CRITICAL PARTITION REGULAR FUNCTIONS FOR COMPACT SPACES

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Abstract. We study ideal-based refinements of sequential compactness arising from the class FinBW($\mathcal{I}$), consisting of topological spaces in which every sequence admits a convergent subsequence indexed by a set outside a given ideal $\mathcal{I}$. A central theme of this work is the existence of critical ideals whose position in the Katětov order determines the relationship between a fixed class of spaces and the corresponding FinBW($\mathcal{I}$) classes. Building on earlier results characterizing several classical topological classes via such ideals, we extend this theory to a broader framework based on partition regular functions, which unifies ordinary convergence with other non-classical convergence notions such as IP- and Ramsey-type convergence. Furthermore, we investigate the existence of critical ideals associated with function classes motivated by Mazurkiewicz's theorem on uniformly convergent subsequences.

1. Introduction

See Section 2 for notions and notations used in the introduction.

In recent years, various ideal-based refinements of classical convergence and topological properties have been intensively studied. One such notion is the class FinBW($\mathcal{I}$) (associated to an ideal $\mathcal{I}$ on ω) which consists of all topological spaces X having the property that for every sequence $(x_n)_{n\in\omega}$ in X there exists $A \notin \mathcal{I}$ such that the subsequence $(x_n)_{n\in A}$ is convergent in X ([14, Definition 1.1]). For instance, FinBW(Fin) coincides with the class of all sequentially compact spaces, where Fin is the ideal of all finite subsets of ω.

A natural problem in this area is to determine whether a given class of topological spaces can be characterized as a FinBW($\mathcal{I}$) for a suitable ideal $\mathcal{I}$.

A particularly interesting question concerns the existence of a *critical ideal* $\mathcal{I}_{\mathcal{C}}$ for a given class $\mathcal{C}$ of topological spaces with the following property: the position of a given ideal $\mathcal{J}$ relative to $\mathcal{I}_{\mathcal{C}}$ in the Katětov order determines the relationship between the classes $\mathcal{C}$ and FinBW($\mathcal{J}$). Problems of this type have already been investigated in the literature, see for instance [4], [14] and [16]. Some of these results are as follows.

Theorem 1.1.

- $\mathsf{Fin}\otimes\mathsf{Fin} \leq_K \mathcal{I} \iff$ FinBW($\mathcal{I}$) *coincides with the class of all finite spaces.*
- *The following conditions are equivalent.*
  (1) $\mathsf{Bl} \leq_K \mathcal{I}$ *and* $\mathsf{Fin}\otimes\mathsf{Fin} \not\leq_K \mathcal{I}$.
  (2) FinBW($\mathcal{I}$) *coincides with the class of all boring spaces.*
- $\mathsf{conv} \not\leq_K \mathcal{I} \iff$ FinBW($\mathcal{I}$) *coincides with the class of all compact metric spaces in the realm of metric spaces.*

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The definition of $\mathrm{FinBW}(\mathcal{I})$ is based on the notion of ordinary convergence. However, there exist other types of convergence that fall outside this classical framework. Examples include convergences arising in ergodic theory and other areas, for instance, *IP-convergence* (introduced by Furstenberg and Weiss [9]) and *R-convergence* (considered in [1], [2] and [10]). Such convergences have been employed, for example, in [11], [12] and [13] to define some classes of topological spaces, namely *Hindman spaces* and *Ramsey spaces*. In general, these forms of convergence cannot be adequately described using the language of ordinary convergence alone.

More recently, in the paper [5], the authors introduced a unifying framework (*partition regular functions*) which, in one formulation, can be interpreted in terms of ordinary convergence, while simultaneously encompassing these other, non-ordinary types of convergence.

Motivated by these developments, the aim of the present paper is to generalize the results of Theorem 1.1 for partition regular functions. This task is addressed in Sections 3, 4 and 6.

In the final section, we address the problem of identifying the critical ideal for the class of functions possessing the property motivated by Mazurkiewicz's theorem [15], which states that every uniformly bounded sequence of continuous functions has a uniformly convergent subsequence once restricted to some perfect set.

2. Preliminaries

All topological spaces considered in the paper are assumed to be Hausdorff.

Recall that an ordinal number α is equal to the set of all ordinal numbers less than α. In particular, the smallest infinite ordinal number $\omega = \{0, 1, \dots\}$ is equal to the set of all natural numbers $\mathbb{N}$, and each natural number $n = \{0, \dots, n-1\}$ is equal to the set of all natural numbers less than n. Using this identification, we can, for instance, write $n \in k$ instead of $n < k$, $n < \omega$ instead of $n \in \omega$, or $A \cap n$ instead of $A \cap \{0, 1, \dots, n-1\}$.

We write $[A]^{<\omega}$ to denote the family of all finite subsets of A, $[A]^{\omega}$ to denote the family of all infinite countable subsets of A and $\mathcal{P}(A)$ to denote the family of all subsets of A. We say that $\mathcal{A} \subseteq [\omega]^{\omega}$ is an *almost disjoint family on* ω if $A \cap B$ is finite for any distinct $A, B \in \mathcal{A}$.

2.1. **Ideals and partition regular functions.** An *ideal* on a nonempty set X is a family $\mathcal{I} \subseteq \mathcal{P}(X)$ that satisfies the following properties:

(1) $\emptyset \in \mathcal{I}$ and $X \notin \mathcal{I}$;
(2) if $A, B \in \mathcal{I}$ then $A \cup B \in \mathcal{I}$;
(3) if $A \subseteq B$ and $B \in \mathcal{I}$ then $A \in \mathcal{I}$;
(4) $\mathcal{I}$ contains all finite subsets of X.

For an ideal $\mathcal{I}$, we write $\mathcal{I}^+ = \{A \subseteq X : A \notin \mathcal{I}\}$ and call it the *coideal of* $\mathcal{I}$. Sets belonging to $\mathcal{I}^+$ will be called $\mathcal{I}$*-positive* sets. An ideal $\mathcal{I}$ is *tall* (also called *dense*) if for every infinite $A \subseteq X$ there is an infinite $B \in \mathcal{I}$ such that $B \subseteq A$.

Example 2.1.

(1) Fin is the ideal of all finite subsets of ω.
(2) $\mathsf{Fin}^2 = \mathsf{Fin} \otimes \mathsf{Fin}$ is the ideal on $\omega^2 = \omega \times \omega$ defined by
$$A \in \mathsf{Fin}^2 \iff \exists i_0 \in \omega \, \forall i \geq i_0 \, (|\{j \in \omega : (i,j) \in A\}| < \omega).$$
(3) BI is the ideal on $\omega^3 = \omega \times \omega \times \omega$ introduced in [14, Definition 4.1] and defined by
$$A \in \mathsf{BI} \iff \exists i_0 \in \omega \, [\forall i < i_0 \, (\{(j,k) \in \omega^2 : (i,j,k) \in A\} \in \mathsf{Fin}^2) \wedge \forall i \geq i_0 \, (|\{(j,k) \in \omega^2 : (i,j,k) \in A\}| < \omega)].$$

The ideal BI is isomorphic to the ideal $\mathcal{WT}$ introduced in [3], where the authors showed its connection with weakly tight almost disjoint families.

(4) conv is the ideal on $\mathbb{Q}\cap[0,1]$ consisting of all subsets of $\mathbb{Q}\cap[0,1]$ which can be covered by ranges of finitely many sequences in $\mathbb{Q}\cap[0,1]$ which are convergent in $[0,1]$.

Definition 2.2 ([5])**.** Let Λ and Ω be countable infinite sets. Let $\mathcal{F}$ be a nonempty family of infinite subsets of Ω such that $F\setminus K\in\mathcal{F}$ for every $F\in\mathcal{F}$ and every finite set $K\subseteq\Omega$. We say that a function $\rho\colon\mathcal{F}\to[\Lambda]^\omega$ is *partition regular* if it satisfies the following conditions:

(M): $\forall E,F\in\mathcal{F}\ (E\subseteq F\implies\rho(E)\subseteq\rho(F))$;
(R): $\forall F\in\mathcal{F}\forall A,B\subseteq\Lambda\,(\rho(F)=A\cup B\implies\exists E\in\mathcal{F}\,(\rho(E)\subseteq A\vee\rho(E)\subseteq B))$;
(S): $\forall E\in\mathcal{F}\,\exists F\in\mathcal{F}\,(F\subseteq E\wedge\forall a\in\rho(F)\,\exists K\in[\Omega]^{<\omega}\,(a\notin\rho(F\setminus K)))$.

The following proposition reveals relationships between partition regular functions and ideals.

Proposition 2.3 ([5, Proposition 3.3])**.**

(1) If $\rho\colon\mathcal{F}\to[\Lambda]^\omega$ is partition regular then $\mathcal{I}_\rho=\{A\subseteq\Lambda:\forall F\in\mathcal{F}\,(\rho(F)\not\subseteq A)\}$ is an ideal on Λ.

(2) If $\mathcal{I}$ is an ideal on Λ then the function $\rho_\mathcal{I}\colon\mathcal{I}^+\to[\Lambda]^\omega$ given by $\rho_\mathcal{I}(A)=A$ is partition regular and $\mathcal{I}=\mathcal{I}_{\rho_\mathcal{I}}$.

The above proposition shows that every ideal can be coded using partition regular functions, namely, for an ideal $\mathcal{I}$ we have a partition regular function $\rho=\rho_\mathcal{I}$ with $\mathcal{I}=\mathcal{I}_\rho$. On the other hand, not every partition regular function can be coded using ideals. For instance, $\mathsf{FS}\neq\rho_\mathcal{I}$ for any ideal $\mathcal{I}$ ([5, Propositions 6.5 and 6.7]), where the partition regular function $\mathsf{FS}:[\omega]^\omega\to[\omega]^\omega$ is given by

$$\mathsf{FS}(D)=\{d_0+\cdots+d_n:n\in\omega,d_0,\dots,d_n\in D,d_i\neq d_j\text{for }i\neq j\},$$

and the ideal

$$\mathcal{H}=\mathcal{I}_\mathsf{FS}=\{A\subseteq\omega:\forall D\in[\omega]^\omega\,(\mathsf{FS}(D)\not\subseteq A)\}$$

is called *the Hindman ideal* [8, p. 109] (see also [11]).

A set $F\in\mathcal{F}$ has *small accretions* if $\rho(F)\setminus\rho(F\setminus K)\in\mathcal{I}_\rho$ for every finite set K. We say that ρ has *small accretions* if for every $E\in\mathcal{F}$, there is $F\in\mathcal{F}$ with $F\subseteq E$ such that F has small accretions.

2.2. **P-like properties.** In this section $\mathcal{I}$ is an ideal on ω, and $\rho\colon\mathcal{F}\to[\Lambda]^\omega$ with $\mathcal{F}\subseteq[\Omega]^\omega$ is a partition regular function.

We write:

- $\rho\in P^-(\Lambda)$ if for every decreasing sequence $\Lambda=A_0\supseteq A_1\supseteq\dots$ such that $A_n\setminus A_{n+1}\in\mathcal{I}_\rho$ for each n there exists $F\in\mathcal{F}$ such that for each $n\in\omega$ there exists a finite set K with $\rho(F\setminus K)\subseteq A_n$.
- $\rho\in P^-$ if for every decreasing sequence $A_0\supseteq A_1\supseteq\dots$ such that $A_0\in\mathcal{I}_\rho^+$ and $A_n\setminus A_{n+1}\in\mathcal{I}_\rho$ for each n there exists $F\in\mathcal{F}$ such that $\rho(F)\subseteq A_0$ and for each $n\in\omega$ there exists a finite set K with $\rho(F\setminus K)\subseteq A_n$.

If $\rho_\mathcal{I}\in P^-(\omega)$ ($\rho_\mathcal{I}\in P^-$) then we write $\mathcal{I}\in P^-(\omega)$ ($\mathcal{I}\in P^-$, respectively). Thus, equivalently, $\mathcal{I}\in P^-(\omega)$ if for every decreasing sequence $\omega=A_0\supseteq A_1\supseteq\dots$ such that $A_n\setminus A_{n+1}\in\mathcal{I}_\rho$ for each n there exists $B\in\mathcal{I}^+$ such that for each $n\in\omega$ there exists a finite set K with $B\setminus K\subseteq A_n$. Similarly one can express $\mathcal{I}\in P^-$.

It is not difficult to show the following characterizations of P^--like properties which we will sometimes used in the paper.

Proposition 2.4.

(1) $\mathcal{I} \in P^-(\omega) \iff$ *for every partition* $\{A_n : n \in \omega\}$ *of* ω *such that* $A_n \in \mathcal{I}$ *for each* n*, there exists* $B \in \mathcal{I}^+$ *such that* $B \cap A_n$ *is finite for each* $n \in \omega$*.*
(2) $\rho \in P^-(\Lambda) \iff$ *for every partition* $\{A_n : n \in \omega\}$ *of* Λ *such that* $A_n \in \mathcal{I}_\rho$ *for each* n*, there exists* $F \in \mathcal{F}$ *such that for each* $n \in \omega$ *there exists a finite set* K *with* $\rho(F \setminus K) \cap A_n = \emptyset$*.*
(3) $\mathcal{I} \in P^- \iff$ *for every family* $\{A_n : n \in \omega\}$ *of pairwise disjoint subsets of* ω *such that* $\bigcup\{A_n : n \in \omega\} \notin \mathcal{I}$ *and* $A_n \in \mathcal{I}$ *for each* n*, there exists* $B \subseteq \bigcup\{A_n : n \in \omega\}$ *such that* $B \in \mathcal{I}^+$ *and* $B \cap A_n$ *is finite for each* $n \in \omega$*.*
(4) $\rho \in P^- \iff$ *for every family* $\{A_n : n \in \omega\}$ *of pairwise disjoint subsets of* Λ *such that* $\bigcup\{A_n : n \in \omega\} \notin \mathcal{I}_\rho$ *and* $A_n \in \mathcal{I}_\rho$ *for each* n*, there exists* $F \in \mathcal{F}$ *such that for each* $n \in \omega$ *there exists a finite set* K *with* $\rho(F \setminus K) \cap A_n = \emptyset$*.*

2.3. **Katětov order.** Let $\mathcal{I}_1$ and $\mathcal{I}_2$ be ideals on X and Y, respectively. We say that $\mathcal{I}_1$ *is above* $\mathcal{I}_2$ *in the Katětov order* (in short: $\mathcal{I}_2 \leq_K \mathcal{I}_1$) if there exists a function $f\colon X \to Y$ such that $f[A] \in \mathcal{I}_2^+$ for each $A \in \mathcal{I}_1^+$.

Let $\rho_i\colon \mathcal{F}_i \to [\Lambda_i]^\omega$ be partition regular with $\mathcal{F}_i \subseteq [\Omega_i]^\omega$ for each $i = 1, 2$. We say that ρ_1 *is above* ρ_2 *in the Katětov order* (in short: $\rho_2 \leq_K \rho_1$) if there is a function $f\colon \Lambda_1 \to \Lambda_2$ such that

$$\forall F_1 \in \mathcal{F}_1\, \exists F_2 \in \mathcal{F}_2\, \forall K_1 \in [\Omega_1]^{<\omega}\, \exists K_2 \in [\Omega_2]^{<\omega}\, (\rho_2(F_2 \setminus K_2) \subseteq f[\rho_1(F_1 \setminus K_1)]).$$

We write $\rho_2 \approx_K \rho_1$ if both $\rho_2 \leq_K \rho_1$ and $\rho_1 \leq_K \rho_2$ hold.

2.4. **Stronger sparseness of partition regular functions.** In this section, we present a strengthening of the property (S) from Definition 2.2 which will be required in some results of this paper.

Definition 2.5. Let $\rho\colon \mathcal{F} \to [\Lambda]^\omega$ with $\mathcal{F} \subseteq [\Omega]^\omega$ be a partition regular function. We say that

(1) $\rho \in (S_1)$ if for every $E \in \mathcal{F}$ there exists $F \in \mathcal{F}$ with $F \subseteq E$ such that for every $A \in \mathcal{I}_\rho$ there exists $G \in \mathcal{F}$ with $\rho(G) \subseteq \rho(F) \setminus A$ such that
$$\forall K \in [\Omega]^{<\omega}\, \exists L \in [\Omega]^{<\omega}\, (\rho(G \setminus L) \subseteq \rho(F \setminus K));$$
(2) $\rho \in (S_2)$ if for every $E \in \mathcal{F}$ there exists $F \in \mathcal{F}$ with $F \subseteq E$ such that for every $B \notin \mathcal{I}_\rho$ with $B \subseteq \rho(F)$ there exists $G \in \mathcal{F}$ with $\rho(G) \subseteq B$ such that
$$\forall K \in [\Omega]^{<\omega}\, \exists L \in [\Omega]^{<\omega}\, (\rho(G \setminus L) \subseteq \rho(F \setminus K)).$$

Proposition 2.6. *Let* $\rho\colon \mathcal{F} \to [\Lambda]^\omega$ *with* $\mathcal{F} \subseteq [\Omega]^\omega$ *be a partition regular function.*

(1) *If* $\rho \in (S_2)$ *then* $\rho \in (S_1)$*.*
(2) $\rho_{\mathcal{I}} \in (S_2)$ *for every ideal* $\mathcal{I}$*.*
(3) *If* $\rho \in P^-$ *and has small accretions, then* $\rho \in (S_2)$*.*

Proof. (1) Take any $E \in \mathcal{F}$. Let $F \in \mathcal{F}$ be a witness for $\rho \in (S_2)$. If $A \in \mathcal{I}_\rho$ then $B = \rho(F) \setminus A \notin \mathcal{I}_\rho$, so the set G from the definition of (S_2) works for (S_1).

(2) Take any $E \in \mathcal{I}^+$, put $F = E$ and fix any $\mathcal{I}$-positive subset B of E. Then $G = B$ works, as $B \setminus K \in \mathcal{I}^+$ and $B \setminus K \subseteq E \setminus K$ for all finite sets K.

(3) Enumerate the set $\Omega = \{k_i : i \in \omega\}$ and define $K_n = \{k_i : i < n\}$. Take any $E \in \mathcal{F}$. Since ρ has small accretions, there exists some $F \in \mathcal{F}, F \subseteq E$ such that $\rho(F) \setminus \rho(F \setminus K_n) \in \mathcal{I}_\rho$ for every $n \in \omega$. Take $B \notin \mathcal{I}_\rho$ with $B \subseteq \rho(F)$ and define $A_n = \rho(F \setminus K_n) \cap B$ for each $n \in \omega$. Then, $(A_n)_{n\in\omega}$ is a decreasing sequence such that $A_n \setminus A_{n+1} \in \mathcal{I}_\rho$ and $A_0 \notin \mathcal{I}_\rho$. Since $\rho \in P^-$, there exists a set $G \in \mathcal{F}$ such that $\rho(G) \subseteq A_0$ and for every $n \in \omega$ there exists a finite set M_n with $\rho(G \setminus M_n) \subseteq A_n$. Then, $\rho(G) \subseteq B$ and for every $i \in \omega$, we get that $\rho(G \setminus M_i) \subseteq \rho(F \setminus K_i)$. □

2.5. **Some classes of topological spaces.** A sequentially compact space X is called *boring* ([14, Definition 3.21]) if there exists a finite set $F \subseteq X$ such that each one-to-one sequence in X converges to some point of F. In case of infinite Hausdorff spaces, X is boring if and only if it is a disjoint union of finitely many one-point compactifications of discrete spaces.

For an ideal $\mathcal{I}$ on ω, we write $\mathrm{FinBW}(\mathcal{I})$ to denote the class of all topological spaces X such that for every sequence $f\colon \omega \to X$ there exists $A \in \mathcal{I}^+$ such that $f \restriction A$ converges. We say that an ideal $\mathcal{I}$ has the FinBW *property* if $[0,1] \in \mathrm{FinBW}(\mathcal{I})$.

For a partition regular function $\rho\colon \mathcal{F} \to [\Lambda]^\omega$, we define $\mathrm{FinBW}(\rho)$ to be the class of all topological spaces X such that for every sequence $f\colon \Lambda \to X$ there exists $F \in \mathcal{F}$ such that $f \restriction \rho(F)$ *ρ-converges* i.e. there exists $x \in X$ such that for every neighborhood U of x there is a finite set K with $f[\rho(F \setminus K)] \subseteq U$.

3. Critical ρ for the class of all finite spaces

Theorem 3.1. *Let $\rho : \mathcal{F} \to [\Lambda]^\omega$ be partition regular with $\mathcal{F} \subseteq [\Omega]^\omega$. If $\rho \in (S_1)$, then the following conditions are equivalent:*

(1) $\mathrm{FinBW}(\rho)$ *coincides with the class of all finite spaces,*
(2) $\omega + 1 \notin \mathrm{FinBW}(\rho)$*, where the ordinal $\omega + 1$ is considered as a topological space with the order topology,*
(3) $\rho \notin P^-(\Lambda)$*,*
(4) $\rho_{\mathsf{Fin}^2} \leq_K \rho$*.*

The assumption $\rho \in (S_1)$ is only used to prove the implication (3) $\implies$ (4).

Proof. (1) $\iff$ (3) This is proved in [5, Theorem 10.5(3)].

(1) $\implies$ (2) This is obvious.

(2) $\implies$ (3) Let $f : \Lambda \to \omega + 1$ be a witness for $\omega + 1 \notin \mathrm{FinBW}(\rho)$. We define $A_\alpha = f^{-1}(\alpha)$ for each $\alpha \in \omega + 1$. Once, we show that the family $\{A_\alpha : \alpha \in \omega + 1\}$ is a witness for $\rho \notin P^-(\Lambda)$, the proof will be finished. Observe that $A_\alpha \in \mathcal{I}_\rho$ for all $\alpha \in \omega + 1$ as otherwise we could find $F \in \mathcal{F}$ and $\alpha \in \omega + 1$ with $\rho(F) \subseteq A_\alpha$ such that $f \restriction \rho(F)$ would be constant, so ρ-convergent. Suppose for the sake of contradiction that there is $F \in \mathcal{F}$ such that for each α there is a finite set K_α with $\rho(F \setminus K_\alpha) \cap A_\alpha = \emptyset$. We claim that $f \restriction \rho(F)$ is ρ-convergent to the point ω (and that will finish the proof). Let U be a neighborhood of the point ω. Then, there is $n \in \omega$ with $(n, \omega] \subseteq U$. Then, $K = \bigcup_{\alpha \leq n} K_\alpha$ is finite and $\rho(F \setminus K) \cap A_\alpha = \emptyset$ for each $\alpha \leq n$, so $f[\rho(F \setminus K)] \subseteq U$.

(3) $\implies$ (4) Since $\rho \notin P^-(\Lambda)$, there exists a partition $\{A_n : n \in \omega\}$ of Λ such that $A_n \in \mathcal{I}_\rho$ for each $n \in \omega$ and for every $F \in \mathcal{F}$ there exists $n \in \omega$ such that $\rho(F \setminus L) \cap A_n \neq \emptyset$ for any finite set L. Let $f : \Lambda \to \omega^2$ be an injective function such that $f[A_n] \subseteq \{n\} \times \omega$ for each n. Once we show that f is a witness for $\rho_{\mathsf{Fin}^2} \leq_K \rho$, the proof will be finished.

Take any $F \in \mathcal{F}$. To finish the proof, we need to find $B \notin \mathsf{Fin}^2$ such that for each finite K there is a finite L with $B \setminus L \subseteq f[\rho(F \setminus K)]$.

Since $\rho \in (S_1)$, there exists $E \in \mathcal{F}$ with $E \subseteq F$ such that for every $A \in \mathcal{I}_\rho$ there exists $H_A \in \mathcal{F}$ with $\rho(H_A) \subseteq \rho(E) \setminus A$ such that

$$\forall K \in [\Omega]^{<\omega}\ \exists L_{K,A} \in [\Omega]^{<\omega}\ (\rho(H_A \setminus L_{K,A}) \subseteq \rho(E \setminus K)).$$

Let $\Omega = \{k_i : i \in \omega\}$ and $K_n = \{k_i : i < n\}$ for each $n \in \omega$. We will construct $G_l \in \mathcal{F}$, $\{a^l_n : n \in \omega\} \subseteq \Lambda$ and $n_l \in \omega$ by the induction on $l \in \omega$ in such a way that the following conditions are satisfied:

(1) $G_l \subseteq H_{\bigcup_{i<l} A_{n_i}}$ for each l,
(2) $a^l_n \in \rho(G_l \setminus L_{K_{l+n}, \bigcup_{i<l} A_{n_i}}) \cap A_{n_l}$ for each l and n,
(3) $a^l_n \neq a^l_k$ for each l and distinct n, k,

(4) $n_l \neq n_k$ for distinct l, k.

Assume $G_j \in \mathcal{F}$, $\{a^j_n : n \in \omega\} \subseteq \Lambda$ and $n_j \in \omega$ have already been constructed for $j < l$. Since $H_{\bigcup_{i<l} A_{n_i}} \in \mathcal{F}$, we can find $G_l \in \mathcal{F}$ such that $G_l \subseteq H_{\bigcup_{i<l} A_{n_i}}$ and for every $a \in \rho(G_l)$ there is a finite set K with $a \notin \rho(G_l \setminus K)$. Let n_l be such that $\rho(G_l \setminus L) \cap A_{n_l} \neq \emptyset$ for every finite L. Note that $n_l \neq n_i$ for all $i < l$ as $\rho(G_l \setminus L) \subseteq \rho(G_l) \subseteq \rho(H_{\bigcup_{i<l} A_{n_i}}) \subseteq \rho(E) \setminus \bigcup_{i<l} A_{n_i}$. Now, we will inductively pick elements a^l_n for $n \in \omega$. Suppose we have already chosen a^l_i for $i < n$. For each $i < n$, we can find a finite set M^l_i such that $a^l_i \notin \rho(G_l \setminus (L_{K_{l+n}, \bigcup_{i<l} A_{n_i}} \cup M^l_i)$. Now we pick any

$$a^l_n \in \rho\left(G_l \setminus \left(L_{K_{l+n}, \bigcup_{i<l} A_{n_i}} \cup \bigcup_{i<n} M^l_i\right)\right) \cap A_{n_l}.$$

That ends the inductive construction.

Let $B = \{f(a^l_n) : l, n \in \omega\}$. Since f is injective and $f(a^l_n) \in \{n_l\} \times \omega$ for each $l, n \in \omega$ (by $a^l_n \in A_{n_l}$), we obtain that $B \cap (\{n_l\} \times \omega)$ is infinite for each $l \in \omega$. Thus, $B \notin \mathsf{Fin}^2$. Let's take any finite set K. Let j_0 be such that $K \subseteq K_{j_0}$. Let $L = \{f(a^j_n) : n, j < j_0\}$. The set L is finite, so the proof will be finished once we show that $B \setminus L \subseteq f[\rho(F \setminus K)]$. Take any $f(a^l_n) \in B \setminus L$. Then, $l \geq j_0$ or $n \geq j_0$, so $l + n \geq j_0$, and consequently we have

$$\begin{aligned} a^l_n &\in \rho(G_l \setminus L_{K_{l+n}, \bigcup_{i<l} A_{n_i}}) \subseteq \rho(H_{\bigcup_{i<l} A_{n_i}} \setminus L_{K_{l+n}, \bigcup_{i<l} A_{n_i}}) \\ &\subseteq \rho(E \setminus K_{l+n}) \subseteq \rho(E \setminus K_{j_0}) \subseteq \rho(F \setminus K), \end{aligned}$$

so $f(a^l_n) \in f[\rho(F \setminus K)]$.

(4) $\implies$ (1) It is easy to see that each finite space is $\mathrm{FinBW}(\rho)$ for every partition regular function ρ. If $\rho_{\mathsf{Fin}^2} \leq_K \rho$, then $\mathrm{FinBW}(\rho) \subseteq \mathrm{FinBW}(\rho_{\mathsf{Fin}^2}) = \mathrm{FinBW}(\mathsf{Fin}^2)$ by [5, Theorem 11.1 and Proposition 10.2], and $\mathrm{FinBW}(\mathsf{Fin}^2)$ coincides with the class of all finite spaces by [14, Theorem 6.5]. □

Corollary 3.2 ([14, Theorem 6.5 and Proposition 2.5])**.** *Let $\mathcal{I}$ be an ideal on ω. The following conditions are equivalent.*

(1) $\mathrm{FinBW}(\mathcal{I})$ *coincides with the class of all finite spaces.*
(2) $\mathcal{I} \notin P^-(\omega)$.
(3) $\mathsf{Fin}^2 \leq_K \mathcal{I}$.

Proof. Take $\rho = \rho_{\mathcal{I}}$ and apply Theorem 3.1 and [5, Propositions 6.5(2), 7.5(2b) and 10.2(4)]. □

Question 3.3. Is there a partition regular function η such that the following conditions are equivalent

(1) $\mathrm{FinBW}(\rho)$ coincides with the class of all finite spaces,
(2) $\eta \leq_K \rho$

for every partition regular function ρ?

4. Critical ρ for the class of all boring spaces

First, we introduce a strengthening of the property $P^-(\Lambda)$ which will be used to characterize the class of all boring spaces.

Definition 4.1. Let $\rho : \mathcal{F} \to [\Lambda]^\omega$ be partition regular with $\mathcal{F} \subseteq [\Omega]^\omega$. We say that

(1) $\rho \in P^-_1(\Lambda)$ if $\rho \in P^-(\Lambda)$;
(2) $\rho \in P^-_2(\Lambda)$ if for every partition $\{A_{i,j} : i, j \in \omega\}$ of Λ such that $A_{i,j} \in \mathcal{I}_\rho$ for each $i, j \in \omega$ there exists $F \in \mathcal{F}$ such that
   (a) $\forall i, j\, \exists K \in [\Omega]^{<\omega}\, (\rho(F \setminus K) \cap A_{i,j} = \emptyset)$,
   (b) $\exists i_0\, \forall i > i_0\, \exists K \in [\Omega]^{<\omega}\, (\rho(F \setminus K) \cap \bigcup_{j<\omega} A_{i,j} = \emptyset)$.

It is not difficult to notice that $\rho \in P_2^-(\Lambda)$ implies $\rho \in P_1^-(\Lambda)$.

Theorem 4.2. *Let $\rho : \mathcal{F} \to [\Lambda]^\omega$ be partition regular with $\mathcal{F} \subseteq [\Omega]^\omega$.*

(1) If $\rho \in (S_1)$ then

$$\rho_{\mathsf{BI}} \leq_K \rho \iff \rho \notin P_2^-(\Lambda).$$

The assumption $\rho \in (S_1)$ is only used to prove the implication "$\Longleftarrow$".

(2) If $\rho \in (S_2)$, then

$$\rho \in P_2^-(\Lambda) \iff \omega^2 + 1 \in \mathrm{FinBW}(\rho),$$

where the ordinal $\omega^2 + 1$ is considered as a topological space with the order topology. The assumption $\rho \in (S_2)$ is only used to prove the implication "$\Longrightarrow$".

Proof. $(1, \implies)$ Let $f : \Lambda \to \omega^3$ be a witness for $\rho_{BI} \leq_K \rho$. We define $A_{i,j} = f^{-1}[\{i\} \times \{j\} \times \omega]$ for each $i, j \in \omega$. Once we show that $\{A_{i,j} : i, j \in \omega\}$ is a witness for $\rho \notin P_2^-(\Lambda)$, the proof will be finished.

First, we observe that $\{A_{i,j} : i, j \in \omega\}$ is a partition of Λ and $A_{i,j} \in \mathcal{I}_\rho$ because the function f is a witness for $\rho_{BI} \leq_K \rho$. Now, we take any $F \in \mathcal{F}$. Since f is a witness for $\rho_{BI} \leq_K \rho$, there exists $G \notin \mathsf{BI}$ such that for every finite K there is a finite L with $G \setminus L \subseteq f[\rho(F \setminus K)]$. Since $G \notin \mathsf{BI}$, we have two cases.

Case (1). $\{(j,k) : (i_0, j, k) \in G\} \notin \mathsf{Fin}^2$ for some $i_0 \in \omega$.

Then there exists j_0 such that $D = \{k : (i_0, j_0, k) \in G\} \notin \mathsf{Fin}$. Once we show that $\rho(F \setminus K) \cap A_{i_0,j_0} \neq \emptyset$ for any finite K, this case will be finished. Take any finite K. Then, we can find a finite L with $G \setminus L \subseteq f[\rho(F \setminus K)]$. Since D is infinite, we can find k_0 such that $(i_0, j_0, k_0) \in G \setminus L$. Then, $(i_0, j_0, k_0) \in f[\rho(F \setminus K)]$, so $A_{i_0,j_0} \cap \rho(F \setminus K) \supseteq f^{-1}[\{(i_0, j_0, k_0)\}] \cap \rho(F \setminus K) \neq \emptyset$.

Case (2). $|\{(j,k) : (i,j,k) \in G\}| = \omega$ for infinitely many $i \in \omega$.

Once we show that for every i there is $i_0 > i$ such that $\rho(F \setminus K) \cap \bigcup_{j<\omega} A_{i_0,j} \neq \emptyset$ for every finite K, this case will be finished. Take any $i \in \omega$. Then, there is $i_0 > i$ such that $D = \{(j,k) : (i_0, j, k) \in G\}$ is infinite. Take any finite K. Then, we can find a finite L with $G \setminus L \subseteq f[\rho(F \setminus K)]$. Since $\{i_0\} \times D$ is infinite, we can find $(j_0, k_0) \in D$ with $(i_0, j_0, k_0) \in G \setminus L$. Then, $(i_0, j_0, k_0) \in f[\rho(F \setminus K)]$, so $\bigcup_{j<\omega} A_{i_0,j} \cap \rho(F \setminus K) \supseteq f^{-1}[\{(i_0, j_0, k_0)\}] \cap \rho(F \setminus K) \neq \emptyset$.

$(1, \impliedby)$ Let $\{A_{i,j} : i, j \in \omega\}$ be a witness for $\rho \notin P_2^-(\Lambda)$. Let $f : \Lambda \to \omega^3$ be an injective function such that $f[A_{i,j}] \subseteq \{(i,j)\} \times \omega$. Once we show that f is a witness for $\rho_{\mathsf{BI}} \leq_K \rho$, the proof will be finished. Take any $F \in \mathcal{F}$. We will find $G \notin \mathsf{BI}$ such that for every finite set K there exists a finite set L with $G \setminus L \subseteq f[\rho(F \setminus K)]$. Let $E \in \mathcal{F}$ be such that $E \subseteq F$ and for every $a \in \rho(E)$ there exists a finite set K_a with $a \notin \rho(E \setminus K_a)$. Since $\{A_{i,j} : i, j \in \omega\}$ is a witness for $\rho \notin P_2^-(\Lambda)$, we have the following cases.

Case (1). For every $i \in \omega$ there is $i_0 > i$ such that $\rho(E \setminus K) \cap \bigcup_{j<\omega} A_{i_0,j} \neq \emptyset$ for every finite K.

In this case, we can find a strictly increasing sequence (i_n) such that for every $n \in \omega$ and every finite K we have

$$\rho(E \setminus K) \cap \bigcup_{j<\omega} A_{i_n,j} \neq \emptyset.$$

Let $g : \omega \to \omega$ be any function such that $g^{-1}(\{n\})$ is infinite for each $n \in \omega$. Let $\Omega = \{m_i : i \in \omega\}$ and $M_n = \{m_i : i < n\}$ for each $n \in \omega$. Now, for each $n \in \omega$ we pick any

$$a_n \in \rho\left(E \setminus \left(\bigcup_{k<n} K_{a_k} \cup M_n\right)\right) \cap \bigcup_{j<\omega} A_{i_{g(n)},j}.$$

Let $G = \{f(a_n) : n \in \omega\}$. Since a_n are pairwise distinct and f is injective, we obtain that $G \cap (\{i_n\} \times \omega \times \omega)$ is infinite for each $n \in \omega$. Consequently, $G \notin \mathsf{BI}$. Take any finite set K and let i be such that $K \subseteq M_i$. Let $L = \{f(a_n) : n < i\}$. Then, L is finite and for every $f(a_n) \in G \setminus L$ we have

$$f(a_n) \in f[\rho(E \setminus M_i)] \subseteq f[\rho(F \setminus K)],$$

so $G \setminus L \subseteq f[\rho(F \setminus K)]$.

Case (2). Case (1) does not hold, and consequently there exist i, j such that $\rho(E \setminus K) \cap A_{i,j} \neq \emptyset$ for every finite set K.

Claim. There exists an infinite set $\{(i_n, j_n) : n \in \omega\}$ of pairs of integers such that $\rho(E \setminus K) \cap A_{i_n,j_n} \neq \emptyset$ for every n and every finite set K.

Proof of claim. Since $\rho \in (S_1)$, there exists $H \in \mathcal{F}$ with $H \subseteq E$ such that for every $A \in \mathcal{I}_\rho$ there exists $P \in \mathcal{F}$ with $\rho(P) \subseteq \rho(H) \setminus A$ such that for every finite set K there exists a finite set L with $\rho(P \setminus L) \subseteq \rho(H \setminus K)$. Since $H \subseteq E$, it is enough to show the claim for the set H instead of E.

Now, we construct $i_n, j_n \in \omega$ and $P_n \in \mathcal{F}$ such that for every n

(i) $\rho(H \setminus K) \cap A_{i_0,j_0} \neq \emptyset$ for every finite set K,
(ii) $\rho(P_n) \subseteq \rho(H) \setminus \bigcup\{A_{i_l,j_l} : l \leq n\}$,
(iii) for every finite set K there exists a finite set L with $\rho(P_n \setminus L) \subseteq \rho(H \setminus K)$,
(iv) $\rho(P_n \setminus K) \cap A_{i_{n+1},j_{n+1}} \neq \emptyset$ for every finite set K.

Since $H \subseteq E$ and Case (1) does not hold for E, we find i_0, j_0 such that $\rho(H \setminus K) \cap A_{i_0,j_0} \neq \emptyset$ for every finite set K. Since $A_{i_0,j_0} \in \mathcal{I}_\rho$, there is $P_0 \in \mathcal{F}$ which satisfies items (ii) and (iii).

Now, suppose that i_l, j_l and P_l have been constructed for $l \leq n$. Since $H \subseteq E$ and Case (1) does not hold for E, there is k_0 such that for every $k > k_0$ there is a finite set K with $\rho(H \setminus K) \cap \bigcup\{A_{k,j} : j \in \omega\} = \emptyset$, so, by item (iii), there is a finite set L with $\rho(P_n \setminus L) \cap \bigcup\{A_{k,j} : j \in \omega\} = \emptyset$. Since $\{A_{i,j} : i, j \in \omega\}$ is a witness for $\rho \notin P_2^-(\Lambda)$, there are i_{n+1}, j_{n+1} such that $\rho(P_n \setminus K) \cap A_{i_{n+1},j_{n+1}} \neq \emptyset$ for every finite set K. Since $\bigcup\{A_{i_l,j_l} : l \leq n\} \in \mathcal{I}_\rho$, there is $P_{n+1} \in \mathcal{F}$ which satisfies items (ii) and (iii). This finishes the construction of i_n, j_n and P_n.

Now, we notice that $\{(i_n, j_n) : n \in \omega\}$ is infinite by items (ii) and (iv). To see that the sequence (i_n, j_n) is as required in the claim, suppose for the sake of contradiction that there is n and a finite set K with $\rho(H \setminus K) \cap A_{i_n,j_n} = \emptyset$. By item (i), we know that $n \geq 1$, By item (iii), there is a finite set L with $\rho(P_{n-1} \setminus L) \cap A_{i_n,j_n} = \emptyset$ which is a contradiction with item (iv). □

Now, the proof proceeds similarly to the proof of the previous case. However, for the sake of completeness, we present it here with minor modifications. Let $g : \omega \to \omega$ be any function such that $g^{-1}(\{n\})$ is infinite for each $n \in \omega$. Let $\Omega = \{m_i : i \in \omega\}$ and $M_n = \{m_i : i < n\}$ for each $n \in \omega$. Now, for each $n \in \omega$ we pick any

$$a_n \in \rho\left(E \setminus \left(\bigcup_{k<n} K_{a_k} \cup M_n\right)\right) \cap A_{i_{g(n)},j_{g(n)}}.$$

Let $G = \{f(a_n) : n \in \omega\}$. Since a_n are pairwise distinct and f is injective, we obtain that $G \cap (\{(i_n, j_n)\} \times \omega)$ is infinite for each $n \in \omega$. We have two cases.

Case (a). If $\{i_n : n \in \omega\}$ is finite, then there is i such that $C = \{n : i_n = i\}$ is infinite. Then, $\{j_n : n \in C\}$ is infinite, so $G \cap (\{(i, j_n)\} \times \omega)$ is infinite for every $n \in C$. Consequently, $G \notin \mathsf{BI}$.

Case (b). If $\{i_n : n \in \omega\}$ is infinite, then $G \cap (\{i_n\} \times \omega \times \omega)$ is infinite for each $n \in \omega$. Consequently, $G \notin \mathsf{BI}$.

Take any finite set K and let i be such that $K \subseteq M_i$. Let $L = \{f(a_n) : n < i\}$. Then, L is finite and for every $f(a_n) \in G \setminus L$ we have

$$f(a_n) \in f[\rho(E \setminus M_i)] \subseteq f[\rho(F \setminus K)],$$

so $G \setminus L \subseteq f[\rho(F \setminus K)]$.

$(2, \Longrightarrow)$ Let $f : \Lambda \to \omega^2 + 1$. We need to find some $F \in \mathcal{F}$ such that $f \restriction \rho(F)$ is ρ-convergent. If there exists some $\alpha \in \omega^2 + 1$ with $f^{-1}(\alpha) \notin \mathcal{I}_\rho$, then there exists $F \in \mathcal{F}$ such that $\rho(F) \subseteq f^{-1}(\alpha)$, so $f \restriction \rho(F)$ is constant therefore ρ-convergent. Now, we assume that $f^{-1}(\alpha) \in \mathcal{I}_\rho$ for each $\alpha \in \omega^2 + 1$. Observe that if an ordinal $\alpha \in \omega^2$ then there exists some $i, k \in \omega$ for which $\alpha = \omega \cdot i + k$. We define a partition $(A_{i,k})_{i,k\in\omega}$ of Λ by $A_{0,0} = f^{-1}(0) \cup f^{-1}(\omega^2)$ and $A_{i,k} = f^{-1}(\omega \cdot i + k)$ if either $k \neq 0$ or $l \neq 0$. Then, $A_{i,k} \in \mathcal{I}_\rho$ for each i, k. Since $\rho \in P_2^-(\Lambda)$, there exists $E \in \mathcal{F}$ such that

(1) $\forall i, k\, \exists K \in [\Omega]^{<\omega}\, (\rho(E \setminus K) \cap A_{i,k} = \emptyset)$,
(2) $\exists i_0\, \forall i > i_0\, \exists K \in [\Omega]^{<\omega}\, (\rho(E \setminus K) \cap \bigcup_{k<\omega} A_{i,k} = \emptyset)$.

Since $\rho \in (S_2)$, there exists $F \in \mathcal{F}, F \subseteq E$ such that for every $B \notin \mathcal{I}_\rho$ with $B \subseteq \rho(F)$ there exists $G \in \mathcal{F}$ with $\rho(G) \subseteq B$ such that

$$\forall K \in [\Omega]^{<\omega}\, \exists L \in [\Omega]^{<\omega}\, (\rho(G \setminus L) \subseteq \rho(F \setminus K)).$$

Case (1). $B = \rho(F) \cap \bigcup_{k<\omega} A_{i_0,k} \notin \mathcal{I}_\rho$ for some $i_0 \in \omega$.

Let $G \in \mathcal{F}$ be such that $\rho(G) \subseteq B$ and

$$\forall K \in [\Omega]^{<\omega}\, \exists L \in [\Omega]^{<\omega}\, (\rho(G \setminus L) \subseteq \rho(F \setminus K)).$$

We will show that $f \restriction \rho(G)$ is ρ-convergent to $\omega \cdot (i_0 + 1)$. Let U be a neighborhood of $\omega \cdot (i_0 + 1)$. Then, there exists k_0 such that $[\omega \cdot i_0 + k_0, \omega \cdot (i_0 + 1)] \subseteq U$. For each $k < k_0$, we take a finite set K_k such that $\rho(E \setminus K_k) \cap A_{i_0,k} = \emptyset$. Since $K = \bigcup_{k<k_0} K_k$ is finite, there exists a finite set L such that $\rho(G \setminus L) \subseteq \rho(F \setminus K)$. Then,

$$\rho(G \setminus L) \cap \bigcup_{k<k_0} A_{i_0,k} \subseteq \rho(F \setminus K) \cap \bigcup_{k<k_0} A_{i_0,k} \subseteq \rho(E \setminus K) \cap \bigcup_{k<k_0} A_{i_0,k} = \emptyset,$$

so $f[\rho(G \setminus L)] \subseteq [\omega \cdot i_0 + k_0, \omega \cdot (i_0 + 1)] \subseteq U$.

Case (2). $\rho(F) \cap \bigcup_{k<\omega} A_{i,k} \in \mathcal{I}_\rho$ for every $i \in \omega$.

For every $i > i_0$ there is a finite set K_i such that $\rho(E \setminus K_i) \cap \bigcup_{j<\omega} A_{i,j} = \emptyset$.

Let

$$B = \bigcup_{i>i_0} \left(\rho(F) \cap \bigcup_{k<\omega} A_{i,k} \right).$$

Since $B \notin \mathcal{I}_\rho$ (as $B = \rho(F) \setminus \bigcup_{i\leq i_0} (\rho(F) \cap \bigcup_{k<\omega} A_{i,k})$) and $B \subseteq \rho(F)$, there is $G \in \mathcal{F}$ such that $\rho(G) \subseteq B$ and

$$\forall K \in [\Omega]^{<\omega}\, \exists L \in [\Omega]^{<\omega}\, (\rho(G \setminus L) \subseteq \rho(F \setminus K)).$$

We will show that $f \restriction \rho(G)$ is ρ-convergent to ω^2. Let U be the neighborhood of ω^2. Then, there exists j_0 such that $[\omega \cdot j_0, \omega^2] \subseteq U$. If $j_0 < i_0$, then $\rho(G) \subseteq \bigcup_{i>i_0} \bigcup_{k<\omega} A_{i,k}$, so $f[\rho(G)] \subseteq [\omega \cdot i_0, \omega^2] \subseteq [\omega \cdot j_0, \omega^2] \subseteq U$. Now, assume that $j_0 \geq i_0$. Since $K = \bigcup_{i_0<i<j_0} K_i$ is finite, there exists a finite set L such that $\rho(G \setminus L) \subseteq \rho(F \setminus K)$. Then,

$$\begin{aligned} f[\rho(G \setminus L)] &\subseteq f[\rho(F \setminus K)] \subseteq f[\rho(E \setminus K)] \\ &\subseteq f\left[\rho(E \setminus K) \cap \bigcup_{i_0<i<j_0} \bigcup_{k<\omega} A_{i,k}\right] \cup f\left[\rho(E \setminus K) \cap \bigcup_{i\geq j_0} \bigcup_{k<\omega} A_{i,k}\right] \\ &= \emptyset \cup f\left[\rho(E \setminus K) \cap \bigcup_{i\geq j_0} \bigcup_{k<\omega} A_{i,k}\right] \subseteq [\omega \cdot j_0, \omega^2] \subseteq U. \end{aligned}$$

(2, $\Longleftarrow$) Take any partition $\{A_{i,j} : i,j \in \omega\}$ of Λ such that $A_{i,j} \in \mathcal{I}_\rho$ for each i,j. Let $f : \Lambda \to \omega^2+1$ be a function such that $f[A_{i,j}] \subseteq \{\omega \cdot i + j\}$ for every i,j. Since $\omega^2+1 \in \mathrm{FinBW}(\rho)$, there exists $F \in \mathcal{F}$ such that $f \restriction \rho(F)$ is ρ-convergent to $p \in \omega^2+1$. We have three cases.

Case (1). $p = \omega^2$.

Take any $i \in \omega$. Then, $U = (\omega \cdot (i+1), \omega^2]$ is an open neighborhood of p, so there is a finite K with $f[\rho(F \setminus K)] \subseteq U$. Thus, $\rho(F \setminus K) \cap \bigcup_{j<\omega} A_{i,j} = \emptyset$.

Case (2). $p = \omega \cdot i_0$ for some $i_0 \in \omega \setminus \{0\}$.

Take any $i \neq i_0 - 1$ and $j \in \omega$. Then, $U = (\omega \cdot (i_0-1) + j, \omega \cdot i_0]$ is an open neighborhood of p, so there is a finite K with $f[\rho(F \setminus K)] \subseteq U$. Thus, $\rho(F \setminus K) \cap \bigcup_{j<\omega} A_{i,j} = \emptyset$ and $\rho(F \setminus K) \cap A_{i_0-1,j} = \emptyset$.

Case (3). $p = 0$ or $p = \omega \cdot i_0 + j_0$ for some $i_0 \in \omega$ and $j_0 \in \omega \setminus \{0\}$.

In this case, $U = \{p\}$ is an open neighborhood of p, so there is a finite K with $f[\rho(F \setminus K)] \subseteq U$. If $p = 0$ we obtain $\rho(F \setminus K) \subseteq A_{0,0}$, so $A_{0,0} \notin \mathcal{I}_\rho$, a contradiction. Similarly, if $p = \omega \cdot i_0 + j_0$ for some $i_0 \in \omega$ and $j_0 \in \omega \setminus \{0\}$, we obtain $\rho(F \setminus K) \subseteq A_{i_0,j_0}$, so $A_{i_0,j_0} \notin \mathcal{I}_\rho$, again a contradiction. □

Recall that a sequentially compact space X is called *boring* ([14, Definition 3.21]) if there exists a finite set $F \subseteq X$ such that each one-to-one sequence in X converges to some point of F.

Theorem 4.3. *Let $\rho : \mathcal{F} \to [\Lambda]^\omega$ be partition regular. If $\rho \in (S_2)$, then the following conditions are equivalent.*

(1) $\rho_{\mathsf{BI}} \leq_K \rho$ *and* $\rho_{\mathsf{Fin}^2} \not\leq_K \rho$.
(2) $\rho \in P_1^-(\Lambda)$ *and* $\rho \notin P_2^-(\Lambda)$.
(3) $\mathrm{FinBW}(\rho)$ *coincides with the class of all boring spaces.*

Proof. (1) $\Longleftrightarrow$ (2) It follows from Theorems 3.1 and 4.2.

(2) $\Longrightarrow$ (3) First, we show that no unboring space is in $\mathrm{FinBW}(\rho)$. Let X be an unboring space. Then, using [14, Proposition 3.2], we can pick distinct points $x_i \in X$ for $i \in \Lambda$ and injective sequences $(x_{i,j})_{j\in\Lambda}$ that are convergent to x_i, respectively. Since $\rho \notin P_2^-(\Lambda)$, there exists a partition $(A_{i,j})_{i,j\in\omega}$ of Λ witnessing the lack of $P_2^-(\Lambda)$. Define a function $f : \Lambda \to \omega^2$ by $f(\lambda) = (i,j)$ for each $\lambda \in A_{i,j}$ and a function $g : \Lambda \to X$ by $g(\lambda) = x_{f(\lambda)}$. Suppose for the sake of contradiction that there exists some $F \in \mathcal{F}$ such that $g \restriction \rho(F)$ is ρ-convergent to some $p \in X$. Since $\rho \notin P_2^-(\Lambda)$, we have two cases.

Case (1). For every l there is some $i_l \geq l$ such that for all finite K,

$$\rho(F \setminus K) \cap \bigcup_{j\in\omega} A_{i_l,j} \neq \emptyset$$

(in particular, $f[\rho(F \setminus K)] \cap (\{i_l\} \times \omega) \neq \emptyset$ as $\bigcup_{j\in\omega} A_{i_l,j} = f^{-1}[\{i_l\} \times \omega]$). Then we can pick l_1, l_2 such that $i_{l_1} \neq i_{l_2}$. Hence, $x_{i_{l_1}} \neq x_{i_{l_2}}$, so one of them has to be distinct from p. Without loss of generality let us assume $x_{i_{l_1}} \neq p$. Since X is Hausdorff, we can take neighborhoods U of $x_{i_{l_1}}$ and V of p that are disjoint and assume without loss of generality that $V \cap \{x_{i_{l_1},j} : j \in \omega\} = \emptyset$. From ρ-convergence of $g \restriction \rho(F)$ to p there is some finite K such that $g[\rho(F \setminus K)] \subseteq V$. However, $f[\rho(F \setminus K)] \cap (\{i_{l_1}\} \times \omega)) \neq \emptyset$, so there is some $\lambda_0 \in \rho(F \setminus K)$ with $f(\lambda_0) \in \{i_{l_1}\} \times \omega$. Let $j \in \omega$ be such that $f(\lambda_0) = (i_{l_1}, j)$. Then, $g(\lambda_0) = x_{f(\lambda_0)} = x_{i_{l_1},j} \notin V$, but on the other hand, $g(\lambda_0) \in g[\rho(F \setminus K)] \subseteq V$, a contradiction.

Case (2). Case (1) does not hold, and consequently there are i_0, j_0 such that $\rho(F \setminus K) \cap A_{i_0,j_0} \neq \emptyset$ for every finite set K.

Claim. There are i_1, j_1 such that $(i_0,j_0) \neq (i_1,j_1)$ and $\rho(F \setminus K) \cap A_{i_1,j_1} \neq \emptyset$ for every finite set K.

Proof of claim. Since $\rho \in (S_1)$, there exist $H \in \mathcal{F}$ with $H \subseteq F$ and $P \in \mathcal{F}$ with $\rho(P) \subseteq \rho(H) \setminus A_{i_0,j_0}$ such that for every finite set K there exists a finite set L with $\rho(P \setminus L) \subseteq \rho(H \setminus K)$. Since $H \subseteq F$ and Case (1) does not hold, there is k_0 such that for every $k > k_0$ there is a finite set K with $\rho(H \setminus K) \cap \bigcup\{A_{k,j} : j \in \omega\} = \emptyset$, so there is a finite set L with $\rho(P \setminus L) \cap \bigcup\{A_{k,j} : j \in \omega\} = \emptyset$. Since $\{A_{i,j} : i, j \in \omega\}$ is a witness for $\rho \notin P_2^-(\Lambda)$, there are i_1, j_1 such that $\rho(P \setminus K) \cap A_{i_1,j_1} \neq \emptyset$ for every finite set K. Since $\rho(P) \subseteq \rho(H) \setminus A_{i_0,j_0}$, we obtain $(i_0, j_0) \neq (i_1, j_1)$. Moreover, $\rho(F \setminus K) \cap A_{i_1,j_1} \neq \emptyset$ for every finite set K. Indeed, suppose there is a finite set K with $\rho(F \setminus K) \cap A_{i_1,j_1} = \emptyset$. Then there exists a finite set L with $\rho(P \setminus L) \subseteq \rho(H \setminus K) \subseteq \rho(F \setminus K)$, so $\rho(P \setminus K) \cap A_{i_1,j_1} = \emptyset$, a contradiction. □

Now, the proof proceeds similarly to the proof of the previous case. However, for the sake of completeness, we present it here with minor modifications. Since $x_{i_0,j_0} \neq x_{i_1,j_1}$, one of them has to be distinct from p. Without loss of generality let us assume $x_{i_0,j_0} \neq p$. Take a neighborhoods U of p with $x_{i_0,j_0} \notin U$. From ρ-convergence of $g \restriction \rho(F)$ to p there is some finite K such that $g[\rho(F \setminus K)] \subseteq U$. However, $\rho(F \setminus K) \cap A_{i_0,j_0} \neq \emptyset$, so there is some $\lambda_0 \in \rho(F \setminus K)$ with $f(\lambda_0) = (i_0, j_0)$. Then, $g(\lambda_0) = x_{f(\lambda_0)} = x_{i_0,j_0} \notin U$, but on the other hand, $g(\lambda_0) \in g[\rho(F \setminus K)] \subseteq U$, a contradiction.

Now, we show that every boring space is in $\mathrm{FinBW}(\rho)$ (in the proof we will only use the property $P_1^-(\Lambda)$). Let X be a boring space. By [14, Proposition 3.2], there are $X_i \subseteq X$ for $i \leq n$ such that $X = \bigcup_{i \leq n} X_i$, $X_i \cap X_j = \emptyset$ whenever $i \neq j$, and for each i there is an $x_i \in X$ which is a limit point of each injective convergent sequence in X_i. Take any $f : \Lambda \to X$.

If there exists some $x \in X$ such that $f^{-1}[\{x\}] \notin \mathcal{I}_\rho$, there exists some $F \in \mathcal{F}$ with $\rho(F) \subseteq f^{-1}[\{x\}]$, so $f \restriction \rho(F)$ is ρ-convergent to x.

Now, assume $f^{-1}[\{x\}] \in \mathcal{I}_\rho$ for every $x \in X$. Since ρ is partition regular, there is $i_0 \leq n$ such that $f^{-1}[X_{i_0}] \notin \mathcal{I}_\rho$. Then, $\{f^{-1}[\{x\}] : x \in X_{i_0}\}$ is a partition of $f^{-1}[X_{i_0}]$ into sets belonging to $\mathcal{I}_\rho$. Since $\rho \in P_1^-(\Lambda)$, there is $E \in \mathcal{F}$ such that $\rho(E) \subseteq f^{-1}[X_{i_0}]$ and for each $x \in X_{i_0}$ there is some finite K_x with $\rho(E \setminus K_x) \cap f^{-1}[\{x\}] = \emptyset$.

We claim that $f \restriction \rho(E)$ is ρ-convergent to x_{i_0}. Let U be a neighborhood of x_{i_0}. Then $X_{i_0} \setminus U$ has to be finite (otherwise we would obtain an injective sequence in X_{i_0} not convergent to x_{i_0}). Then for $K = \bigcup_{x \in X_{i_0} \setminus U} K_x$ we have $\rho(E \setminus K) \cap f^{-1}[X_{i_0} \setminus U] = \emptyset$, so $f[\rho(E \setminus K)] \subseteq U$.

(3) $\implies$ (2) Suppose for the sake of contradiction that $\rho \notin P_1^-(\Lambda)$ or $\rho \in P_2^-(\Lambda)$. If $\rho \notin P_1^-(\Lambda)$ then by Theorem 3.1, $\mathrm{FinBW}(\rho)$ contains only finite spaces, a contradiction. If $\rho \in P_2^-(\Lambda)$, then by Theorem 4.2, $\omega^2 + 1 \in \mathrm{FinBW}(\rho)$. But $\omega^2 + 1$ is not a boring space, a contradiction. □

Corollary 4.4 ([14, Theorem 6.5 and Proposition 4.4])**.** *Let $\mathcal{I}$ be an ideal on ω. The following conditions are equivalent.*

(1) $\mathrm{FinBW}(\mathcal{I})$ *coincides with the class of all boring spaces.*
(2) $\mathsf{BI} \leq_K \mathcal{I}$ *and* $\mathsf{Fin}^2 \not\leq_K \mathcal{I}$.

Proof. Take $\rho = \rho_{\mathcal{I}}$ and apply Theorem 4.3 and [5, Propositions 6.5(2), 7.5(2b) and 10.2(4)]. □

Question 4.5. Are there partition regular functions η_1 and η_2 such that the following conditions are equivalent

(1) $\mathrm{FinBW}(\rho)$ coincides with the class of all boring spaces.
(2) $\eta_2 \leq_K \rho$ and $\eta_1 \not\leq_K \rho$

for every partition regular function ρ?

5. Brand-new class of partition regular functions

If for every partition regular function ρ there would be an ideal $\mathcal{J}$ with $\rho \approx_K \rho_{\mathcal{J}}$, then one could deduce Theorem 3.1 (Theorem 4.3, resp.) from Corollary 3.2 (Corollary 4.4, resp.). However, as was mentioned in the preliminaries, the partition regular function FS satisfies $\mathsf{FS} \neq \rho_{\mathcal{J}}$ for any ideal $\mathcal{J}$, and from the proof it follows that "$\neq$" can be replaced by "$\not\approx_K$". Below, we provide a collection of additional examples of partition regular functions of that sort (Proposition 5.4).

Definition 5.1. Let $\mathcal{I}$ be an ideal on ω, $\mathcal{A} = \{A_\alpha : \alpha < \mathfrak{c}\}$ be an almost disjoint family on ω, $\overline{\mathcal{A}} = \{A \setminus K : A \in \mathcal{A}, K \in [\omega]^{<\omega}\}$, and $\mathcal{I}^+ = \{B_\alpha : \alpha < \mathfrak{c}\}$. Let $\{P_n : n \in \omega\}$ be a partition of ω such that $P_n \in \mathcal{I}$ for each $n \in \omega$. We define a function $\rho^{(\mathcal{I})} : \overline{\mathcal{A}} \to [\omega]^\omega$ by

$$\rho^{(\mathcal{I})}(A_\alpha \setminus K) = B_\alpha \setminus \bigcup\{P_n : n < \max(K \cap A_\alpha)\}$$

with the convention that $\max \emptyset = 0$. The definition of $\rho^{(\mathcal{I})}$ depends on the family $\mathcal{A}$, the enumeration of $\{B_\alpha : \alpha < \mathfrak{c}\}$ and the partition $\{P_n : n \in \omega\}$, but for simplicity we just write $\rho^{(\mathcal{I})}$.

Proposition 5.2. *For every ideal $\mathcal{I}$,*

(1) $\rho^{(\mathcal{I})}$ is partition regular,
(2) $\mathcal{I}_{\rho^{(\mathcal{I})}} = \mathcal{I}$,
(3) $\rho^{(\mathcal{I})}$ has small accretions,
(4) if $\mathcal{I} \in P^-$, then $\rho^{(\mathcal{I})} \in P^-$ (in Proposition 5.3, we show that in general one cannot reverse this implication),
(5) $\rho^{(\mathcal{I})} \in (S_2)$.

Proof. (1) Since $\overline{\mathcal{A}}$ is a nonempty family of infinite subsets of ω such that $F \setminus K \in \overline{\mathcal{A}}$ for every $F \in \overline{\mathcal{A}}$ and a finite set K, we only need to show that $\rho^{(\mathcal{I})}$ satisfies properties (M), (R) and (S) from Definition 2.2.

(M) Take any two distinct elements of $\overline{\mathcal{A}}$, say A_α and A_β for some α, β and some finite K_1, K_2, such that $A_\alpha \setminus K_1 \subseteq A_\beta \setminus K_2$. Since $\mathcal{A}$ is an almost disjoint family, $\alpha = \beta$. Note that if $A_\alpha \setminus K_1 \subseteq A_\alpha \setminus K_2$ then $\max(K_1 \cap A_\alpha) \geq \max(K_2 \cap A_\alpha)$. It follows that

$$B_\alpha \setminus \bigcup_{n<\max(K_1\cap A_\alpha)} P_n \subseteq B_\alpha \setminus \bigcup_{n<\max(K_2\cap A_\alpha)} P_n.$$

(R) Take any α and any finite set K such that $C \cup D = B_\alpha \setminus \bigcup_{n<\max(K\cap A_\alpha)} P_n$ for some $C, D \in [\omega]^\omega$. Since $B_\alpha \notin \mathcal{I}$ and $P_n \in \mathcal{I}$ for all n, either $C \notin \mathcal{I}$ or $D \notin \mathcal{I}$. Without loss of generality, assume $D \notin \mathcal{I}$. Then, $D = B_\beta$ for some β and $\rho^{(\mathcal{I})}(A_\beta) = B_\beta \subseteq D$.

(S) Take any α and any finite set K_1. We need to find β and a finite set K_2 with $A_\beta \setminus K_2 \subseteq A_\alpha \setminus K_1$ such that for any element $x \in B_\beta \setminus \bigcup_{n<\max(K_2\cap A_\beta)} P_n$ there is some finite set L for which $x \notin B_\beta \setminus \bigcup_{n<\max((K_2\cup L)\cap A_\beta)} P_n$. As in the proof above, we know that for the first inclusion to be true, α must equal β. Let $K_2 = K_1$. Take any $x \in B_\beta \setminus \bigcup_{n<\max(K_2\cap A_\beta)} P_n$. Since $(P_n)_n$ is a partition of ω there is some n_0 with $x \in P_{n_0}$. Choose $n_1 \in A_\beta$ with $\max(K_2 \cup \{n_0\}) < n_1$ and let $L = n_1 + 1$. Then, L is as required.

(2) Straightforward.

(3) Take any α and any finite set L. We will show that the set $A_\alpha \setminus L$ has small accretions. Take any finite set K and note that

$$\left(B_\alpha \setminus \bigcup_{n<\max(L\cap A_\alpha)} P_n\right) \setminus \left(B_\alpha \setminus \bigcup_{n<\max((L\cup K)\cap A_\alpha)} P_n\right)$$
$$\subseteq B_\alpha \cap \bigcup_{n<\max((K\setminus L)\cap A_\alpha)} P_n \in \mathcal{I} = \mathcal{I}_{\rho^{(\mathcal{I})}}.$$

(4) Take any disjoint sets $(A_n)_{n\in\omega}$ belonging to $\mathcal{I}_{\rho^{(\mathcal{I})}}$ such that $\bigcup_{n\in\omega} A_n \notin \mathcal{I}_{\rho^{(\mathcal{I})}}$. Since $\mathcal{I} = \mathcal{I}_{\rho^{(\mathcal{I})}}$ and $\mathcal{I} \in P^-$, there exists some set $C \notin \mathcal{I}$ such that $C \subseteq \bigcup_{n\in\omega} A_n$ and for each $n \in \omega$ there is some finite set K_n such that $(C \setminus K_n) \cap A_n = \emptyset$. Let α be such that $B_\alpha = C$. Fix any $n \in \omega$. Let $n_0 \in \omega$ be such that $K_n \subseteq \bigcup_{i<n_0} P_i$. Let $L = \{\min(A_\alpha \setminus n_0)\}$. Then,

$$B_\alpha \setminus \bigcup_{i<\max(L\cap A_\alpha)} P_i \subseteq B_\alpha \setminus K_n = C \setminus K_n.$$

(5) Take any α, a finite set M and $B \notin \mathcal{I}$ such that $B \subseteq B_\alpha \setminus \bigcup_{n<\max(M\cap A_\alpha)} P_n$. Then, $B = B_\beta$ for some β. Take any finite set K. We want to find some finite set L such that

$$B_\beta \setminus \bigcup_{n<\max(L\cap A_\beta)} P_n \subseteq B_\alpha \setminus \bigcup_{n<\max((M\cup K)\cap A_\alpha)} P_n.$$

Since $B_\beta \subseteq B_\alpha$, it is enough to satisfy the condition that $\max(L\cap A_\beta) \geq \max((M\cup K)\cap A_\alpha)$. It suffices to take $L = \{a\}$ for some $a \in A_\beta$ with $a \geq \max((M \cup K) \cap A_\alpha)$. □

For an ideal $\mathcal{I}$ on ω, we define the ideal $\mathsf{Fin}\otimes\mathcal{I}$ on $\omega\times\omega$ by

$$A \in \mathsf{Fin}\otimes\mathcal{I} \iff \{n \in \omega : \{k \in \omega : (n,k) \in A\} \notin \mathcal{I}\} \in \mathsf{Fin}.$$

For an ideal $\mathcal{I}$ on ω, an almost disjoint family $\mathcal{A} = \{A_\alpha : \alpha < \mathfrak{c}\}$ on ω and an enumeration $\mathcal{I}^+ = \{B_\alpha : \alpha < \mathfrak{c}\}$, by $\rho^{(\mathsf{Fin}\otimes\mathcal{I})}$ we denote the function as defined above with the partition given by $P_n = \{n\} \times \omega$ for each $n \in \omega$.

Proposition 5.3. *For any ideal $\mathcal{I}$ on ω,*

(1) $\mathsf{Fin}\otimes\mathcal{I} \notin P^-$,
(2) $\rho^{(\mathsf{Fin}\otimes\mathcal{I})} \in P^-$.

Proof. (1) It is enough to take $C_n = (\omega \setminus n) \times \omega$ for each $n \in \omega$ as a witness for the lack of the property P^-.

(2) Take any decreasing sequence $(C_n)_{n\in\omega}$ of $\mathsf{Fin}\otimes\mathcal{I}$-positive sets such that the difference $C_n \setminus C_{n+1}$ is in $\mathsf{Fin}\otimes\mathcal{I}$ for each n. We need to find α and a finite set L such that for each n there is a finite set K_n with

$$B_\alpha \setminus \bigcup_{k<\max((K_n\cup L)\cap A_\alpha)} (\{k\} \times \omega) = B_\alpha \setminus (\max((K_n \cup L) \cap A_\alpha) \times \omega) \subseteq C_n.$$

For each n, $C_n \notin \mathsf{Fin}\otimes\mathcal{I}$, so there is $i_n > n$ such that

$$(C_n)_{(i_n)} = \{k : (i_n, k) \in C\} \notin \mathcal{I}.$$

Take L to be an empty set and let α be such that

$$B_\alpha = \bigcup_{n<\omega} \{i_n\} \times (C_n)_{(i_n)} \quad \text{and} \quad K = \{\min(A_\alpha \setminus i_n)\}.$$

Then, using the fact that $C_m \subseteq C_n$ for $m > n$, we get $B_\alpha \setminus (\max(K_n\cap A_\alpha)\times\omega) \subseteq C_n$ for every n. □

Proposition 5.4. $\rho^{(\mathsf{Fin}\otimes\mathcal{I})} \not\approx_K \rho_{\mathcal{J}}$ *for any ideals $\mathcal{I}$ and $\mathcal{J}$ on ω.*

Proof. We have two cases: (1) $\mathcal{J} \notin P^-(\omega)$, (2) $\mathcal{J} \in P^-(\omega)$.

Case (1). If $\mathcal{J} \notin P^-(\omega)$ then there is no infinite space $X \in \mathrm{FinBW}(\mathcal{J})$ by Corollary 3.2. However $\rho^{(\mathsf{Fin}\otimes\mathcal{I})} \in P^-(\omega\times\omega)$ and $\rho^{(\mathsf{Fin}\otimes\mathcal{I})} \in (S_1)$, so there is some infinite space $X \in \mathrm{FinBW}(\rho^{(\mathsf{Fin}\otimes\mathcal{I})})$ by Theorem 3.1. Therefore $\mathrm{FinBW}(\rho^{(\mathsf{Fin}\otimes\mathcal{I})}) \nsubseteq \mathrm{FinBW}(\rho_{\mathcal{J}})$, so $\rho_{\mathcal{J}} \not\leq_K \rho^{(\mathsf{Fin}\otimes\mathcal{I})}$ by [5, Theorem 11.1 and Proposition 10.2].

Case (2). We show that $\rho^{(\mathsf{Fin}\otimes\mathcal{I})} \not\leq_K \rho_{\mathcal{J}}$. Let us take any function $f : \omega \to \omega\times\omega$.

Case (2a). There is some $i \in \omega$ with $f^{-1}[\{i\}\times\omega] \notin \mathcal{J}$.

Then, $f[f^{-1}[\{i\}\times\omega]] \in \mathsf{Fin}\otimes\mathcal{I}$. Therefore for any α and any finite K, it cannot be true that $B_\alpha \setminus (\max((K\cup L)\cap A_\alpha)\times\omega \subseteq f[f^{-1}[\{i\}\times\omega]]$ for any finite L, so f cannot witness the Katětov reduction.

Case (2b). For all $i \in \omega$, $f^{-1}[\{i\}\times\omega] \in \mathcal{J}$.

Then, $(f^{-1}[\{i\}\times\omega])$ defines a partition of ω into sets belonging to $\mathcal{J}$ and since $\mathcal{J} \in P^-(\omega)$ there exists some $F \notin \mathcal{J}$ such that for every i there is some finite set K_i with $(F\setminus K_i)\cap f^{-1}[\{i\}\times\omega] = \emptyset$. Then, $f[F\setminus K_i]\cap(\{i\}\times\omega) = \emptyset$, so $f[F] \in \mathsf{Fin}\otimes\mathcal{I}$. Therefore for any α and any finite set K, it cannot be true that $B_\alpha \setminus (\max((K\cup L)\cap A_\alpha)\times\omega \subseteq f[F]$ for any finite set L, so again f cannot witness the Katětov reduction. □

6. Critical ρ for the class of all metric compact spaces

Lemma 6.1. *For every partition regular ρ and every $f\colon \Lambda \to [0,1]$ there is $g\colon \Lambda \to [0,1]\cap\mathbb{Q}$ such that g is an injection and*

$$\forall F \in \mathcal{F}\,\forall p \in [0,1](f \restriction \rho(F) \text{ is } \rho\text{-convergent to } p \iff g \restriction \rho(F) \text{ is } \rho\text{-convergent to } p).$$

Proof. Without loss of generality, we assume that $\Omega = \Lambda = \omega$.

For each $n \in \omega$ choose $g(n) \in [0,1]\cap\mathbb{Q}\setminus\{g(i) : i < n\}$ such that $|f(n) - g(n)| < 1/2^n$. We will show that g is as needed. Fix $F \in \mathcal{F}$.

Assume that $f \restriction \rho(F)$ is ρ-convergent to p. Fix an open neighborhood $(p-r, p+r)$ of p. Since $f \restriction \rho(F)$ is ρ-convergent to p, there exists a finite set K such that $f[\rho(F\setminus K)] \subseteq (p - r/2, p + r/2)$. Let k be chosen in a way that $1/2^k < r/2$. Since ρ satisfies (S), there exists a finite set K' such that $\rho(F\setminus K')\cap\{0,1,\dots,k\} = \emptyset$. Then, $g[\rho(F\setminus(K\cup K'))] \subseteq (p-r, p+r)$. Thus $g \restriction \rho(F)$ is ρ-convergent to p.

Similarly, one can show that if $g \restriction \rho(F)$ is ρ-convergent to p then $f \restriction \rho(F)$ is also ρ-convergent to p. □

Theorem 6.2. *Let $\rho\colon \mathcal{F} \to [\Lambda]^\omega$ be partition regular with $\mathcal{F} \subseteq [\Omega]^\omega$. If $\rho \in P^-$ and has small accretions, then the following conditions are equivalent.*

1. *(1)* $\mathrm{FinBW}(\rho)$ *coincides with the class of all compact metric spaces in the realm of metric spaces.*
2. *(2)* $[0,1] \in \mathrm{FinBW}(\rho)$.
3. *(3)* $\rho_{\mathsf{conv}} \not\leq_K \rho$.

The assumption "$\rho \in P^-$" is only used to prove the implication "(3) $\implies$ (2)".

Proof. (1) $\iff$ (2) The implication "$\implies$" is obvious, and to show the reverse implication it is enough to notice that every compact metric space is a continuous image of the Cantor space, and the latter space is homeomorphic to a closed subset of $[0,1]$.

(2) $\implies$ (3) Let us assume that $\rho_{\mathsf{conv}} \leq_K \rho$. Then, it follows from [5, Theorem 11.1(1)] that $\mathrm{FinBW}(\rho) \subseteq \mathrm{FinBW}(\rho_{\mathsf{conv}})$. From [5, Proposition 10.2(4)] we also see that $\mathrm{FinBW}(\mathsf{conv}) = \mathrm{FinBW}(\rho_{\mathsf{conv}})$. Since $[0,1] \notin \mathrm{FinBW}(\mathsf{conv})$ [16, Section 2.7], we obtain $[0,1] \notin \mathrm{FinBW}(\rho)$.

(3) $\implies$ (2) Without loss of generality, we can assume $\Omega = \Lambda = \omega$. Suppose that $[0,1] \notin \mathrm{FinBW}(\rho)$. Then, there is $f\colon \omega \to [0,1]$ such that $f \restriction \rho(F)$ is not ρ-convergent for any $F \in \mathcal{F}$. By Lemma 6.1, without loss of generality, we can assume

that $f\colon \omega \to [0,1] \cap \mathbb{Q}$. We are going to show that f is a witness for $\rho_{\mathsf{conv}} \leq_K \rho$. Let $E_1 \in \mathcal{F}$. Since ρ has small accretions, there is $F_1 \subseteq E_1$, $F_1 \in \mathcal{F}$ such that: $\rho(F_1) \setminus \rho(F_1 \setminus K) \in \mathcal{I}_\rho$ for every finite K.

Define $Z^{-1} = \rho(F_1)$ and observe that $\rho(F_1 \setminus K) \subseteq Z^{-1} \cap \rho(E_1 \setminus K)$ for every finite K (as $F_1 \subseteq E_1$). We begin with a construction of $x^n_m \in \omega$, $A^n \subseteq \omega$, $y^n \in [0,1]$, open sets $U^n, V^n \subseteq [0,1]$ and $Z^n \subseteq \omega$ for all $n, m \in \omega$ that will satisfy the following conditions:

(1) $y^n \in V^n \subseteq U^n$,
(2) $Z^n \subseteq Z^{n-1}$ and $Z^n \notin \mathcal{I}_\rho$,
(3) $f[Z^n] \cap V^n = \emptyset$,
(4) $(f(x^n_m))_{m \in A^n}$ converges to y^n,
(5) $x^n_m \in \rho(F_1 \setminus [0,m]) \cap Z^{n-1})$; in particular, $\rho(F_1 \setminus [0,m]) \cap Z^n \neq \emptyset$ for all $n \in \omega$,
(6) $y^i \neq y^j$ for $i \neq j$.

In the n-th step, for every $m \in \omega$ we choose $x^n_m \in \rho(F_1 \setminus [0,m]) \cap Z^{n-1}$ (this is possible by (5) or, in the case of $n = 0$, by $\rho(F_1 \setminus [0,m]) \subseteq Z^{-1}$). The sequence $f(x^n_m)_{m \in \omega}$ has a subsequence $f(x^n_m)_{m \in A^n}$ convergent to some y^n. Note that by (2) for every m we have

$$Z^{n-1} \setminus \rho(F_1 \setminus [0,m]) \subseteq \rho(F_1) \setminus \rho(F_1 \setminus [0,m]) \in \mathcal{I}_\rho.$$

Since ρ is P^-, there is $G \in \mathcal{F}$ such that $\rho(G) \subseteq Z^{n-1}$ and for every m there exists k such that $\rho(G \setminus [0,k]) \subseteq \rho(F_1 \setminus [0,m])$. However, $f[\rho(G)]$ is not ρ-convergent, so it is not ρ-convergent to y^n. Hence, there exists an open interval U^n containing y^n such that for all k we have $f[\rho(G \setminus [0,k])] \not\subseteq U^n$. Recall that for every m there exists k such that $\rho(G \setminus [0,k]) \subseteq \rho(F_1 \setminus [0,m]) \cap Z^{n-1}$. So also for all m we have $f[\rho(F_1 \setminus [0,m]) \cap Z^{n-1}] \not\subseteq U^n$. Since $\rho \in P^-$, we know that there exists n_0 such that the set

$$\{i \in Z^{n-1} : f(i) \notin (y^n - 1/n_0, y^n + 1/n_0)\} \in \mathcal{I}^+_\rho$$

(if not, $\rho \in P^-$ would give us a ρ-sequence ρ-converging to y^n). Put $V^n = U^n \cap (y^n - 1/n_0, y^n + 1/n_0)$ and $Z^n = \{i \in Z^{n-1} : f(i) \notin V^n\}$.

Now, we will verify that this construction meets all required conditions. Items (1), (3), (4) and condition $Z^n \subseteq Z^{n-1}$ from (2) are clear. To show that $Z^n \notin \mathcal{I}_\rho$, observe that $Z^n \supseteq \{i \in Z^{n-1} : f(i) \notin (y^n - 1/n_0, y^n + 1/n_0)\} \in \mathcal{I}^+_\rho$. Thus, (2) is satisfied. Now we consider the second part of item (5) (as the first part is clear). Since $f[\rho(F_1 \setminus [0,m]) \cap Z^{n-1}] \not\subseteq U^n$, there is $x \in \rho(F_1 \setminus [0,m]) \cap Z^{n-1}$ such that $f(x) \notin U^n$. Since $V^n \subseteq U^n$ (by (1)), $f(x) \notin V^n$, so $x \in \rho(F_1 \setminus [0,m]) \cap Z^n$. Finally, we verify (6). Assume, for the sake of contradiction, that $y^k = y^n$ for some $k < n$. Since V^k is an open neighborhood of y^k (by (1)) and $y^n = y^k$ is the limit of the sequence $(f(x^n_m))_{m \in A^n}$, for almost all $m \in A^n$ we have $f(x^n_m) \in V^k$. On the other hand, by (2) and (5), $x^n_m \in Z^{n-1} \subseteq Z^k$ for all $m \in A^n$, so $f[Z^k] \cap V^k \neq \emptyset$ which contradicts (3).

We define a set

$$X = \{x^n_m : n \in \omega, m \in A^n \setminus [0,n]\}.$$

Let $F_2 = f[X]$. Then, $F_2 \notin \mathsf{conv}$ because X contains elements of infinitely many sequences convergent to different values. Now, let $K_1 \in \mathsf{Fin}$ and find $n \in \omega$ such that $K_1 \subseteq [0,n]$. We let

$$K_2 = f[\{x^k_m : k \leq n, m \in A^k \setminus [0,k], m \leq n\}] \in \mathsf{Fin}.$$

Then

$$\begin{aligned} F_2 \setminus K_2 =& f[\{x^k_m : k > n, m \in A^k \setminus [0,k]\}] \\ &\cup f[\{x^k_m : k \leq n, m > n, m \in A^k \setminus [0,k]\}] \subseteq f[\{x^k_m : k \in \omega, m > n\}]. \end{aligned}$$

Since $\{x^k_m : k \in \omega, m > n\} \subseteq \rho(F_1 \setminus [0,n])$ by (5), we conclude that $F_2 \setminus K_2 \subseteq f[\rho(F_1 \setminus [0,n])] \subseteq f[\rho(F_1 \setminus K_1)]$. □

Definition 6.3. Let $\mathcal{A} = \{A_\alpha : \alpha < \mathfrak{c}\}$ be an almost disjoint family and X be the family of all $x \in ([0,1] \cap \mathbb{Q})^{\omega\times\omega}$ such that

- for every $p \in [0,1]$ there is its open neighborhood U such that $x[(\omega \setminus [0,n]) \times \omega] \not\subseteq U$ for all $n \in \omega$,
- x is an injection,
- $x[(\omega \setminus [0,n]) \times \omega] \notin \mathsf{conv}$ for all $n \in \omega$.

Fix an enumeration $X = \{x_\alpha : \alpha < \mathfrak{c}\}$.

Now we define $\overline{\mathcal{A}} = \{A \setminus K : A \in \mathcal{A}, K \in [\omega]^{<\omega}\}$ and a map $\rho^{[0,1]} \colon \overline{\mathcal{A}} \to [[0,1] \cap \mathbb{Q}]^\omega$ by

$$\rho^{[0,1]}(A_\alpha \setminus K) = x_\alpha[(\omega \setminus [0, \max(A_\alpha \cap K)]) \times \omega].$$

Proposition 6.4. *$\rho^{[0,1]}$ is a partition regular function with $\mathsf{conv} = \mathcal{I}_{\rho^{[0,1]}}$.*

Proof. Since $\overline{\mathcal{A}}$ is a nonempty family of infinite subsets of ω such that $F \setminus K \in \overline{\mathcal{A}}$ for every $F \in \overline{\mathcal{A}}$ and a finite set K, we only need to show that $\rho^{[0,1]}$ satisfies properties (M), (R) and (S) from Definition 2.2.

(M): Take $E = A_\alpha \setminus K_1$ and $F = A_\beta \setminus K_2$ for some α, β and finite sets K_1, K_2 such that $E \subseteq F$. Since $\mathcal{A}$ is an almost disjoint family, $\alpha = \beta$, hence we just need to check whether it is true that $[0, \max(A_\alpha \cap K_2)] \subseteq [0, \max(A_\alpha \cap K_1)]$. Let $m = \max(A_\alpha \cap K_2)$. Then, $m \in A_\alpha \cap K_2$, so $m \notin A_\alpha \setminus K_2 = F \supseteq E = A_\alpha \setminus K_1$. Hence $m \notin A_\alpha \setminus K_1$, and consequently $m \in A_\alpha \cap K_1$.

(S): Fix $E \in \overline{\mathcal{A}}$. Then, there is α and finite set L such that $E = A_\alpha \setminus L$. Let $F = E$. Fix $a \in \rho^{[0,1]}(F)$. We need to find a finite set K such that $a \notin x_\alpha[(\omega \setminus [0, \max(A_\alpha \cap (K \cup L))]) \times \omega]$. Since x_α is an injection, we know there exist unique $i \in \omega \setminus [0, \max(A_\alpha \cap L)]$ and $j \in \omega$ such that $x_\alpha(i,j) = a$. Then, we find $i_0 \in A_\alpha$ with $i_0 \geq i$, put $K = \{i_0\}$ and observe that it is as needed.

(R): Fix α, a finite set K and $A, B \subseteq \mathbb{Q} \cap [0,1]$ such that $\rho^{[0,1]}(A_\alpha \setminus K) = A \cup B$. Since $\rho^{[0,1]}(A_\alpha \setminus K) = x_\alpha[(\omega \setminus [0, \max(A_\alpha \cap K)]) \times \omega] \notin \mathsf{conv}$, we have $A \notin \mathsf{conv}$ or $B \notin \mathsf{conv}$. Without loss of generality, we assume $A \notin \mathsf{conv}$. The set A has infinitely many distinct limit points $\{p_n : n \in \omega\}$. Let $(q_{n,i})_{i\in\omega}$ for $n \in \omega$ be sequences in A such that $\lim_{i\to\infty} q_{n,i} = p_n$ and $q_{n,i} \neq q_{n',i'}$ whenever $(n,i) \neq (n',i')$. Define $x : \omega \times \omega \to [0,1] \cap \mathbb{Q}$ by $x(i,n) = q_{n,i}$ (note the exchange of the order of indices). Then, $x \in X$, so there exists β such that $x_\beta = x$, so $\rho^{[0,1]}(A_\beta) = x_\beta[\omega \times \omega] \subseteq A$.

($\mathcal{I}_{\rho^{[0,1]}} = \mathsf{conv}$) From the proof of (R), it follows that $\mathsf{conv}^+ \subseteq \mathcal{I}^+_{\rho^{[0,1]}}$, and the reversed inclusion is straightforward. □

Lemma 6.5. $[0,1] \notin \mathrm{FinBW}(\rho^{[0,1]})$.

Proof. Define $f \colon [0,1] \cap \mathbb{Q} \to [0,1]$ by $f(q) = q$. Take any $A_\alpha \setminus K \in \overline{\mathcal{A}}$ and $p \in [0,1]$. Since $x_\alpha \in X$, there exists some open set $U \ni p$ such that for all finite sets L,

$$f[\rho^{[0,1]}(A_\alpha \setminus (K \cup L))] = \rho^{[0,1]}(A_\alpha \setminus (K \cup L)) = x_\alpha[(\omega \setminus [0, \max(A_\alpha \cap (K \cup L))]) \times \omega] \not\subseteq U.$$

Hence, $f \restriction \rho^{[0,1]}(A_\alpha \setminus K)$ is not $\rho^{[0,1]}$-convergent to p. □

Theorem 6.6. *For a partition regular function ρ, the following conditions are equivalent.*

(1) $\mathrm{FinBW}(\rho)$ *coincides with the class of all compact metric spaces in the realm of metric spaces.*
(2) $[0,1] \in \mathrm{FinBW}(\rho)$.
(3) $\rho^{[0,1]} \not\leq_K \rho$.

Proof. (1) $\iff$ (2) Repeat the argument from the proof of Theorem 6.2.

(2) $\implies$ (3) Assume that $\rho^{[0,1]} \leq_K \rho$. Then, by [5, Theorem 11.1(1)] we know that $\mathrm{FinBW}(\rho) \subseteq \mathrm{FinBW}(\rho^{[0,1]})$. By Lemma 6.5, we know that $[0,1] \notin \mathrm{FinBW}(\rho^{[0,1]})$, therefore $[0,1] \notin \mathrm{FinBW}(\rho)$.

(3) $\implies$ (2) Without loss of generality, we assume that $\Omega = \Lambda = \omega$. Assume that $[0,1] \notin \mathrm{FinBW}[\rho]$. Then, there exists $f : \omega \to [0,1]$ such that $f \restriction \rho(F)$ is not ρ-convergent for any $F \in \mathcal{F}$. Define g as in Lemma 6.1. We claim that g is a witness for $\rho^{[0,1]} \leq_K \rho$. Fix $F \in \mathcal{F}$.

Claim. $g[\rho(F \setminus [0,n])] \notin \mathsf{conv}$ for every $n \in \omega$.

Proof of claim. Assume otherwise. Then, $g[\rho(F \setminus [0,n])] = \{z^i_j : j \in \omega, i < m\}$ for finitely many convergent sequences $(z^i_j)_{j\in\omega}$, $i < m$, with distinct limits. Define

$$Z_i = g^{-1}[\{z^i_j : j \in \omega\}] \cap \rho(F \setminus [0,n])$$

and observe that $\rho(F \setminus [0,n]) = \bigcup_{i<m} Z_i$, so there exists $i < m$ and $G \in \mathcal{F}$ such that $\rho(G) \subseteq Z_i$ since ρ is partition regular. But then, $g \restriction \rho(G)$ is convergent, so it also is ρ-convergent, which is a contradiction. □

Using the above claim, for every $n \in \omega$ inductively find sequences $(x^n_k)_{k\in\omega}$ convergent to some y_n such that the following conditions hold for every $i, j, k, n, m \in \omega$:

- $x^n_k \in g[\rho(F \setminus [0,n])]$,
- $n \neq m \implies y_n \neq y_m$,
- $(n,k) \neq (i,j) \implies x^n_k \neq x^i_j$.

Now we consider two cases.

Case 1. For every $p \in [0,1]$ there is a neighborhood U of p such that for any $n \in \omega$,

$$\{x^i_k : k \in \omega, i > n\} \nsubseteq U.$$

Define $x : \omega \times \omega \to [0,1] \cap \mathbb{Q}$ by $x(n,k) = x^n_k$. Then, there exists α such that $x_\alpha = x$. We fix K_1 and let K_2 be a finite set such that $\max(A_\alpha \cap K_2) \geq \max K_1$. Then,

$$\begin{aligned}\rho^{[0,1]}(A_\alpha \setminus K_2) &= x[(\omega \setminus [0, \max(A_\alpha \cap K_2)]) \times \omega] \\ &\subseteq x[(\omega \setminus [0, \max(K_1)]) \times \omega] \subseteq g[\rho(F \setminus [0, \max K_1])] \subseteq g[\rho(F \setminus K_1)].\end{aligned}$$

Case 2. There is $p \in [0,1]$ such that for every neighborhood U of p there is $n_U \in \omega$ with

$$\{x^i_k : k \in \omega, i \geq n_U\} \subseteq U.$$

Since $g \restriction \rho(F)$ is not ρ-convergent to p, there exists a neighborhood U of p such that $g[\rho(F \setminus [0,n])] \nsubseteq U$ for all $n \in \omega$. For every $n \geq n_U$, we pick $a_n \in \rho(F \setminus [0,n])$ such that $g(a_n) \notin U$. Since ρ is partition regular, without loss of generality, we assume that a_n are pairwise distinct. We define $x : \omega \times \omega \to [0,1] \cap \mathbb{Q}$ by

$$x(n,k) = \begin{cases} g(a_{n+n_U}), & \text{if } k = 0, \\ x^{n+n_U}_k, & \text{if } k \neq 0. \end{cases}$$

Claim. $x \in X$.

Proof of claim. It is clear that $x[(\omega \setminus [0,n]) \times \omega] \notin \mathsf{conv}$ and that x is injective (as g is injective, $g(a_{n+n_U}) \notin U$ for all $n \in \omega$, while $x^{n+n_U}_k \in U$ for all $n, k \in \omega$). Let $q \in [0,1]$ and consider two cases.

Case I: $q = p$. Then, $q \in U$ and for every $n \in \omega$,

$$g(a_{n+1+n_U}) = x(n+1, 0) \in x[(\omega \setminus [0,n]) \times \omega] \quad \text{and} \quad g(a_{n+1+n_U}) \notin U,$$

hence $x[(\omega \setminus [0,n]) \times \omega] \nsubseteq U$.

Case II: $q \neq p$. There exist disjoint open neighborhoods V, W of p and q, respectively. Then, for all $n \in \omega$:

$$x_1^{n_V+n+1+n_U} = x(n_V + n + 1, 1) \in x[(\omega \setminus [0, n]) \times \omega] \quad \text{and} \quad x_1^{n_V+n+1+n_U} \notin W. \ \square$$

Since $x \in X$, there exists α such that $x_\alpha = x$. Fix K_1 and let K_2 be finite such that $\max(A_\alpha \cap K_2) \geq \max K_1$. Then,

$$\begin{aligned}\rho^{[0,1]}(A_\alpha \setminus K_2) &= x[(\omega \setminus [0, \max(A_\alpha \cap K_2)]) \times \omega] \\ &\subseteq x[(\omega \setminus [0, \max(K_1)]) \times \omega] \subseteq g[\rho(F \setminus [0, \max K_1])] \subseteq g[\rho(F \setminus K_1)].\end{aligned}$$

$\square$

Corollary 6.7 ([16])**.** *Let $\mathcal{I}$ be an ideal on ω. The following conditions are equivalent.*

(1) $\mathrm{FinBW}(\mathcal{I})$ *coincides with the class of all compact metric spaces in the realm of metric spaces.*
(2) $\mathsf{conv} \not\leq_K \mathcal{I}$.

Proof. Take $\rho = \rho_{\mathcal{I}}$ and apply Theorem 6.6 and [5, Propositions 7.5(2b) and 10.2(4)]. $\square$

Theorem 6.8. $\rho^{[0,1]} \leq_K \rho_{\mathsf{conv}}$ *and* $\rho_{\mathsf{conv}} \not\leq_K \rho^{[0,1]}$.

Proof. Assume for the sake of contradiction that $\rho^{[0,1]} \not\leq_K \rho_{\mathsf{conv}}$. Then, $[0,1] \in \mathrm{FinBW}(\rho_{\mathsf{conv}}) = \mathrm{FinBW}(\mathsf{conv})$ by Theorem 6.6 and [5, Proposition 10.2], a contradiction with Corollary 6.7.

Now, we prove $\rho_{\mathsf{conv}} \not\leq_K \rho^{[0,1]}$. Take any $f : \mathbb{Q} \cap [0,1] \to \mathbb{Q} \cap [0,1]$. Define

$$I_n^L = \left(\frac{1}{2^{n+2}}, \frac{1}{2^{n+1}}\right) \quad \text{and} \quad I_n^R = \left(1 - \frac{1}{2^{n+1}}, 1 - \frac{1}{2^{n+2}}\right)$$

for all $n \in \omega$. Now, for all $n \in \omega$, we

(1) choose a sequence $Z_n^L \subseteq I_n^L \cap \mathbb{Q}$ convergent to some $z_n^L \in I_n^L$,
(2) take an infinite $Y_n^L \subseteq Z_n^L$ such that $f[Y_n^L]$ is convergent to some y_n^L,
(3) define $H_n^L = \{h \in Y_n^L : |f(h) - y_n^L| \leq 1/n\}$,
(4) take a subsequence $(y_n^L)_{n \in G^L}$ convergent to u^L, where G^L is infinite.

Similarly, we follow the construction of $Z_n^R, z_n^R, Y_n^R, y_n^R, H_n^R, G^R$ and u^R.

For $S \in \{L, R\}$, we enumerate

$$G^S = \{g_n^S : n \in \omega\} \quad \text{and} \quad H_n^S = \{h_i^{n,S} : i \in \omega\}.$$

Define $x : \omega \times \omega \to [0,1] \cap \mathbb{Q}$ by

$$x(2n, i) = h_i^{g_n^L, L} \quad \text{and} \quad x(2n+1, i) = h_i^{g_n^R, R}.$$

Notice the following properties of the sequence x.

(1) x is an injection since $H_n^L \subseteq Y_n^L \subseteq Z_n^L \subseteq I_n^L$ (similarly when replacing L with R) and the intervals I_n^L and I_n^R are pairwise disjoint.
(2) $x[(\omega \setminus [0, n]) \times \omega] \notin \mathsf{conv}$ since for all $k > n$ we can find a sequence in $x[(\omega \setminus [0, n]) \times \omega]$ convergent to $z_{g_k^L}^L$.
(3) If $p = 0$ then $U = (-1/2, 1/2)$ is a neighborhood of p and for all $n \in \omega$ we have $x[(\omega \setminus [0, n]) \times \omega] \not\subseteq U$ since $h_0^{g_n^R, R} \in x[(\omega \setminus [0, n]) \times \omega]$, however $h_0^{g_n^R, R} \notin U$.
(4) If $p \in (0, 1]$ then there exists an open neighborhood U of p such that there exists $n \in \omega$ with $I_{g_k^L}^L \cap U = \emptyset$ for all $k > n$. So $x[(\omega \setminus [0, n]) \times \omega] \not\subseteq U$ because $h_0^{g_k^L, L} \in x[(\omega \setminus [0, n]) \times \omega]$, however $h_0^{g_k^L, L} \notin U$.

Therefore $x \in X$, so there exists α such that $x = x_\alpha$.

Take any $B \subseteq \mathbb{Q} \cap [0,1]$ such that for every finite set F there exists a finite set K such that $B \setminus K = \rho_{\mathsf{conv}}(B \setminus K) \subseteq f[\rho^{[0,1]}(A_\alpha \setminus F)]$. Once we show that $B \in \mathsf{conv}$, the proof will be finished.

There exists a finite set K such that

$$B \setminus K \subseteq f[\rho^{[0,1]}(A_\alpha \setminus \emptyset)] = f\left[\bigcup_{n\in\omega} H^L_{g^L_n}\right] \cup f\left[\bigcup_{n\in\omega} H^R_{g^R_n}\right].$$

Let

$$B^L = (B \setminus K) \cap f\left[\bigcup_{n\in\omega} H^L_{g^L_n}\right] \quad \text{and} \quad B^R = (B \setminus K) \cap f\left[\bigcup_{n\in\omega} H^R_{g^R_n}\right].$$

We are going to show that B^L is convergent to u^L and similarly one can show that B^R converges to u^R, so $B = B^L \cup B^R \in \mathsf{conv}$.

Let $\varepsilon > 0$. Then, there exists $k \in \omega$ such that for all $n > k$

$$|y^L_{g^L_n} - u^L| < \frac{\varepsilon}{2} \quad \text{and} \quad \frac{1}{g^L_n} < \frac{\varepsilon}{2}.$$

Let F be a finite set such that $\max(F \cap A_\alpha) > 2k$. Then, there exists $M \in [[0,1] \cap \mathbb{Q}]^{<\omega}$ such that $B \setminus M \subseteq f[\rho^{[0,1]}(A_\alpha \setminus F)]$. We are going to show that $|b - u^L| < \varepsilon$ for all $b \in B^L \setminus M$.

Fix $b \in B^L \setminus M \subseteq f[\rho^{[0,1]}(A_\alpha \setminus F)] \cap f[\bigcup_{n\in\omega} H^L_{g^L_n}]$. There exist $m, i \in \omega$ such that $b = f(x_\alpha(2m,i))$ and $2m > 2k$. Since $x_\alpha(2m,i) = h_i^{g^L_m, L} \in H^L_{g^L_m}$, we obtain that

$$|b - u^L| \leq |b - y^L_{g^L_m}| + |y^L_{g^L_m} - u^L| < \frac{1}{g^L_m} + \frac{\varepsilon}{2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \qquad \square$$

7. Ideal version of Mazurkiewicz theorem

Mazurkiewicz's theorem [15] states that every uniformly bounded sequence of continuous functions has a uniformly convergent subsequence once restricted to some perfect set. The question whether the subsequence can be enumerated by $\mathcal{I}$-positive set for a given ideal $\mathcal{I}$, was investigated in [7] where the authors proved that the answer is positive for F_σ-ideals. The following theorem shows that this result can be extended to every ideal with the FinBW property which in turn means that the ideal conv is critical for the ideal version of Mazurkiewicz's theorem.

Theorem 7.1. *Let $\mathcal{I}$ be an ideal on ω. The following conditions are equivalent.*

1. *$\mathcal{I}$ has the* FinBW *property.*
2. *For every sequence $(f_n)_{n\in\omega}$ of uniformly bounded continuous real-valued functions defined on $\mathbb{R}$ there exists $A \in \mathcal{I}^+$ and a nonempty perfect set $P \subseteq \mathbb{R}$ such that the subsequence $(f_n \restriction P)_{n\in A}$ is uniformly convergent.*

Proof. (2) $\implies$ (1) Assume that $\mathcal{I}$ does not have the FinBW property. Then, there is a sequence $(x_n)_{n\in\omega}$ of real numbers from the interval $[0,1]$ such that for every $A \in \mathcal{I}^+$ the subsequence $(x_n)_{n\in A}$ is not convergent. For each $n \in \omega$, let $f_n \colon \mathbb{R} \to \mathbb{R}$ be the constant function with the value x_n. Fix any $A \in \mathcal{I}^+$ and a nonempty perfect set $P \subseteq \mathbb{R}$. Assume, for the sake of contradiction, that the subsequence $(f_n \restriction P)_{n\in A}$ is uniformly convergent. Let $x \in P$. Then, $(f_n(x))_{n\in A}$ should converge, but $f_n(x) = x_n$ for every $n \in A$ and $(x_n)_{n\in A}$ is not convergent, a contradiction.

(1) $\implies$ (2) Fix $M > 0$ such that $|f_n(x)| \leq M$ for every $x \in \mathbb{R}$ and $n \in \omega$. Let $\{I_s : s \in 2^{<\omega}\}$ be a family of nonempty open subintervals of $(-M-1, M+1)$ satisfying:

(C1) $I_\emptyset = (-M-1, M+1)$,
(C2) $\{I_s : s \in 2^n\}$ is an open cover of $(-M-1, M+1)$ for every $n \in \omega$,
(C3) for every $\varepsilon > 0$ there exists $n \in \omega$ such that for every $s \in 2^n$, $\operatorname{diam}(I_s) < \varepsilon$,
(C4) $I_s = I_{s^\frown 0} \cup I_{s^\frown 1}$ for every $s \in 2^{<\omega}$.

For each $m, n \in \omega$ and $x \in \mathbb{R}$, let $p(m, n, x)$ be the least, with respect to the lexicographic order, $s \in 2^n$ such that $f_m(x) \in I_s$.

Let $\mathbb{Q} = \{q_n : n \in \omega\}$ be an enumeration of the rationals such that $q_i \neq q_j$ for all distinct $i, j \in \omega$. For $t_0, \dots, t_{n-1} \in 2^n$, define

$$B(t_0, \dots, t_{n-1}) = \{m \in \omega : p(m, n, q_i) = t_i \text{ for every } i \in n\},$$

and set

$$\mathcal{B}^n = \{B(t_0, \dots, t_{n-1}) : t_0, \dots, t_{n-1} \in 2^n\}.$$

Then, $\mathcal{B}^n$ is a finite partition of ω, and $\mathcal{B}^{n+1}$ refines $\mathcal{B}^n$. Thus, $\bigcup_{n\in\omega} \mathcal{B}^n$ is a finite-branching tree under inclusion.

Since $\mathcal{I}$ has the FinBW property, we can use [6, Proposition 2.8] to obtain a sequence $(Z_n)_{n\in\omega}$ and a set $Z \in \mathcal{I}^+$ such that for every $n \in \omega$

- $Z_n \in \mathcal{B}^n$,
- $Z_n \supseteq Z_{n+1}$,
- $Z \setminus Z_n \in \mathsf{Fin}$.

Let $(x_i)_{i\in\omega}$ be a sequence of pairwise distinct elements such that

$$Z \cap \bigcap_{n\in\omega} Z_n \subseteq \{x_i : i \in \omega\} \subseteq Z$$

and $x_i \in Z_i$ for each $i \in \omega$. Define for each $n \in \omega$,

$$A_n = (\{x_n\} \cup (Z \setminus Z_{n+1})) \setminus \bigcup_{i<n} A_i.$$

Then, $\bigcup_{n\in\omega} A_n = Z$ and $A_n \cap A_m = \emptyset$ whenever $n \neq m$. Moreover, $A_n \in \mathsf{Fin}$ and $A_n \subseteq Z \setminus Z_n$ for all $n \in \omega$.

For each $n \in \omega$, let $z(n, 0), \dots, z(n, n-1) \in 2^n$ be such that

$$Z_n = B\left(z(n, 0), \dots, z(n, n-1)\right).$$

Define $f : \mathbb{Q} \to \mathbb{R}$ by $\{f(q_i)\} = \bigcap_{n>i} \operatorname{cl}(I_{z(n,i)})$. Note that f is well-defined since $\lim_{n\to\infty} \operatorname{diam}(\operatorname{cl}(I_{z(n,i)})) = 0$ (by (C3), (C4) and the fact that $z(n, i) \in 2^n$ for all $n > i$) and $I_{z(n,i)} \supseteq I_{z(n+1,i)}$ for all $n > i$ (by $B\left(z(n, 0), \dots, z(n, n-1)\right) = Z_n \supseteq Z_{n+1} = B\left(z(n+1, 0), \dots, z(n+1, n)\right)$). Also note that $(f_m \restriction \mathbb{Q})_{m\in Z}$ is pointwise convergent to f. Indeed, if U is a neighborhood of some $f(q_i)$ then we can find $n > i$ such that $\operatorname{cl}(I_{z(n,i)}) \subseteq U$. Then for all $m \in Z \cap Z_n$, we have $f_m(q_i) \in \operatorname{cl}(I_{z(n,i)}) \subseteq U$. Since $Z \cap Z_n = Z \setminus (Z \setminus Z_n)$ and $Z \setminus Z_n$ is finite, $(f_m(q_i))_{m\in Z}$ converges to $f(q_i)$.

For each $\delta \in \{0, 1\}$, we will construct

- a family of finite sequences $\{a^\delta(s) \in 2^{<\omega} : s \in 2^{<\omega}\}$,
- a family of nonempty open intervals $\{U^\delta_s : s \in 2^{<\omega}\}$,
- a family $\{D^\delta_s \subseteq \mathbb{Q} : s \in 2^{<\omega}\}$,
- a family of rationals $\{q^\delta_s \in \mathbb{Q} : s \in 2^{<\omega}\}$,
- a sequence $(k^\delta_n)_{n\in\omega}$ in ω.

Additionally, for each $n \in \omega$, we define

$$G^0_n = A_{k^0_n} \cup \ldots \cup A_{k^1_n - 1} \quad \text{and} \quad G^1_n = A_{k^1_n} \cup \ldots \cup A_{k^0_{n+1} - 1}.$$

These objects will be chosen so that the following conditions hold for all $\delta \in \{0, 1\}$ and $s \in 2^{<\omega}$ (where $\operatorname{lh}(s)$ means the length of a finite sequence s):

(P1) $\operatorname{lh}(a^\delta(s)) = \operatorname{lh}(s) + 1$,
(P2) $U^\delta_{s^\frown 0}, U^\delta_{s^\frown 1} \subseteq U^\delta_s$ and $\operatorname{cl}(U^\delta_{s^\frown 0}) \cap \operatorname{cl}(U^\delta_{s^\frown 1}) = \emptyset$,
(P3) $\operatorname{diam}\left(U^\delta_s\right) \leq 1/2^{\operatorname{lh}(s)}$,

(P4) $D^\delta_{s^\frown 0}, D^\delta_{s^\frown 1} \subseteq D^\delta_s$,
(P5) $f[D^\delta_{s^\frown j}] \subseteq I_{a^\delta(s)}$ for each $j \in \{0,1\}$,
(P6) $D^\delta_s \subseteq U^\delta_s \subseteq \mathrm{cl}(D^\delta_s)$,
(P7) $f_i\left[\mathrm{cl}\left(U^\delta_{s^\frown j}\right)\right] \subseteq I_{a^\delta(s\restriction(\mathrm{lh}(s)-1))}$ for each $j \in \{0,1\}$, $i \in G^\delta_{\mathrm{lh}(s)}$ unless $s = \emptyset$,
(P8) $q^\delta_s \in D^\delta_s$,
(P9) $k^0_0 = 0$ and $k^0_n < k^1_n < k^0_{n+1}$ for each $n \in \omega$,
(P10) $f_i(q^\delta_{s^\frown j}) \in I_{a^\delta(s)}$ for each $j \in \{0,1\}$ and $i \in A_m$ for some $m \geq k^\delta_{\mathrm{lh}(s)+1}$.

Set $U^0_\emptyset = U^1_\emptyset = (0,1)$ and $D^0_\emptyset = D^1_\emptyset = (0,1) \cap \mathbb{Q}$. Pick any $q^0_\emptyset, q^1_\emptyset \in (0,1) \cap \mathbb{Q}$. Observe that conditions (P3), (P6) and (P8) are met. At this point there is nothing to check in (P2), (P4), (P5), (P7) and (P10) (as $U^\delta_{(j)}$, $D^\delta_{(j)}$ and $q^\delta_{(j)}$ for $\delta, j \in \{0,1\}$ are not defined yet). Put $k^0_0 = 0$ and $k^0_0 = 1$ (in particular, $k^0_0 < k^1_0$, so (P9) is satisfied). The sequences $a^\delta(\emptyset)$ will be defined in the next step, so we do not have to check (P1). This ends the initial step of our construction.

Suppose now that $n \in \omega$ and that we have already defined (accordingly to (P1)-(P10)) the following objects

- $a^\delta(t)$ for all $t \in 2^{<n}$ and $\delta \in \{0,1\}$,
- U^δ_s, D^δ_s and q^δ_s for all $s \in 2^{\leq n}$ and $\delta \in \{0,1\}$,
- k^0_i and k^1_i for all $i \leq n$.

Note that at this point G^0_n is defined. Let

$$V^0_s = U^0_s \cap \bigcap_{i \in G^0_n} f_i^{-1}[I_{a^0(s\restriction(n-1))}]$$

for all $s \in 2^n$ (in the case of $n = 0$ put $a^0(s \restriction (n-1)) = \emptyset$). Observe that $q^0_s \in V^0_s$, as $q^0_s \in D^0_s \subseteq U^0_s$ (by (P6) and (P8)) and $f_i(q^0_s) \in I_{a^0(s\restriction(n-1))}$ for all $i \in G^0_n$ (by $G^0_n \subseteq \bigcup_{m \geq k^0_n} A_m$ and item (P10); in the case of $n = 0$ the condition $f_i(q^0_s) \in I_{a^0(s\restriction(n-1))} = I_\emptyset$ is trivially satisfied). In particular, $V^0_s \neq \emptyset$. Moreover, by the fact that G^0_n is finite and by the continuity of all f_i's, the set V^0_s is open.

Observe that

$$D^0_s = D^0_s \cap f^{-1}[I_{a^0(s\restriction(n-1))}] = (D^0_s \cap f^{-1}[I_{a^0(s\restriction(n-1))^\frown 0}]) \cup (D^0_s \cap f^{-1}[I_{a^0(s\restriction(n-1))^\frown 1}])$$

(by (P5) together with (C4); in the case of $n = 0$ instead of (P5) use the fact that $D^0_\emptyset \subseteq \mathbb{Q} = f^{-1}[I_\emptyset]$). Since D^0_s is dense in V^0_s (by $V^0_s \subseteq U^0_s$ together with (P6)), we can find a nonempty open interval $W^0_s \subseteq V^0_s$ and a sequence $a^0(s)$ (equal either to $a^0(s \restriction (n-1))^\frown 0$ or to $a^0(s \restriction (n-1))^\frown 1$) such that $D^0_s \cap f^{-1}[I_{a^0(s)}]$ is dense in W^0_s. Note that (P1) is satisfied as $\mathrm{lh}(a^0(s)) = \mathrm{lh}(a^0(s \restriction (n-1))) + 1 = n + 1 = \mathrm{lh}(s) + 1$ (in the case of $n = 0$ we get $\mathrm{lh}(a^0(\emptyset)) = 1$).

Find any nonempty open intervals $U^0_{s^\frown j}$, for all $s \in 2^n$ and $j \in \{0,1\}$, satisfying $\mathrm{cl}(U^0_{s^\frown j}) \subseteq W^0_s$, (P2) and (P3). Also, define $D^0_{s^\frown j} = U^0_{s^\frown j} \cap D^0_s \cap f^{-1}[I_{a^0(s)}]$. Note that (P4) and (P5) are met. Since $D^0_s \cap f^{-1}[I_{a^0(s)}]$ is dense in W^0_s and $\mathrm{cl}(U^0_{s^\frown j}) \subseteq W^0_s$, (P6) is also satisfied. Since $\mathrm{cl}(U^0_{s^\frown j}) \subseteq W^0_s \subseteq V^0_s$, we get (P7).

Pick any $q^0_{s^\frown j} \in D^0_{s^\frown j}$ for all $s \in 2^n$ and $j \in \{0,1\}$. Then, $f(q^0_{s^\frown j}) \in I_{a^0(s)}$ (by (P5)). Since $(f_i \restriction \mathbb{Q})_{i \in Z}$ is pointwise convergent to f and each $I_{a^0(s)}$ is open, there is $k^0_{n+1} > k^1_n$ such that $f_i(q^0_{s^\frown j}) \in I_{a^0(s)}$ for all $i \in \bigcup_{m \geq k^0_{n+1}} A_m$, $s \in 2^n$ and $j \in \{0,1\}$. Observe that (P8), (P9) and (P10) are met.

Note that at this point G^1_n is already defined. Repeat the above construction for $\delta = 1$ instead of $\delta = 0$ to get $a^1(s), U^1_{s^\frown j}, D^1_{s^\frown j}$ and $q^1_{s^\frown j}$ for all $j \in \{0,1\}$ and $s \in 2^n$, as well as k^1_{n+1}. Hence, we have defined

- $a^\delta(t)$ for all $t \in 2^n$ and $\delta \in \{0,1\}$,
- U^δ_s, D^δ_s and q^δ_s for all $s \in 2^{n+1}$ and $\delta \in \{0,1\}$,
- k^0_{n+1} and k^1_{n+1}.

This finishes the construction.

Observe that

$$Z = \bigcup_{n\in\omega} A_n = \left(\bigcup_{n\in\omega} G^0_n\right) \cup \left(\bigcup_{n\in\omega} G^1_n\right).$$

Since $Z \in \mathcal{I}^+$, there is $\delta \in \{0,1\}$ such that $\bigcup_{n\in\omega} G^\delta_n \in \mathcal{I}^+$. Let

$$A = \bigcup_{n\in\omega} G^\delta_n \quad \text{and} \quad P = \bigcap_{n\in\omega} \bigcup_{s\in 2^n} \mathrm{cl}(U^\delta_s).$$

Thanks to (P2) and (P3), P is a perfect set and for each $x \in P$ there is a unique $\alpha_x \in 2^\omega$ such that $\{x\} = \bigcap_{n\in\omega} \mathrm{cl}(U^\delta_{\alpha_x\restriction n})$. Let $g\colon P \to \mathbb{R}$ be given by $g(x) = \limsup_{n\to\infty} f(q^\delta_{\alpha_x\restriction n})$ for all $x \in P$.

We will show that $(f_n \restriction P)_{n\in A}$ is uniformly convergent to g. Fix any $\varepsilon > 0$ and take $N \in \omega$ such that $\mathrm{diam}(I_{a^\delta(s)}) < \varepsilon$ for all $s \in 2^N$ (such N exists thanks to conditions (C3) and (P1)).

Fix any $x \in P$ and $i \in A \setminus \bigcup_{n\le N} G^\delta_n$. Since $\bigcup_{n\le N} G^\delta_n$ is a finite set, to finish the proof we only need to show that $|f_i(x) - g(x)| < \varepsilon$.

As $i \in A \setminus \bigcup_{n\le N} G^\delta_n$, we have $i \in G^\delta_{n+1}$ for some $n \ge N$. Then, we have $x \in \mathrm{cl}(U^\delta_{\alpha_x\restriction(n+2)})$, so by (P7) we get $f_i(x) \in I_{a^\delta(\alpha_x\restriction n)}$. Moreover, by conditions (P4) and (P8) we obtain $q^\delta_{\alpha_x\restriction(n+1+k)} \in D^\delta_{\alpha_x\restriction(n+1)}$ for all $k \in \omega$. Thus, $g(x) \in \mathrm{cl}(I_{a^\delta(\alpha_x\restriction n)})$ by (P5). Since $n \ge N$, $\mathrm{diam}(\mathrm{cl}(I_{a^\delta(\alpha_x\restriction n)})) = \mathrm{diam}(I_{a^\delta(\alpha_x\restriction n)}) \le \mathrm{diam}(I_{a^\delta(\alpha_x\restriction N)}) < \varepsilon$ (by (C4) and the choice of N). Finally, we conclude that $|f_i(x) - g(x)| \le \mathrm{diam}(\mathrm{cl}(I_{a^\delta(\alpha_x\restriction n)})) < \varepsilon$. □

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