are no Fréchet $\mathfrak{b}$-scales in $V[G]$. Aiming towards a contradiction, assume that there is a Fréchet $\mathfrak{b}$-scale $\mathcal{B} \in V[G]$. By a standard closing off argument, there is a set $C \subseteq S_{\omega_1}(\omega_2)$ which is a club relative to $S_{\omega_1}(\omega_2)$ such that if $\alpha \in C$, then $V[G_\alpha] \models \mathcal{B}_\alpha$ is a dominating Fréchet weak $\mathfrak{b}$-scale, where $G_\alpha = G \cap \mathbb{P}_\alpha$ and $\mathcal{B}_\alpha = \mathcal{B} \cap V[G_\alpha]$.

It follows that there is $\alpha \in C$ such that D_α codes $\mathcal{B}_\alpha$. In this way, we have that $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha * \mathbb{L}(\dot{\mathcal{B}}_\alpha)$. Let $A_{\mathsf{gen}} \in V[G_{\alpha+1}]$ be the image of the generic real added by $\mathbb{L}(\dot{\mathcal{B}}_\alpha)$.

Claim 44. $\mathbb{D}(\mathcal{B}) \models A_{gen}$ *accumulates to* $\emptyset$ *(in* $V[G]$*).*

It is enough to prove that for every $U \in V[G_\alpha]$ basic open set of $\emptyset$ and $g_1, ..., g_m \in \mathcal{B} \setminus \mathcal{B}_\alpha$, it is the case that $U \cap U(\dot{g}_1) \cap ... \cap U(\dot{g}_m) \cap \dot{A}_{\mathsf{gen}} \setminus \{\emptyset\} \neq \emptyset$. Note that since $V[G_\alpha] \models \mathcal{B}_\alpha$ is a dominating family and $g_1, ..., g_m \in \mathcal{B} \setminus \mathcal{B}_\alpha$, then they are dominating over $V[G_\alpha]$. Let $\beta > \alpha$ such that $g_1, ..., g_m \in V[G_\beta]$.

We go to $V[G_\alpha]$. Let $\mathbb{B}$ be the Boolean completion of the quotient $\mathbb{P}_\beta / G_\alpha$. Propositions 14 and 15 imply that $\mathbb{B}$ is σ-filtered. Proposition 37 implies that $U \cap U(\dot{g}_1) \cap ... \cap U(\dot{g}_m) \cap \dot{A}_{\mathsf{gen}} \setminus \{\emptyset\} \neq \emptyset$.

Claim 45. $\mathbb{D}(\mathcal{B}) \models A_{\mathsf{gen}}$ *does not contain a sequence converging to* $\emptyset$ *(in* $V[G]$*).*

Let Y be an infinite subset of A_{gen}. Since $\mathcal{I}_{\mathbb{D}(\mathcal{B}_\alpha)}(\emptyset) \restriction A_{\mathsf{gen}}$ is an ω-hitting ideal, there is $B \in \mathcal{I}_{\mathbb{D}(\mathcal{B}_\alpha)}(\emptyset)$ such that $Y \cap B$ is infinite. Let U be a basic neighborhood of $\emptyset$ (in $\mathbb{D}(\mathcal{B}_\alpha)$) such that $B \cap U = \emptyset$. It follows that $U \setminus Y$ is infinite. Since U is still an open set in $\mathbb{D}(\mathcal{B})$, the result follows. $\square$

12. The Category Dichotomy for definable ideals

In [30] the third author proved that if $\mathcal{I}$ is a Borel ideal, then either $\mathcal{I} \leq_{\mathsf{K}} \mathsf{nwd}$ or there is $X \in \mathcal{I}^+$ such that $\mathcal{ED} \leq_{\mathsf{K}} \mathcal{I} \restriction X$. This statement is known as the *Category Dichotomy*.[4] It is natural to ask for which other classes of ideals the Category Dichotomy can be extended. The article [16] contains many results in this direction. Here we are interested in how far the dichotomy can hold within the projective hierarchy. Using results from the previous sections, we will provide (consistently) a new example of an ideal that does not satisfy the Category Dichotomy with the least possible complexity.

In [25] the second author and Navarro proved that the Category Dichotomy holds for analytic ideals, as well as for all ideals in the Solovay model. We will now prove that the dichotomy also holds for co-analytic ideals.

Let $\mathcal{I}$ be an ideal on a countable set. The game $\mathcal{G}_{\mathsf{Cat}}(\mathcal{I})$ (which was first played by Laflamme, see [46] and [47]) is played in the following way:

I	A_0		A_1		...	A_0		...
II		b_0		b_1			b_n	

The game lasts ω rounds. In round n, Player I plays $A_n \in \mathcal{I}$ and Player II responds with $b_n \notin A_n$. *Player* II *wins the game* if the $\{b_n \mid n \in \omega\} \in \mathcal{I}^+$. In this game and the one below, Player I will be a man and Player II a woman. In the proof of [30, Theorem 3.1], the third author obtained the following result.

Proposition 46 (H. [30])**.** *Let* $\mathcal{I}$ *be an ideal on* ω*.*

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[4] Note that the two alternatives of the Category Dichotomy are not mutually exclusive, so it is not a dichotomy in the traditional sense.

(1) *If for every* $X \in \mathcal{I}^+$, *the Player* II *has a winning strategy in* $\mathcal{G}_{\mathsf{Cat}}(\mathcal{I} \restriction X)$, *then* $\mathcal{I} \leq_K \mathsf{nwd}$.
(2) *If Player* I *has a winning strategy in* $\mathcal{G}_{\mathsf{Cat}}(\mathcal{I})$, *then there is* $X \in \mathcal{I}^+$ *such that* $\mathcal{ED} \leq_K \mathcal{I} \restriction X$.

The following is a trivial consequence of this last proposition:

Corollary 47. *Let* Γ *be a class of ideals such that if* $\mathcal{I} \in \Gamma$ *and* $X \in \mathcal{I}^+$, *then* $\mathcal{I} \restriction X \in \Gamma$. *If for every* $\mathcal{I} \in \Gamma$ *the game* $\mathcal{G}_{\mathsf{Cat}}(\mathcal{I})$ *is determined, then every ideal in* Γ *satisfies the Category Dichotomy.*

In [45] the fourth author and Sabok defined an "unfolded" version of the previous game, which we will now review. We need to introduce some notation:

(1) Let R be the set of all functions $f : \omega \longrightarrow \omega \cup \{-1\}$ such that $f(n) < n$ for all $n \in \omega$ and the set $\{n \mid f(n) \neq -1\}$ is infinite.
(2) Let $f \in R$. Define $\widetilde{f} : \omega \longrightarrow \omega$ as the function obtained in the following way: enumerate $\{n \mid f(n) \neq -1\} = \{n_i \mid i \in \omega\}$ in an increasing way and let $\widetilde{f}(i) = f(n_i)$.

Intuitively, if we imagine $f \in R$ as an infinite sequence of elements from $\omega \cup \{-1\}$, then $\widetilde{f}$ is the sequence obtained by deleting all the occurrences of -1 and then "compressing" to get rid of the empty spaces.

Let $\mathcal{I}$ be an ideal on a countable set and $F : \omega^\omega \longrightarrow \mathcal{I}^+$ any function. The game $\mathcal{H}_{\mathsf{Cat}}(\mathcal{I}, F)$ is played in the following way:

I	A_0		A_1		...	A_n		...
II		(b_0, m_0)		(b_1, m_1)			(b_n, m_n)	

The game lasts ω rounds. In round n, Player I plays $A_n \in \mathcal{I}$ and Player II responds with (b_n, m_n) such that $b_n \notin A_n$ and $m_n \in n \cup \{-1\}$ (intuitively, $m_n = -1$ can be interpreted as Player II refraining from making a move). *Player* II *wins the game* if the following conditions are met:

(1) The set $\{n \mid m_n \neq -1\}$ is infinite.
(2) $F(\widetilde{g}) = \{b_n \mid n \in \omega\}$, where $g : \omega \longrightarrow \omega \cup \{-1\}$ is the function such that $g(n) = m_n$.

Note that if Player II was the winner of the game, then $\{b_n \mid n \in \omega\} \in \mathcal{I}^+$. The first point of the following proposition was proved by the fourth author and Sabok as part of the proof of Theorem 1.6 of [45]. We provide a proof for completeness, but first we introduce some more notation. Let $n \in \omega$. Define T_n as the set of all $t : n \longrightarrow n \cup \{-1\}$ such that $t(i) < i$ for all $i < n$. Note that each T_n is a finite set. Let $s \in \omega^n$ and $t \in T_n$. Define $s * t : n \longrightarrow \omega \times (\omega \cup \{-1\})$ given by $(s * t)(i) = (s(i), t(i))$.

Proposition 48 (K., Sabok [45])**.** *Let* $\mathcal{I}$ *be a co-analytic ideal on* ω *and* $F : \omega^\omega \longrightarrow \mathcal{I}^+$ *a continuous surjection*[5].

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[5] Note that the existence of such function is guaranteed by the fact that $\mathcal{I}$ is co-analytic.

(1) *If Player* I *has a winning strategy in* $\mathcal{H}_{\mathsf{Cat}}(\mathcal{I}, F)$*, then he has a winning strategy in* $\mathcal{G}_{\mathsf{Cat}}(\mathcal{I})$*.*
(2) *If Player* II *has a winning strategy in* $\mathcal{H}_{\mathsf{Cat}}(\mathcal{I}, F)$*, then she has a winning strategy in* $\mathcal{G}_{\mathsf{Cat}}(\mathcal{I})$*.*

Proof. Let $\sigma : (\omega \times (\omega \cup \{-1\}))^{<\omega} \longrightarrow \mathcal{I}$ be a a winning strategy for the Player I in the game $\mathcal{H}_{\mathsf{Cat}}(\mathcal{I}, F)$. We will use σ to define a strategy $\pi : \omega^{<\omega} \longrightarrow \mathcal{I}$ for him in the game $\mathcal{G}_{\mathsf{Cat}}(\mathcal{I})$. Both games start with $\pi(\emptyset) = \sigma(\emptyset)$. Assume we are at round $n+1$ and Player I has to make a move (in $\mathcal{G}_{\mathsf{Cat}}(\mathcal{I})$) after Player II played $s \in \omega^{n+1}$. Player I plays (in $\mathcal{G}_{\mathsf{Cat}}(\mathcal{I})$) the set $\pi(s) = \bigcup\limits_{t \in T_{n+1}} \sigma(s * t) \in \mathcal{I}$. We claim that this is a winning strategy for Player I in $\mathcal{G}_{\mathsf{Cat}}(\mathcal{I})$.

Assume this is not the case, so there is a run of the game $\mathcal{G}_{\mathsf{Cat}}(\mathcal{I})$ where Player I followed π, but Player II was declared the winner. Let $(b_i)_{i\in\omega}$ be the sequence played by Player II. Since she was the winner, we know that $Y = \{b_i \mid i \in \omega\} \in \mathcal{I}^+$. Recall F is onto, so Y is in the range of F. Moreover, we can find $g \in R$ such that $F(\widetilde{g}) = Y$. Consider the run of the game $\mathcal{H}_{\mathsf{Cat}}(\mathcal{I}, F)$ where Player I followed σ and Player II played $(b_n, g(n))$ at round n. This is a valid run of the game since for every $n \in \omega$, it is the case that $g(n) \in n \cup \{-1\}$ and $\sigma((b_0, g(0)), ..., (b_{n,} g(n)))$ is contained in $\pi(b_0, ..., b_n)$, so b_{n+1} is not in $\sigma((b_0, g(0)), ..., (b_{n,} g(n)))$ (it is not even in $\pi(b_0, ..., b_n)$, which is bigger). It follows that Player II won the game, but this was impossible since σ was a winning strategy.

We now prove the second part of the proposition. Player II can easily win $\mathcal{G}_{\mathsf{Cat}}(\mathcal{I})$ by playing the first coordinates of her winning strategy of $\mathcal{H}_{\mathsf{Cat}}(\mathcal{I}, F)$. □

We now prove the determinacy of these games for co-analytic ideals.

Proposition 49. *Let* $\mathcal{I}$ *be a co-analytic ideal on* ω *and* $F : \omega^\omega \longrightarrow \mathcal{I}^+$ *a continuous surjection. The games* $\mathcal{H}_{\mathsf{Cat}}(\mathcal{I}, F)$ *and* $\mathcal{G}_{\mathsf{Cat}}(\mathcal{I})$ *are determined.*

Proof. By Proposition 48, the determinacy of $\mathcal{H}_{\mathsf{Cat}}(\mathcal{I}, F)$ implies the determinacy of $\mathcal{G}_{\mathsf{Cat}}(\mathcal{I})$. We now show that $\mathcal{H}_{\mathsf{Cat}}(\mathcal{I}, F)$ is determined. By Martin's Determinacy Theorem for Borel games (see [40]), it is enough to prove that $\{(B, g) \mid g \in R \wedge F(\widetilde{g}) = B\} \subseteq \mathcal{P}(\omega) \times (\omega \cup \{-1\})^\omega$ (the winning set for Player II) is Borel. This is easy: membership in R is G_δ while the condition $F(\widetilde{g}) = B$ is closed by the continuity of F. □

Since the restriction of a co-analytic ideal is also co-analytic, by Proposition 49 and 47, we obtain the following:

Theorem 50. *Every co-analytic ideal satisfies the Category Dichotomy.*

How far can the Category Dichotomy be extended? If we assume suitable determinacy axioms or instances of the *Closed-sets Covering Property* (see [55], [12] and [25]), we can extend the Category Dichotomy throughout the entire Projective Hierarchy. However, just in ZFC, the class of analytic and co-analytic ideals is the highest we can go, since it is consistent that the Category Dichotomy fails for $\triangle^1_2$ ideals.

We were able to find two consistent examples of $\triangle^1_2$ ideals in the literature for which the Category Dichotomy fails:

(1) *The ideal generated by a co-analytic tight MAD family.* In [6], Bergfalk, Fischer and Switzer proved that $V = L$ implies that there is a co-analytic tight MAD family. The ideal generated by such family is a $\triangle^1_2$ ideal and can not satisfy the Category Dichotomy (see Proposition 24 of [16]).
(2) *The dual of a $\triangle^1_2$ Ramsey ultrafilter.* In [54, Theorem 5.1], Schilhan proved that it is consistent that there is a $\triangle^1_2$ Ramsey ultrafilter. The dual of such ultrafilter cannot satisfy the Category Dichotomy (see [16, Proposition 23]).

We will now provide a new example of a $\triangle^1_2$ ideal for which the Category Dichotomy fails using Dow spaces. The following is [16, Theorem 45]:

Theorem 51 (Dow, F., G., H. [16])**.** *If X is a topological space with the following properties:*

(1) *X is countable.*
(2) *X is zero dimensional.*
(3) *X is Fréchet.*
(4) *X has uncountable π-character everywhere.*

Then $\mathsf{nwd}(X)$ does not satisfy the Category Dichotomy.

It follows that if $\mathcal{B}$ is a Fréchet $\mathfrak{b}$-scale, then the ideal $\mathsf{nwd}(\mathbb{D}(\mathcal{B}))$ does not satisfy the Category Dichotomy. We now prove the following:

Theorem 52. *If $V = L$, then there exists a Fréchet $\mathfrak{b}$-scale $\mathcal{B}$ such that $\mathsf{nwd}\big(\mathbb{D}(\mathcal{B})\big)$ is a Δ^1_2 ideal.*

The following lemma is used in the proof of the preceding theorem.

Lemma 53. *Let (X, τ) be a countable topological space, and suppose that $\mathscr{B}$ is a co-analytic basis for τ. Then the topology τ is Σ^1_2, and the ideal $\mathsf{nwd}(X, \tau)$ is Δ^1_2.*

Proof. As $\mathscr{B}$ is co-analytic, for each $s \in X$ the set

$$X_s = \big\{A \subseteq X : s \in A \ \rightarrow\ \exists U \in \mathscr{B}\big(s \in U \wedge U \subseteq A\big)\big\}$$

is Σ^1_2. Moreover, for any $A \subseteq X$

$$A \in \tau \iff \forall s \in X\,(A \in X_s)\,.$$

Therefore τ belongs to Σ^1_2, as it is a countable intersection of Σ^1_2 sets.

Recall that $\triangle^1_2$ denotes the class $\Sigma^1_2 \cap \Pi^1_2$. We first show that $\mathsf{nwd}(X, \tau)$ is a Π^1_2 subset of $\mathscr{P}(X)$. To that end, it suffices to verify that its complement

$$\mathsf{nwd}(X, \tau)^+ = \mathscr{P}(X) \setminus \mathsf{nwd}(X, \tau)$$

is Σ^1_2. This follows from the following equivalence:

$$A \in \mathsf{nwd}(X,\tau)^+ \iff \exists U \in \mathscr{B} \setminus \{\emptyset\}\ \forall V \in \mathscr{B} \setminus \{\emptyset\}\ (V \subseteq U \to V \cap A \neq \emptyset)$$

which defines a Σ^1_2 condition under the assumption that $\mathscr{B}$ is co-analytic. Since the complement of a Σ^1_2 set is a Π^1_2 set, the result follows.

On the other hand, we show that $\mathsf{nwd}(X,\tau)$ is also a Σ^1_2 subset of $\mathscr{P}(X)$. We begin with the case of closed nowhere dense sets. Let $\overline{\mathsf{nwd}}(X,\tau)$ denote the family of closed nowhere dense subsets of X. We claim that $\overline{\mathsf{nwd}}(X,\tau)$ is Σ^1_2. Indeed, a set $F \subseteq X$ is closed and nowhere dense if and only if its complement $X \setminus F$ is a dense open set. Equivalently,

$$F \in \overline{\mathsf{nwd}}(X,\tau) \iff X \setminus F \in \tau \ \wedge\ \forall U \in \mathscr{B} \setminus \{\emptyset\} \big(U \cap (X \setminus F) \neq \emptyset\big).$$

Since the topology τ is Σ^1_2 and $\mathscr{B}$ is a co-analytic, the universal quantification over $\mathscr{B} \setminus \{\emptyset\}$ defines a co-analytic set. Therefore, the conjunction above is Σ^1_2, and it follows that $\overline{\mathsf{nwd}}(X,\tau)$ is Σ^1_2.

Now, we consider the general case. A subset $A \subseteq X$ is nowhere dense if and only if it is contained in some closed nowhere dense set. That is,

$$A \in \mathsf{nwd}(X,\tau) \iff \exists F \in \overline{\mathsf{nwd}}(X,\tau) \big(A \subseteq F\big)$$

from which follows that $\mathsf{nwd}(X,\tau)$ is also a Σ^1_2 subset of $\mathscr{P}(X)$. □

Proof of Theorem 52. By Lemma 53, it suffices to construct a Fréchet $\mathfrak{b}$-scale such that the topology $\tau_{\mathcal{B}}$ admits a co-analytic basis. Note that by Theorem 28, $V = L$ implies that all $\mathfrak{b}$-scales are Fréchet.

Claim 54. *$V = L$ implies that there is a co-analytic $\mathfrak{b}$-scale.*

Proof. Consider

$$F \subseteq \left(\omega^{\omega^{<\omega}}\right)^{\leq\omega} \times \omega^{\omega^{<\omega}} \times \omega^{\omega^{<\omega}}$$

defined by declaring that $(A, g, f) \in F$ if and only if the following conditions hold:

1. f is increasing from $\omega^{<\omega}$ to ω, that is, for $s, t \in \omega^{<\omega}$ and $i, j \in \omega$:
   - If $s \subset t$, then $f(s) < f(t)$, and
   - If $i < j$, then $f(s^\frown i) < f(s^\frown j)$.
2. For every $h \in \operatorname{ran}(A)$ we have $h <^* f$.
3. $g <^* f$.
4. $f(\emptyset) > g(\emptyset)$.

Note that F is Borel, since it is defined as a finite conjunction of Borel conditions. Moreover, for every $(A, g) \in \left(\omega^{\omega^{<\omega}}\right)^{\leq\omega} \times \omega^{\omega^{<\omega}}$, the section

$$F_{(A,g)} = \left\{f \in \omega^{\omega^{<\omega}} : (A, g, f) \in F\right\}$$

is cofinal in the Turing degrees[6].

---

[6] We refer to elements of 2^ω, ω^ω, or $\omega^{\omega^{<\omega}}$ collectively as *reals*. For x and y reals, we say x is *Turing reducible* to y, denoted by $x \leq_T y$, if there exists an oracle Turing machine

We will prove this for A countably infinite. The argument for A finite is essentially the same. Fix $\{h_i : i \in \omega\}$ an enumeration of A, and an arbitrary real $y \in 2^\omega$. We construct a function $f \in F_{(A,g)}$ such that $y \leq_T f$.

First, define a function $f_0 \in F_{(A,g)}$ by recursion on $\omega^{<\omega}$. For each $s \in \omega^{<\omega}$, choose $f_0(s)$ so large that:

(1) If $t \subset s$, then $f_0(t) < f_0(s)$,
(2) If $i < j$, then $f_0(s^\frown i) < f_0(s^\frown j)$, and
(3) $f_0(s) > \max\{h_i(s) : i \leq |s|\} + g(s)$,

Since only finitely many inequalities must be satisfied at each stage, this construction is possible.

Now, we code y along the branch $\langle 0\rangle, \langle 0,0\rangle, \ldots$. Define f by

$$f(s) = \begin{cases} 2f_0(s) + 1 & \text{if } s \text{ is not of the form } \langle 0^n\rangle \text{ for any } n, \\ 2f_0(\langle 0^n\rangle) + y(n) & \text{if } s = \langle 0^n\rangle \text{ for some } n, \end{cases}$$

where $\langle 0^n\rangle$ denotes the sequence of n many zeros. Note that $f \in F_{(A,g)}$ and y is computable from f since for each n,

$$y(n) = f(\langle 0^n\rangle) \bmod 2,$$

and the parity of f at these nodes can be obtained by an oracle Turing machine with input f. Thus $y \leq_T f$, establishing that $F_{(A,g)}$ is cofinal in the Turing degrees.

Therefore, it follows from [59, Theorem 1.3.] that there exists a co-analytic set that is compatible with F, that is, there is a co-analytic $\mathfrak{b}$-scale □

Now, let $\mathcal{B} = \{f_\alpha : \alpha < \omega_1\}$ be a co-analytic $\mathfrak{b}$-scale. Note that by construction, $\mathcal{B}$ is well-ordered by $<^*$, that is, for any distinct $f, g \in \mathcal{B}$, either $f <^* g$ or $g <^* f$. We shall prove that the topology $\tau_{\mathcal{B}}$ admits a co-analytic basis $\mathscr{B}$. We verify the following statements.

(1) The family $\mathcal{U} = \{U(f) : f \in \mathcal{B}\}$ is co-analytic.

Let $\Phi : \omega^{\omega^{<\omega}} \to 2^{\omega^{<\omega}}$ be the mapping defined by $f \mapsto U(f)$. We claim that Φ is continuous. We need to show that the preimage of any subbasic open set is open.

For $s \in \omega^{<\omega}$, consider $V_s = \{A \subseteq \omega^{<\omega} : s \in A\}$ a subbasic open set. Based on the definition of $U(f)$, we have:

$$f \in \Phi^{-1}(V_s) \iff s \in U(f) \iff \forall t \subsetneq s\,(f(t) \neq s(|t|))\,.$$

This preimage is a finite intersection of open sets:

$$\Phi^{-1}(V_s) = \bigcap_{t \subsetneq s} \{f \in \omega^{\omega^{<\omega}} : f(t) \neq s(|t|)\}.$$

---

that computes x given y as an oracle. A set of reals A is *cofinal* in the Turing degrees if for every $x \in 2^\omega$ there exists some $y \in A$ such that $x \leq_T y$.

Therefore, $\Phi^{-1}(V_s)$ is open. Similarly, for the complement of the subbasic open set:

$$f \in \Phi^{-1}(2^{\omega^{<\omega}} \setminus V_s) \iff s \notin U(f) \iff \exists t \subsetneq s\,(f(t) = s(|t|))\,.$$

This preimage is a finite union of open sets:

$$\Phi^{-1}(2^{\omega^{<\omega}} \setminus V_s) = \bigcup_{t \subsetneq s} \{f \in \omega^{\omega^{<\omega}} : f(t) = s(|t|)\},$$

which is also open. Since the preimages of subbasic sets are open, we conclude that Φ is continuous. Moreover, Φ is injective. Indeed, let $f, g \in \omega^{\omega^{<\omega}}$ with $f \neq g$. For the minimal t such that $f(t) \neq g(t)$, define $s = t^\frown f(t)$. Then $s \in U(g) \setminus U(f)$, so $\Phi(f) \neq \Phi(g)$. By Theorem 2.6 in [37], the injective continuous image of a co-analytic set is co-analytic. Therefore, we conclude that $\mathcal{U} = \{U(f) : f \in \mathcal{B}\}$ is co-analytic.

(2) For each $n \in \omega$, the following family is co-analytic

$$\mathcal{U}^n = \left\{ \bigcap_{f \in C} U(f) : C \in \mathcal{B}^n \right\}.$$

Let $X = \omega^{\omega^{<\omega}}$ and consider the mapping $\Phi_n : X^n \to 2^{\omega^{<\omega}}$ defined by $(f_0, \ldots, f_{n-1}) \mapsto \bigcap_{i<n} U(f_i)$.

First, we show that Φ_n is continuous. For each $j < n$, let $\pi_j : X^n \to X$ be the projection onto the j-th coordinate, and let $\Phi : X \to 2^{\omega^{<\omega}}$ be the continuous map $f \mapsto U(f)$. Then the map $\phi_j = \Phi \circ \pi_j$ is continuous as a composition of continuous functions.

Note that Φ_n is given by the finite intersection:

$$\Phi_n(f_0, \ldots, f_{n-1}) = \bigcap_{j<n} \phi_j(f_0, \ldots, f_{n-1}).$$

Since the intersection of n elements in $2^{\omega^{<\omega}}$ is a continuous operation, it follows that Φ_n is continuous. Next, we establish that Φ_n is injective. Let $A, B \in X^n$ be distinct as sets. Let $t \in \omega^{<\omega}$ be a minimal node such that the sets of values $V_A = \{f(t) : f \in A\}$ and $V_B = \{g(t) : g \in B\}$ are distinct, say $V_A \setminus V_B \neq \emptyset$. Suppose $k \in V_A \setminus V_B$ and let $s = t^\frown k$. Thus $s \in \Phi_n(B) \setminus \Phi_n(A)$, so it follows that $\Phi_n(A) \neq \Phi_n(B)$. Finally, note that the product $\mathcal{B}^n$ is co-analytic in X^n. By Theorem 2.6 in [37], we conclude that $\Phi_n(\mathcal{B}^n) = \left\{\bigcap_{f \in C} U(f) : C \in \mathcal{B}^n\right\}$ is co-analytic.

(3) The family $\mathcal{U}^{<\omega} = \left\{\bigcap_{f \in C} U(f) : C \in \mathcal{B}^{<\omega}\right\}$ is co-analytic.

Since the family $\mathcal{U}^{<\omega}$ can be expressed as the countable union of the families $\mathcal{U}^n$ previously defined:

$$\mathcal{U}^{<\omega} = \bigcup_{n\in\omega} \mathcal{U}^n = \bigcup_{n\in\omega} \left\{ \bigcap_{f\in C} U(f) : C \in \mathcal{B}^n \right\},$$

and each $\mathcal{U}^n$ is co-analytic, it follows that $\mathcal{U}^{<\omega}$ is a co-analytic family.

(4) For any finite sets $A, B \subset \omega^{<\omega}$, the family

$$\mathcal{X}_{A,B} = \left\{ \bigcap_{f\in C} U(f) \cap \bigcap_{s\in A} \langle s\rangle \cap \bigcap_{s\in B} \langle s\rangle^c : C \in [\mathcal{B}]^{<\omega} \right\}$$

is co-analytic.

Let $K_{A,B} = \left(\bigcap_{s\in A}\langle s\rangle\right) \cap \left(\bigcap_{s\in B}\langle s\rangle^c\right)$. Note that if $K_{A,B} = \emptyset$, then $\mathcal{X}_{A,B} = \{\emptyset\}$, which is a closed set and thus co-analytic. We assume therefore that $K_{A,B} \neq \emptyset$. In particular, we assume that A and B are disjoint. Let $\Psi_{A,B} : 2^{\omega^{<\omega}} \to 2^{\omega^{<\omega}}$ be the mapping defined by $\Psi_{A,B}(U) = U \cap K_{A,B}$. The continuity of $\Psi_{A,B}$ follows from the fact that the pre-image of any sub-basic open set, that is, for each $s \in \omega^{<\omega}$ the set $V_s = \{A \subseteq \omega^{<\omega} : s \in A\}$ and its complement V_s^c, is either $V_s, \emptyset, V_s^c$, or $2^{\omega^{<\omega}}$, all of which are open.

Let $\mathcal{U}^{<\omega}$ be the co-analytic family of (3). We partition this family into $\mathcal{U}_\emptyset = \{U \in \mathcal{U}^{<\omega} : U \cap K_{A,B} = \emptyset\}$ and $\mathcal{U}_+ = \mathcal{U}^{<\omega} \setminus \mathcal{U}_\emptyset$. Note that $\mathcal{U}_+ = \mathcal{U}^{<\omega} \cap \{U : U \cap K_{A,B} \neq \emptyset\}$ remains co-analytic, since the condition $U \cap K_{A,B} \neq \emptyset$ defines an open set in $2^{\omega^{<\omega}}$, and $\mathcal{U}^{<\omega}$ is co-analytic.

On the domain $\mathcal{U}_+$, the mapping $\Psi_{A,B}$ is injective. To see that, consider $U_1, U_2 \in \mathcal{U}_+$ such that $U_1 \neq U_2$. Let $C_1, C_2 \in \mathcal{B}^{<\omega}$ be such that $U_1 = \bigcap\{U(f) : f \in C_1\}$ and $U_2 = \bigcap\{U(f) : f \in C_2\}$. Therefore, there is $g \in C_1 \Delta C_2$ and, without loss of generality, we assume $g \in C_1 \setminus C_2$. Since $\mathcal{B}$ is a $\mathfrak{b}$-scale, there exists $N \in \omega$ such that for all $s \in \omega^{<\omega}$ with $|s| > N$, the value $g(s)$ is distinct from $f(s)$ for every $f \in C_2$. This implies that for any such s, if $s \in U_2$ then the successor node $s^* = s^\frown g(s)$ belongs to $U_2 \setminus U_1$. Given that $U_2 \cap K_{A,B}$ is non-empty and the trees in $\mathcal{U}_+$ possess an infinite branching structure, there are infinitely many nodes $s \in U_2 \cap K_{A,B}$ with $|s| > N$. For each such s, the successor s^* remains within the cones generated by A. Furthermore, the finiteness of B ensures that we can choose at least one s such that $s^* = s^\frown g(s) \in K_{A,B}$. Thus, $\Psi_{A,B}(U_1) \neq \Psi_{A,B}(U_2)$.

By Theorem 2.6 in [37], we conclude that $\Psi_{A,B}(\mathcal{U}_+)$ is co-analytic. Since $\Psi_{A,B}(\mathcal{U}_\emptyset) = \{\emptyset\}$ is a closed set, it follows that

$$\mathcal{X}_{A,B} = \Psi_{A,B}(\mathcal{U}_\emptyset) \cup \Psi_{A,B}(\mathcal{U}_+)$$

is co-analytic.

(5) The family $\mathscr{B}$ generated by the cones $\langle s\rangle$ and co-cones $\langle s\rangle^c$ for $s \in \omega^{<\omega}$, together with the trees $U(f)$ for $f \in \mathcal{B}$, is co-analytic.
 Note that for any $U \in 2^{\omega^{<\omega}}$

$$U \in \mathscr{B} \iff \exists A, B \in [\omega^{<\omega}]^{<\omega}\,(U \in \mathcal{X}_{A,B}).$$

As each $\mathcal{X}_{A,B}$ is co-analytic and the class of co-analytic sets is closed under countable unions, it follows that $\mathscr{B}$ is co-analytic.

This completes the proof that $\tau_{\mathcal{B}}$ admits a co-analytic basis.

□

13. Open Questions

In this final section, we present several open questions that we have been unable to solve. We first restate Moore's problem:

Problem 55 (Moore). *Is there a countable, Fréchet, zero dimensional space with π-weight exactly $\mathfrak{b}$?*

In the paper, we proved that it is consistent that every $\mathfrak{b}$-scale is Fréchet and also that none is. However, the following remains unsolved:

Problem 56. *Is it consistent that there are Fréchet $\mathfrak{b}$-scales and a non-Fréchet $\mathfrak{b}$-scales at the same time?*

We conjecture that this happens in the model of Theorem 32.

Problem 57. *Is there a Fréchet $\mathfrak{b}$-scale in the model of Theorem 32?*

We do not know much about the ideal $\mathsf{nwd}(\mathbb{D}\,(\mathcal{D}))$ in case $\mathcal{D}$ is not a Fréchet $\mathfrak{b}$-scale.

Problem 58.

(1) *Is it consistent that there is a $\mathfrak{b}$-scale $\mathcal{D}$ such that $\mathsf{nwd}(\mathbb{D}\,(\mathcal{D}))$ satisfies the Category Dichotomy?*
(2) *Is it consistent that for every $\mathfrak{b}$-scale $\mathcal{D}$, the ideal $\mathsf{nwd}(\mathbb{D}\,(\mathcal{D}))$ satisfies the Category Dichotomy?*

Finally, the following problem may also be of interest.

Problem 59. *How do the combinatorial properties of a scale reflect on the topological properties of its Dow space and vice versa?*

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