$\mathcal{I}$-CLOSED SETS AND $\mathcal{I}$-CONTINUOUS FUNCTIONS

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ABSTRACT. Let $\mathcal{I}$ be an ideal on ω. A sequence $(x_n)_{n\in\omega}$ in a topological space X is said to be $\mathcal{I}$-convergent to a point $x \in X$ if for every neighborhood U of x we have $\{n \in \omega : x_n \notin U\} \in \mathcal{I}$. A set $Y \subseteq X$ is $\mathcal{I}$-closed if for every sequence $(x_n)_{n\in\omega}$ in Y that is $\mathcal{I}$-convergent to some $x \in X$ we have $x \in Y$. We characterize ideals for which finite unions of $\mathcal{I}$-closed sets are also $\mathcal{I}$-closed and we analyze which ideals satisfy this characterization.

The notion of $\mathcal{I}$-closed sets allows one to define $\mathcal{I}$-continuous functions. We compare such functions with those that preserve $\mathcal{I}$-convergence as well as, more generally, with those that preserve $\mathcal{J}$-convergence, where $\mathcal{J}$ is another ideal. We also characterize when preserving $\mathcal{I}$-convergence implies preserving $\mathcal{J}$-convergence. Finally, we attempt to compare $\mathcal{I}$-continuity with $\mathcal{J}$-continuity.

Our results allow us to answer two questions posed by Hang Zhang and Shuguo Zhang in [Top. App. **301** (2021), 107545].

1. INTRODUCTION

The concept of functions preserving $\mathcal{I}$-convergence is as old as the notion of ideal convergence. Both terms were defined in [6], where it was noted that for metric spaces functions preserving $\mathcal{I}$-convergence and continuous functions are one and the same while in [8] it was noted that continuous functions preserve $\mathcal{I}$-convergence in all topological spaces. The notion of $\mathcal{I}$-closed sets, introduced in [11], is a generalization of sequentially closed sets that uses ideal convergence. It was natural for authors of [11] to follow that with the concept of $\mathcal{I}$-continuous functions. Several questions left open in [11] were later answered in [10], where the authors used the Katětov order to show that, unlike ordinary closed sets, for some ideals finite union of $\mathcal{I}$-closed sets may not be $\mathcal{I}$-closed and that there may be $\mathcal{I}$-continuous functions that do not preserve $\mathcal{I}$-convergence.

The main goal of our paper is study these relatively new notions in-depth with the aim to improve the results obtained in [10] and answer the remaining open questions from that article. We will use the Katětov order to fully characterize ideals for which the finite union of $\mathcal{I}$-closed sets is $\mathcal{I}$-closed as well as ideals for which the notions of $\mathcal{I}$-continuous functions and functions preserving $\mathcal{I}$-convergence are the same. In order to do that, we will introduce new concepts of weakly K-uniform ideals and $\mathcal{I}$-closures of sets. We will also compare the concepts of $\mathcal{I}$-continuous functions, $\mathcal{J}$-continuous functions as well as functions preserving $\mathcal{I}$-convergence or $\mathcal{J}$-convergence and attempt to characterize implications between these notions.

The article is organized as follows. In Section 2 we present the definitions used throughout the paper. In Section 3 we study $\mathcal{I}$-closed sets and $\mathcal{I}$-closures of sets.

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We use the new concept of weakly K-uniform ideals to characterize ideals for which finite unions of $\mathcal{I}$-closed are $\mathcal{I}$-closed. In Section 4 we present a number of examples of ideals and check whether they are weakly K-uniform. In Section 5 we characterize when functions preserving $\mathcal{J}$-convergence have to preserve $\mathcal{I}$-convergence as well. In Section 6 we characterize when $\mathcal{J}$-continuous functions have to preserve $\mathcal{I}$-convergence and when functions preserving $\mathcal{I}$-convergence have to be $\mathcal{J}$-continuous. In Section 7 we attempt to characterize when $\mathcal{J}$-continuous functions are $\mathcal{I}$-continuous as well. We provide some necessary and sufficient conditions for that to happen.

2. PRELIMINARIES

2.1. **Ideals.** By ω we denote the set $\{0, 1, \ldots\}$. Given a set Ω, a nonempty subset of $\mathcal{P}(\Omega)$ is called an *ideal* on Ω if it is closed under taking subsets and finite unions of its elements. Additionally, we assume that ideals are *proper* ($\neq \mathcal{P}(\Omega)$) and that all finite subsets of Ω belong to every ideal (note that it gives us $\Omega = \bigcup \mathcal{I}$). In this paper we consider ideals on countable sets, i.e. Ω is always countable. The simplest example of an ideal is

$$\mathrm{Fin} = \{A \subseteq \omega : |A| < \infty\}.$$

By $\mathcal{I}^+$ we denote the family $\{A \subseteq \Omega : A \notin \mathcal{I}\}$. Sets in $\mathcal{I}^+$ are called *$\mathcal{I}$-positive.* For a set $A \subseteq \Omega$ we define

$$\mathcal{I}|A = \{B \cap A : B \in \mathcal{I}\}.$$

It is a proper ideal on A if and only if $A \in \mathcal{I}^+$. We say that a family $\mathcal{B} \subseteq \mathcal{P}(\Omega)$ *generates* the ideal $\mathcal{I}$ if

$$\mathcal{I} = \{A \subseteq \Omega : \exists_{B_0 \ldots B_k \in \mathcal{B}}\ A \subseteq B_0 \cup \ldots B_k\}.$$

For ideals $\mathcal{I}$ and $\mathcal{J}$ we define

$$\mathcal{I} \oplus \mathcal{J} = \{A \times \{0\} \cup B \times \{1\} : A \in \mathcal{I}, B \in \mathcal{J}\}.$$

This is a proper ideal even if only one of $\mathcal{I}$ and $\mathcal{J}$ is proper, so we allow the case in which one of them is not proper.

Ideals $\mathcal{I}$ and $\mathcal{J}$ are said to be *isomorphic* if there exists a bijection $f : \bigcup \mathcal{J} \longrightarrow \bigcup \mathcal{I}$ such that

$$f^{-1}[M] \in \mathcal{J} \iff M \in \mathcal{I}$$

for all $M \subseteq \bigcup \mathcal{I}$. In this case we write $\mathcal{I} \cong \mathcal{J}$. An ideal $\mathcal{I}$ is *homogeneous* if for any $A \in \mathcal{I}^+$ we have $\mathcal{I}|A \cong \mathcal{I}$.

Let $\mathcal{I}$ and $\mathcal{J}$ be ideals. We say that $\mathcal{I}$ is *Katětov below* $\mathcal{J}$ (written $\mathcal{I} \leq_K \mathcal{J}$) if there exists a function $f : \bigcup \mathcal{J} \longrightarrow \bigcup \mathcal{I}$ such that $f^{-1}[M] \in \mathcal{J}$ for any $M \in \mathcal{I}$. Such function may be called a *witness* for $\mathcal{I} \leq_K \mathcal{J}$. Ideals $\mathcal{I}$ and $\mathcal{J}$ are *K-equivalent* if both $\mathcal{I} \leq_K \mathcal{J}$ and $\mathcal{J} \leq_K \mathcal{I}$.

By $K(\mathcal{I})$ we denote the family $\{A \subseteq \bigcup \mathcal{I} : \mathcal{I}|A \leq_K \mathcal{I}\}$. It is known that $\mathcal{I} \leq_K \mathcal{I}|A$ for all $A \in \mathcal{I}^+$, therefore $K(\mathcal{I}) = \{A \subseteq \bigcup \mathcal{I} : \mathcal{I}|A \text{ and } \mathcal{I} \text{ are K-equivalent}\}$. An ideal $\mathcal{I}$ is *K-uniform* if $K(\mathcal{I}) = \mathcal{I}^+$ (the term K-homogeneous is also used).

An ideal $\mathcal{I}$ is *tall* (or *dense*) if for every infinite $A \subseteq \bigcup \mathcal{I}$ there exists an infinite $B \subseteq A$ such that $B \in \mathcal{I}$. Note that $\mathcal{I} \leq_K \mathrm{Fin}$ if and only if $\mathcal{I}$ is not tall. An ideal $\mathcal{I}$ is *nowhere tall* if for any $B \in \mathcal{I}^+$ the ideal $\mathcal{I}|B$ is not tall.

Below we present several examples of ideals.

- The van der Waerden ideal is defined by

$$\mathcal{W} = \{A \subseteq \omega : \exists_{n\in\omega}\, A \text{ does not contain a finite arithmetic progression of length n}\}.$$

  It is an ideal thanks to the van der Waerden's Theorem. $\mathcal{W}$ is tall and homogeneous (by [7, Remark under Example 2.7]), so it is also K-uniform.
- The asymptotic density zero ideal is a tall and K-uniform (see [9, Proposition 2.1.11]) ideal defined by

$$\mathcal{I}_d = \left\{ A \subseteq \omega : \lim_{n\to\infty} \frac{|A \cap \{0,1,\dots,n\}|}{n} = 0 \right\}.$$

- For a given function $f : \omega \to [0,\infty)$ such that $\sum_{n\in\omega} f(n) = \infty$ we can define an ideal

$$\mathcal{I}_f = \left\{ A \subseteq \omega : \sum_{n\in A} f(n) < \infty \right\}.$$

  Such ideals are called *summable ideals*. A summable ideal $\mathcal{I}_f$ is tall if and only if $\lim_{n\to\infty} f(n) = 0$. In particular, in the case when $f(n) = \frac{1}{n+1}$ we get the tall summable ideal

$$\mathcal{I}_{\frac{1}{n}} = \left\{ A \subseteq \omega : \sum_{i\in A} \frac{1}{i+1} < \infty \right\}.$$

Remark. It is known that $\mathcal{I}_{\frac{1}{n}} \subseteq \mathcal{I}_d$ (so therefore $\mathcal{I}_{\frac{1}{n}} \leq_K \mathcal{I}_d$) and that $\mathcal{I}_d \not\leq_K \mathcal{I}_{\frac{1}{n}}$ (see [5]). Moreover, $\mathcal{I}_{\frac{1}{n}} \not\leq_K \mathcal{W}$ ([3, Lemma 3.1], see also [2, Theorem 7.7(10)]), and hence also $\mathcal{I}_d \not\leq_K \mathcal{W}$. By Szemerédi's theorem we have $\mathcal{W} \subseteq \mathcal{I}_d$. It is an open question whether $\mathcal{W} \leq_K \mathcal{I}_{\frac{1}{n}}$ (see [1, Question 1.2]). In fact, this is a weaker version of the famous Erdős-Turán conjecture, which states that $\mathcal{W} \subseteq \mathcal{I}_{\frac{1}{n}}$.

Definition 2.1. An ideal $\mathcal{I}$ on Ω is *weakly K-uniform* if for any $A \subseteq \Omega$ we have $A \in K(\mathcal{I})$ or $\Omega \setminus A \in K(\mathcal{I})$.

If an ideal is K-uniform, then it is weakly K-uniform. In Corollary 4.6 we provide an example of a non-K-uniform ideal that is weakly K-uniform.

Proposition 2.2. *Let $\mathcal{I}$ be an ideal on Ω. The following conditions are equivalent:*

1. *$\mathcal{I}$ is weakly K-uniform;*
2. *For all $B, C \subseteq \Omega$, if $\Omega = B \cup C$, then $B \in K(\mathcal{I})$ or $C \in K(\mathcal{I})$;*
3. *For all $A, B, C \subseteq \Omega$, if $A = B \cup C$ and $A \in K(\mathcal{I})$, then $B \in K(\mathcal{I})$ or $C \in K(\mathcal{I})$.*

Proof. (1) $\implies$ (2): Fix sets B and C such that $B \cup C = \Omega$. Then $C \setminus B$ is the complement of B. Hence, $B \in K(\mathcal{I})$ or $C \setminus B \in K(\mathcal{I})$. In the latter case, we immediately get $C \in K(\mathcal{I})$ as well.

(2) $\implies$ (3): Let A, B, C be such that $A \in K(\mathcal{I})$ and $A = B \cup C$. Since $\mathcal{I}|A \leq_K \mathcal{I}$, there exists a function $f : \Omega \longrightarrow A$ such that $f^{-1}[M] \in \mathcal{I}$ for all $M \in \mathcal{I}|A$. Then $f^{-1}[B] \cup f^{-1}[C] = \Omega$. By (2), $f^{-1}[B] \in K(\mathcal{I})$ or $f^{-1}[C] \in K(\mathcal{I})$. Without loss of generality, assume that $f^{-1}[B] \in K(\mathcal{I})$. Let $g : \Omega \to f^{-1}[B]$ be a witness for that, which means that $g^{-1}[M] \in \mathcal{I}$ for every $M \in \mathcal{I}|(f^{-1}[B])$. Define $h : \Omega \to B$ by $h(x) = f(g(x))$ for all $x \in \Omega$. Fix $M \in \mathcal{I}|B$. Then $h^{-1}[M] = g^{-1}[f^{-1}[M]] \in \mathcal{I}$. Thus $B \in K(\mathcal{I})$.

(3) $\implies$ (1): This is obvious. □

2.2. **Ideal convergence.** For an ideal $\mathcal{I}$ on Ω, a sequence $(x_n)_{n\in\Omega}$ in a topological space X is said to be $\mathcal{I}$-*convergent* to $x \in X$ (written $x_n \xrightarrow{\mathcal{I}} x$) if $\{n \in \Omega : x_n \notin U\} \in \mathcal{I}$ for any neighborhood U of x. For the ideal Fin, this notion coincides with the usual convergence.

Definition 2.3. [11, Definition 3.1] Let X be a topological space and $\mathcal{I}$ be an ideal on Ω. A set $A \subseteq X$ is

- $\mathcal{I}$-*closed* if for any sequence $(x_n)_{n\in\Omega}$ in A $\mathcal{I}$-converging to some $x \in X$, we have $x \in A$;
- $\mathcal{I}$-*open* if $X \setminus A$ is $\mathcal{I}$-closed.

If a set is closed (open), then it is also $\mathcal{I}$-closed ($\mathcal{I}$-open) (see [11, Remark 3.2(3)]).

Definition 2.4. Let X and Y be topological spaces and $\mathcal{I}$ be an ideal on Ω. A function $f : X \longrightarrow Y$

- is $\mathcal{I}$-*continuous* if for any $\mathcal{I}$-closed set $A \subseteq Y$, the preimage $f^{-1}[A]$ is $\mathcal{I}$-closed (see [11, Definition 4.1]);
- preserves $\mathcal{I}$-convergence if for any $x \in X$ and any sequence $(x_n)_{n\in\Omega}$ in X such that $x_n \xrightarrow{\mathcal{I}} x$, we have $f(x_n) \xrightarrow{\mathcal{I}} f(x)$ (see [8]).

Notice that the notion of functions that preserve Fin-convergence is the same as the classical notion of sequentially continuous functions.

If a function f is continuous, then it preserves $\mathcal{I}$-convergence, for every ideal $\mathcal{I}$ (see [8, Theorem 3]), and if it preservers $\mathcal{I}$-convergence, then it is $\mathcal{I}$-continuous (see [11, Theorem 4.2]). It is also known that for every ideal $\mathcal{I}$ there are non-continuous functions that preserve $\mathcal{I}$-convergence (see [11, Example 4.7]).

Proposition 2.5. *Let $\mathcal{I}$ be an ideal. Composition of two $\mathcal{I}$-continuous functions is $\mathcal{I}$-continuous.*

Proof. Straightforward. □

For an ideal $\mathcal{I}$ on Ω, the topological space $X(\mathcal{I})$ is defined as $\Omega \cup \{\omega\}$, where all points of Ω are isolated and all neighborhoods of ω are of the form $\{\omega\} \cup (\Omega \setminus A)$ for $A \in \mathcal{I}$. For the purpose of spaces $X(\mathcal{I})$, we will always assume that $\omega \notin \Omega$, purely to avoid confusion in the notation.

Lemma 2.6 ([10, Lemma 3.3(2)])**.** *Let $\mathcal{I}$ and $\mathcal{J}$ be ideals. Then a set $A \subseteq \bigcup \mathcal{I}$ is $\mathcal{J}$-closed in $X(\mathcal{I})$ if and only if $\mathcal{I}|A \not\leq_K \mathcal{J}$.*

Remark. Observe that for any $A \subseteq X(\mathcal{I})$, if $\omega \in A$, then A is closed (so also $\mathcal{J}$-closed for every ideal $\mathcal{J}$). Moreover, Lemma 2.6 holds also for improper ideals $\mathcal{I} = \mathcal{P}(\Omega)$ as in this case ω is isolated in $X(\mathcal{I})$ and all subsets of $X(\mathcal{I})$ are closed, while $\mathcal{I}|A \not\leq_K \mathcal{J}$ holds for every $A \subseteq \Omega$.

3. $\mathcal{I}$-CLOSED SETS

Theorem 3.1. *Let $\mathcal{I}$ and $\mathcal{J}$ be ideals. The following conditions are equivalent:*

1. *For any topological space X and any $A \subseteq X$, if A is $\mathcal{I}$-closed, then it is $\mathcal{J}$-closed;*
2. *$\mathcal{J} \leq_K \mathcal{I}$.*

Proof. Without loss of generality we may assume that both $\mathcal{I}$ and $\mathcal{J}$ are ideals on ω.

(1) $\implies$ (2): Suppose that $\mathcal{J} \not\leq_K \mathcal{I}$. Then, by Lemma 2.6, the set ω is $\mathcal{I}$-closed in $X(\mathcal{J})$, but it is not $\mathcal{J}$-closed in $X(\mathcal{J})$, contradicting (1).

(2) $\implies$ (1): Assume that $\mathcal{J} \leq_K \mathcal{I}$ and fix a topological space X and an $\mathcal{I}$-closed set $A \subseteq X$. We will show that A is $\mathcal{J}$-closed. Let $(x_n)_{n\in\omega}$ be a sequence in A that is $\mathcal{J}$-convergent to a point $x \in X$. Since $\mathcal{J} \leq_K \mathcal{I}$, there is a function $f : \omega \longrightarrow \omega$ such that $f^{-1}[M] \in \mathcal{I}$ for any $M \in \mathcal{J}$. Define a sequence $(y_n)_{n\in\omega}$ by $y_n = x_{f(n)}$ for all n. We will show that the sequence $(y_n)_{n\in\omega}$ is $\mathcal{I}$-convergent to x. Fix a neighborhood U_x of x. Then

$$\{n \in \omega : y_n \notin U_x\} = \{n \in \omega : x_{f(n)} \notin U_x\} = f^{-1}[\{m \in \omega : x_m \notin U_x\}] \in \mathcal{I},$$

because $\{m \in \omega : x_m \notin U_x\}$ belongs to $\mathcal{J}$.

Since $(y_n)_{n\in\omega} \subseteq A$ and A is $\mathcal{I}$-closed, we obtain $x \in A$. □

In [10, Theorem 2.5], Hang Zhang and Shuguo Zhang proved that if $\mathcal{I}$ is K-uniform, then in any topological space the union of two $\mathcal{I}$-closed sets is $\mathcal{I}$-closed. The following theorem is stronger, although the first part of the proof is very similar.

Theorem 3.2. *Let $\mathcal{I}$ be an ideal. The following conditions are equivalent:*

1. *$\mathcal{I}$ is weakly K-uniform;*
2. *for any topological space, the union of two $\mathcal{I}$-closed sets is $\mathcal{I}$-closed.*

Proof. Without loss of generality we will assume that $\mathcal{I}$ is an ideal on ω.

(1) $\implies$ (2): Fix a topological space X. Let $Y, Z \subseteq X$ be $\mathcal{I}$-closed and $(x_n)_{n\in\omega}$ be a sequence in $Y \cup Z$ such that $x_n \xrightarrow{\mathcal{I}} x \in X$. Define

$$B = \{n \in \omega : x_n \in Y\} \text{ and } C = \{n \in \omega : x_n \in Z\}.$$

Then $B \cup C = \omega$. Since $\mathcal{I}$ is weakly K-uniform, Proposition 2.2 implies that $B \in K(\mathcal{I})$ or $C \in K(\mathcal{I})$. Without loss of generality, we may assume that $B \in K(\mathcal{I})$. Let $f : \omega \to B$ witness this, i.e., $f^{-1}[M] \in \mathcal{I}$ for all $M \in \mathcal{I}|B$. Define a sequence $(y_n)_{n\in\omega}$ by $y_n = x_{f(n)}$ for all $n \in \omega$. Fix a neighborhood U_x of x. Then

$$\{n \in \omega : y_n \notin U_x\} = \{n \in \omega : x_{f(n)} \notin U_x\} = f^{-1}[\{m \in \omega : x_m \notin U_x\} \cap B] \in \mathcal{I},$$

because $\{m \in \omega : x_m \notin U_x\} \in \mathcal{I}$. Thus, $y_n \xrightarrow{\mathcal{I}} x$. Since $y_n \in Y$ for all n and Y is $\mathcal{I}$-closed, we obtain $x \in Y \subseteq Y \cup Z$.

(2) $\implies$ (1): Assume $\mathcal{I}$ is not weakly K-uniform. Then there exists a set $A \subseteq \omega$ such that $\mathcal{I}|A \not\leq_K \mathcal{I}$ and $\mathcal{I}|(\omega \setminus A) \not\leq_K \mathcal{I}$. By Lemma 2.6, both A and $(\omega \setminus A)$ are $\mathcal{I}$-closed in $X(\mathcal{I})$. However, $A \cup (\omega \setminus A) = \omega$ is not $\mathcal{I}$-closed in $X(\mathcal{I})$. □

Definition 3.3. Let X be a topological space and $\mathcal{I}$ be an ideal. For $A \subseteq X$ we define $\mathcal{I}$-closure of A as the smallest $\mathcal{I}$-closed set containing A. We will denote it by $cl_{\mathcal{I}}(A)$.

Proposition 3.4. *Let $\mathcal{I}$ be an ideal and X be a topological space. Then for any $A, B \subseteq X$ we have:*

(a) $A \subseteq B \implies cl_{\mathcal{I}}(A) \subseteq cl_{\mathcal{I}}(B)$;
(b) $cl_{\mathcal{I}}(A) \cup cl_{\mathcal{I}}(B) \subseteq cl_{\mathcal{I}}(A \cup B)$;
(c) $cl_{\mathcal{I}}(A \cap B) \subseteq cl_{\mathcal{I}}(A) \cap cl_{\mathcal{I}}(B)$;
(d) $cl_{\mathcal{I}}(A) \subseteq cl(A)$.
(e) $A \subseteq B \subseteq cl_{\mathcal{I}}(A) \implies cl_{\mathcal{I}}(A) = cl_{\mathcal{I}}(B)$.

Proof. Item (a) follows directly from the definition of $\mathcal{I}$-closure. Items (b) and (c) follow from (a). To obtain (d), take any $A \subseteq X$. Then $cl(A)$ contains A and is a closed set, so it is also $\mathcal{I}$-closed (by [11, Remark 3.2(3)]). Thus, $cl_{\mathcal{I}}(A) \subseteq cl(A)$. It remains to obtain (e). To do that, we only need to combine (a) with the observation that if $B \subseteq cl_{\mathcal{I}}(A)$, then $cl_{\mathcal{I}}(A) \cap cl_{\mathcal{I}}(B)$ is an $\mathcal{I}$-closed set containing B because intersections of $\mathcal{I}$-closed sets are $\mathcal{I}$-closed. $\square$

Remark. In general we cannot replace "$\subseteq$" with "$=$" in Proposition 3.4(d). Indeed, let $\mathcal{I}$ be an ideal on ω such that there is $A \in \mathcal{I}^+$ with $\mathcal{I}|A \not\leq_K \mathcal{I}$. Then, in the space $X(\mathcal{I})$, the set A is $\mathcal{I}$-closed (by Lemma 2.6), but it is not closed. Therefore, $cl_{\mathcal{I}}(A) = A \neq cl(A)$.

Remark. Obviously, in general we cannot replace "$\subseteq$" with "$=$" in Proposition 3.4(c) as for $X = \mathbb{R}$ with the standard topology and $A = (0,1)$, $B = (1,2)$ we have $cl_{\mathcal{I}}(A \cap B) = cl(A \cap B) = \emptyset$ and $cl_{\mathcal{I}}(A) \cap cl_{\mathcal{I}}(B) = cl(A) \cap cl(B) = \{1\}$.

Next result shows that in general we cannot replace "$\subseteq$" with "$=$" in Proposition 3.4(b).

Proposition 3.5. *Let $\mathcal{I}$ be an ideal. Then the following are equivalent:*

1. *$\mathcal{I}$ is weakly K-uniform;*
2. *$cl_{\mathcal{I}}(A) \cup cl_{\mathcal{I}}(B) = cl_{\mathcal{I}}(A \cup B)$, for every topological space X and every $A, B \subseteq X$.*

Proof. (2) $\implies$ (1): If $\mathcal{I}$ is not weakly K-uniform, then by Theorem 3.2 there are a topological space X and two $\mathcal{I}$-closed sets $A, B \subseteq X$ such that $A \cup B$ is not $\mathcal{I}$-closed. Hence, $cl_{\mathcal{I}}(A) \cup cl_{\mathcal{I}}(B) = A \cup B \neq cl_{\mathcal{I}}(A \cup B)$.

(1) $\implies$ (2): It follows from Theorem 3.2 that, in the case of weakly K-uniform ideal $\mathcal{I}$, the union of two $\mathcal{I}$-closed sets is also $\mathcal{I}$-closed. Hence, $cl_{\mathcal{I}}(A) \cup cl_{\mathcal{I}}(B)$ is an $\mathcal{I}$-closed set containing $A \cup B$. Thus, $cl_{\mathcal{I}}(A \cup B) \subseteq cl_{\mathcal{I}}(A) \cup cl_{\mathcal{I}}(B)$. On the other hand, $cl_{\mathcal{I}}(A) \cup cl_{\mathcal{I}}(B) \subseteq cl_{\mathcal{I}}(A \cup B)$ follows from Proposition 3.4(b). $\square$

Lemma 3.6. *Let X be a topological space, $D \subseteq X$ and $\mathcal{I}$ be an ideal on Ω. Then $cl_{\mathcal{I}}(D) = \bigcup_{\alpha<\omega_1} D^{(\alpha)}$, where $D^{(0)} = D$ and*

$$D^{(\alpha)} = \left\{ x \in X : x \text{ is the } \mathcal{I}\text{-limit of some sequence } (x_n)_{n\in\Omega} \text{ in } \bigcup_{\beta<\alpha} D^{(\beta)} \right\},$$

for all $0 < \alpha < \omega_1$.

Proof. Obviously, $D^{(0)} \subseteq cl_{\mathcal{I}}(D)$. Moreover, if $\bigcup_{\beta<\alpha} D^{(\beta)} \subseteq cl_{\mathcal{I}}(D)$ for some $\alpha < \omega_1$, then also $D^{(\alpha)} \subseteq cl_{\mathcal{I}}(D)$ by Proposition 3.4(e). Thus, $\bigcup_{\alpha<\omega_1} D^{(\alpha)} \subseteq cl_{\mathcal{I}}(D)$. To finish the proof, we need to show that $\bigcup_{\alpha<\omega_1} D^{(\alpha)}$ is $\mathcal{I}$-closed. Indeed, if $(x_n)_{n\in\Omega}$ is a sequence in $\bigcup_{\alpha<\omega_1} D^{(\alpha)}$ that is $\mathcal{I}$-convergent to some $x \in X$, then for each $n \in \Omega$ find $\alpha_n < \omega_1$ such that $x_n \in D^{(\alpha_n)}$ and observe that $x \in D^{(\alpha)}$, where $\alpha = (\sup_{n\in\Omega} \alpha_n) + 1 < \omega_1$. $\square$

4. WEAKLY K-UNIFORM IDEALS

In this section we give examples of ideals that are weakly K-uniform as well as ideals that are not weakly K-uniform. Moreover, we answer two questions posed in [2] and [10] and provide an example of a weakly K-uniform ideals which is not K-uniform.

We start by presenting a few examples of ideals that are weakly K-uniform.

Example 4.1. Non-tall ideals are weakly K-uniform.

Proof. Let $\mathcal{I}$ be a non-tall ideal. Without loss of generality we can assume that $\mathcal{I}$ is an ideal on ω. Fix $B \subseteq \omega$ and take an infinite set A that does not contain any infinite subset belonging to $\mathcal{I}$. Then either $|A \cap B| = \omega$ or $|A \cap (\omega \setminus B)| = \omega$. Without loss of generality, assume $|A \cap B| = \omega$. Then $\mathcal{I}|B$ is not tall, hence $\mathcal{I}|B \leq_K \mathrm{Fin} \leq_K \mathcal{I}$. Thus $B \in K(\mathcal{I})$. □

Before presenting further examples, we need to introduce the following lemma.

Lemma 4.2 (Folklore)**.** *Let $\mathcal{I}$, $\mathcal{J}$ and $\mathcal{K}$ be ideals. Then $\mathcal{K} \leq_K \mathcal{I} \oplus \mathcal{J}$ if and only if $\mathcal{K} \leq_K \mathcal{I}$ and $\mathcal{K} \leq_K \mathcal{J}$.*

Proof. Without loss of generality we may assume that all considered ideals are on ω.

If $f : \omega \to \omega$ witnesses $\mathcal{K} \leq_K \mathcal{I}$ and $g : \omega \to \omega$ witnesses $\mathcal{K} \leq_K \mathcal{J}$, then it is easy to see that the function $h : \omega \times \{0,1\} \to \omega$ given by:

$$h(n,i) = \begin{cases} f(n), & \text{if } i = 0, \\ g(n), & \text{if } i = 1, \end{cases}$$

for all $(n,i) \in \omega \times \{0,1\}$, is a witness for $\mathcal{K} \leq_K \mathcal{I} \oplus \mathcal{J}$.

On the other hand, if some function $h : \omega \times \{0,1\} \to \omega$ witnesses $\mathcal{K} \leq_K \mathcal{I} \oplus \mathcal{J}$, then $f : \omega \to \omega$ given by $f(n) = h(n,0)$ for all $n \in \omega$, is a witness for $\mathcal{K} \leq_K \mathcal{I}$. Similarly one can show that $\mathcal{K} \leq_K \mathcal{I} \oplus \mathcal{J}$ implies $\mathcal{K} \leq_K \mathcal{J}$. □

Example 4.3. Let $\mathcal{I} = \mathcal{J} \oplus \mathcal{J}'$, where $\mathcal{J}, \mathcal{J}'$ are ideals such that $\mathcal{J} \leq_K \mathcal{J}'$ and $\mathcal{J}$ is weakly K-uniform. Then $\mathcal{I}$ is weakly K-uniform. Recall that, for example, $\mathcal{J} = \mathcal{W}$ and $\mathcal{J}' = \mathcal{I}_d$ satisfy these conditions.

Proof. For simplicity of notation, we will assume that both $\mathcal{J}$ and $\mathcal{J}'$ are ideals on ω.

Fix $B \subseteq \omega \times \{0,1\}$. Denote $B_0 = \{n \in \omega : (n,0) \in B\}$. Then $B_0 \cup (\omega \setminus B_0) = \omega$. Since $\mathcal{J}$ is weakly K-uniform, either $B_0 \in K(\mathcal{J})$ or $(\omega \setminus B_0) \in K(\mathcal{J})$. Without loss of generality, assume that $B_0 \in K(\mathcal{J})$. We will show that $B \in K(\mathcal{I})$.

It is easy to see that $\mathcal{I}|B \leq_K \mathcal{J}|B_0$ (it is witnessed by the function $f : B_0 \to B$ given by $f(n) = (n,0)$ for all $n \in B_0$). Observe that $\mathcal{J}|B_0 \leq_K \mathcal{J} \leq_K \mathcal{J}'$. Hence, from Lemma 4.2 we get $\mathcal{I}|B \leq_K \mathcal{J}|B_0 \leq_K \mathcal{I}$.

□

Now we want to show that some of the ideals presented in Example 4.3 are weakly K-uniform, but not K-uniform. Let us note that, by Proposition 2.2, existence of such ideals gives a negative answer to the following question posed by Hang Zhang and Shuguo Zhang.

Question 4.4. [10, Question 4] *Let $\mathcal{I}$ be a non-K-uniform ideal. Does there exist $A \in K(\mathcal{I})$ such that $B \notin K(\mathcal{I})$ and $C \notin K(\mathcal{I})$ for some B and C with $A = B \cup C$?*

Proposition 4.5. *Let $\mathcal{I} = \mathcal{J} \oplus \mathcal{J}'$, where $\mathcal{J}, \mathcal{J}'$ are ideals such that $\mathcal{J}' \not\leq_K \mathcal{J}$. Then $\mathcal{I}$ is not K-uniform.*

Proof. Since $\mathcal{J}' \not\leq_K \mathcal{J}$, we have $\mathcal{J}' \not\leq_K \mathcal{I}$ (by Lemma 4.2). This finishes the proof as $\mathcal{I}|(\omega \times \{1\})$ is isomorphic to $\mathcal{J}'$. $\square$

Corollary 4.6. *Let $\mathcal{I} = \mathcal{J} \oplus \mathcal{J}'$, where $\mathcal{J}, \mathcal{J}'$ are ideals such that $\mathcal{J} \leq_K \mathcal{J}'$, $\mathcal{J}' \not\leq_K \mathcal{J}$ and $\mathcal{J}$ is K-uniform. Then $\mathcal{I}$ is weakly K-uniform, but not K-uniform. In particular, $\mathcal{W} \oplus \mathcal{I}_d$ is weakly K-uniform, but not K-uniform.*

Proof. The ideal $\mathcal{I}$ is weakly K-uniform by Example 4.3, but it is not K-uniform by Proposition 4.5. $\square$

Remark. The notions of weakly K-uniform ideals and K-uniform ideals differ both among tall and non-tall ideals. Indeed, non-tall ideal $\mathrm{Fin} \oplus \mathcal{W}$ is weakly K-uniform by Example 4.1, but it is not K-uniform (as $\mathcal{W} \not\leq_K \mathrm{Fin} \oplus \mathcal{W}$). $\mathcal{W} \oplus \mathcal{I}_d$ is an example of tall ideal that is weakly K-uniform, but not K-uniform (by Corollary 4.6).

Next, we present some examples of ideals that are not weakly K-uniform.

Example 4.7. Let $\mathcal{I} = \mathcal{J} \oplus \mathcal{J}'$, where $\mathcal{J}, \mathcal{J}'$ are ideals such that $\mathcal{J} \not\leq_K \mathcal{J}'$ and $\mathcal{J}' \not\leq_K \mathcal{J}$. Then $\mathcal{I}$ is not weakly K-uniform.

Proof. The assumptions $\mathcal{J} \not\leq_K \mathcal{J}'$ and $\mathcal{J}' \not\leq_K \mathcal{J}$ give us $\mathcal{J} \not\leq_K \mathcal{I}$ and $\mathcal{J}' \not\leq_K \mathcal{I}$ (by Lemma 4.2). This finishes the proof, since it implies that $\bigcup \mathcal{J} \times \{0\} \notin K(\mathcal{I})$ and $\bigcup \mathcal{J}' \times \{1\} \notin K(\mathcal{I})$. $\square$

We will end this section by showing that tall summable ideals $\mathcal{I}_f$ are not weakly K-uniform. The proof of this fact is a bit more complex. We will need the following lemma and a characterization of the Katětov order among summable ideals.

Lemma 4.8. *Let $f : \omega \to [0, \infty)$ be such that $\sum_{n\in\omega} f(n) = \infty$ and $\lim_{n\to\infty} f(n) = 0$. Then there are sets $B, C \subseteq \omega$ such that:*

1. *$\omega = B \cup C$,*
2. *$\sum_{i\in B} f(i) = \infty$ and $\sum_{i\in C} f(i) = \infty$,*
3. *for every $M \in \omega$ there exist numbers $n, m \in \omega$ such that:*

$$\frac{\sum_{i=0}^{n} f(i)}{\sum_{i=0}^{n} f(b_i)} > M \text{ and } \frac{\sum_{i=0}^{m} f(i)}{\sum_{i=0}^{m} f(c_i)} > M,$$

where $(b_n)_{n\in\omega}$ and $(c_n)_{n\in\omega}$ are the increasing enumerations of B and C, respectively.

Proof. We will construct sets $A_1, A_2, \ldots$ and numbers $a_0, n_0, a_1, n_1, a_2, n_2 \ldots$ such that:

- $a_i + n_i < a_{i+1}$ for all $i \in \omega$;
- $A_1 = \{a_1, a_1 + 1, \ldots, a_1 + n_1\}$;
- $A_i = \{a_i, a_i + 1, \ldots, a_i + n_i\} \cup \{a_{i-2} + n_{i-2} + 1, \ldots, a_{i-1} - 1\}$ for $i \geq 2$;
- for each $i \geq 1$ we have:

$$\frac{\sum_{m=0}^{n_i} f(m)}{(\sum_{m=0}^{a_{i-1}} f(m)) + 1} > i + 1;$$

- for each $i \geq 1$ we have:

$$\sum_{m=a_i}^{a_i+n_i} f(m) < 1 \text{ and } \sum_{m=a_{i-1}+n_{i-1}+1}^{a_i-1} f(m) > 1.$$

Let $a_0 = -1$ and $n_0 = 0$. There exists n_1 such that:

$$\sum_{i=0}^{n_1} f(i) > 2 = 2 \cdot \left(\left(\sum_{i=0}^{a_0} f(i) \right) + 1 \right).$$

Let $a_1 \geq 1$ be such that:

$$\sum_{i=a_1}^{a_1+n_1} f(i) < 1 \text{ and } \sum_{i=0}^{a_1-1} f(i) > 1.$$

Define the set A_1 as $\{a_1, a_1+1, \ldots, a_1+n_1\}$.

Now, let us assume that $k \geq 1$ and we have already defined sets $A_1, \ldots, A_k$ and numbers $a_0, n_0, a_1, n_1, \ldots, a_k, n_k$. There exists n_{k+1} such that:

$$\frac{\sum_{i=0}^{n_{k+1}} f(i)}{(\sum_{i=0}^{a_k} f(i)) + 1} > k + 2.$$

Moreover, there exists a number $a_{k+1} > a_k + n_k$ such that:

$$\sum_{i=a_{k+1}}^{a_{k+1}+n_{k+1}} f(i) < 1 \text{ and } \sum_{i=a_k+n_k+1}^{a_{k+1}-1} f(i) > 1.$$

Let $A_{k+1} = \{a_{k+1}, \ldots, a_{k+1}+n_{k+1}\} \cup \{a_{k-1}+n_{k-1}+1, \ldots, a_k - 1\}$. This finishes our construction.

In the following, we will use the observation that $\max(A_i) = a_i + n_i < a_i + n_i + 1 = \min(A_{i+2})$ for each $i \geq 1$. Define

$$B = \bigcup_{n \in \omega \setminus \{0\}} A_{2n} \text{ and } C = \bigcup_{n \in \omega} A_{2n+1}.$$

Below, we show that those sets are as needed.

(1) Let $n \in \omega$. If $n < a_1$, then $n \in A_2$. On the other hand, if $n \geq a_1$, denote $m = \max\{k \in \omega : a_k \leq n\}$. Then $n \in A_m \cup A_{m+2}$. Hence, $n \in B \cup C$.

(2) We have $\sum_{i \in B} f(i) = \infty$, because $\sum_{m \in A_{2i}} f(i) > 1$ for all $i \geq 1$ and $A_{2i} \cap A_{2j} = \emptyset$ for $i \neq j$ (by the above observation). In a similar way it is possible to show that $\sum_{i \in C} f(i) = \infty$.

(3) Let $M \in \omega$. There exists $i \geq 1$ such that $2i + 1 > M$. Observe that $B \cap \{a_{2i-1}, \ldots, a_{2i} - 1\} = A_{2i} \cap \{a_{2i-1}, \ldots, a_{2i} - 1\} = \emptyset$ (by the above observation). Then

$$\begin{aligned} \frac{\sum_{m=0}^{n_{2i}} f(m)}{\sum_{m=0}^{n_{2i}} f(b_m)} &> \frac{\sum_{m=0}^{n_{2i}} f(m)}{\sum_{m=0}^{a_{2i-1}} f(m) + \sum_{m=a_{2i}}^{a_{2i}+n_{2i}} f(m)} \\ &> \frac{\sum_{m=0}^{n_{2i}} f(m)}{(\sum_{m=0}^{a_{2i-1}} f(m)) + 1} > 2i + 1 > M. \end{aligned}$$

In a similar way it is possible to show that the second inequality from (3) also holds.

□

Theorem 4.9. [4, Theorem 6.2] *Let $f : \omega \to (0, \infty)$ and $g : \omega \to (0, \infty)$ be two nonincreasing functions such that $\sum_{n \in \omega} f(n) = \sum_{n \in \omega} g(n) = \infty$. The following statements are equivalent:*

(1) *there is $M > 0$ such that*

$$\frac{\sum_{i=0}^{n} f(i)}{\sum_{i=0}^{n} g(i)} \leq M$$

for all $n \in \mathbb{N}$;

(2) $\mathcal{I}_g \leq_K \mathcal{I}_f$.

Theorem 4.10. *All tall summable ideals are not weakly K-uniform.*

Proof. Fix a tall summable ideal $\mathcal{I}_f$. Then $f : \omega \to [0, \infty)$ is such that $\sum_{n\in\omega} f(n) = \infty$ and $\lim_{n\to\infty} f(n) = 0$. Without loss of generality we can assume that $f(n) > 0$ for all $n \in \omega$ (if not, replace f with $\hat{f} : \omega \to (0, \infty)$ given by:

$$\hat{f}(n) = \begin{cases} f(n), & \text{if } f(n) > 0, \\ \frac{1}{2^n}, & \text{if } f(n) = 0, \end{cases}$$

and observe that $\mathcal{I}_{\hat{f}} = \mathcal{I}_f$, $\sum_{n\in\omega} \hat{f}(n) \geq \sum_{n\in\omega} f(n) = \infty$ and $\lim_{n\to\infty} \hat{f}(n) = 0$). Since being not weakly K-uniform is invariant over isomorphisms of ideals, we can additionally assume that f is nonincreasing.

Take sets B and C from Lemma 4.8 and let $(b_n)_{n\in\omega}$ and $(c_n)_{n\in\omega}$ be the increasing enumerations of B and C, respectively. Define function $g : \omega \to (0, \infty)$ as $g(n) = f(b_n)$ for all $n \in \omega$. Similarly, we can define function $h : \omega \to (0, \infty)$ as $h(n) = f(c_n)$ for all $n \in \omega$. Then g and h are also nonincreasing and such that $\sum_{n\in\omega} g(n) = \sum_{i\in B} f(n) = \infty$ and $\sum_{n\in\omega} h(n) = \sum_{i\in C} f(i) = \infty$. Observe that:

$$\mathcal{I}_g = \left\{ M \subseteq \omega : \sum_{n\in M} g(n) < \infty \right\} \cong \left\{ M \subseteq B : \sum_{n\in M} f(n) < \infty \right\} = \mathcal{I}_f | B$$

and similarly $\mathcal{I}_h \cong \mathcal{I}|C$. By Theorem 4.9 and the choice of sets B and C we obtain that $\mathcal{I}_f|B \not\leq_K \mathcal{I}_f$ and $\mathcal{I}_f|C \not\leq_K \mathcal{I}_f$. Since $B \cup C = \omega$, the ideal $\mathcal{I}_f$ is not weakly K-uniform (by Proposition 2.2). □

Corollary 4.11. *Let $\mathcal{I}_f$ be a tall summable ideal. Then it is nowhere weakly K-uniform, i.e. $\mathcal{I}_f|A$ is not weakly K-uniform for any $A \in \mathcal{I}_f^+$.*

Proof. It follows from Theorem 4.10 as all restrictions $\mathcal{I}_f|A$ for $A \in \mathcal{I}_f^+$ are tall summable ideals. □

Corollary 4.12. *Let $\mathcal{I}$ be a summable ideal. The following conditions are equivalent:*

(1) *$\mathcal{I}$ is weakly K-uniform;*
(2) *$\mathcal{I}$ is a non-tall ideal.*

Proof. (1) $\implies$ (2) follows from Corollary 4.11 and (2) $\implies$ (1) follows from Example 4.1. □

The following answers [2, Question 8.8(2)].

Corollary 4.13. *The summable ideal $\mathcal{I}_{\frac{1}{n}}$ is not K-uniform.*

Proof. It follows from Corollary 4.11, since all K-uniform ideals are weakly K-uniform. □

5. Functions preserving $\mathcal{I}$-convergence

Theorem 5.1. *Let $\mathcal{I}$ and $\mathcal{J}$ be ideals. The following conditions are equivalent:*

1. *for any $A \in \mathcal{I}^+$ there exists $B \in \mathcal{J}^+$ such that $\mathcal{I}|A \leq_K \mathcal{J}|B$;*
2. *every function that preserves $\mathcal{J}$-convergence also preserves $\mathcal{I}$-convergence.*

Proof. Without loss of generality we may assume that both $\mathcal{I}$ and $\mathcal{J}$ are ideals on ω.

(1) $\implies$ (2): Let $f : X \longrightarrow Y$ be a function that preserves $\mathcal{J}$-convergence. Suppose to the contrary that f does not preserve $\mathcal{I}$-convergence. Then there exist $x \in X$ and $(x_n)_{n\in\omega} \subseteq X$ such that $x_n \xrightarrow{\mathcal{I}} x$, but $(f(x_n))_{n\in\omega}$ does not $\mathcal{I}$-converge to $f(x)$. Thus, there exists $U_{f(x)}$, a neighborhood of $f(x)$, such that:

$$A = \{n \in \omega : f(x_n) \notin U_{f(x)}\} \in \mathcal{I}^+.$$

Notice that:

$$A = \left\{n \in \omega : f(x_n) \in Y \setminus U_{f(x)}\right\} = \left\{n \in \omega : x_n \in f^{-1}[Y \setminus U_{f(x)}]\right\}.$$

By our assumption, there exists $B \in \mathcal{J}^+$ such that $\mathcal{I}|A \leq_K \mathcal{J}|B$. Let $g : B \longrightarrow A$ be the function witnessing it. Define $y_n = x_{g(n)}$ if $n \in B$ and $y_n = x$ if $n \in \omega\setminus B$. We will show that the sequence $(y_n)_{n\in\omega}$ is $\mathcal{J}$-convergent to x. Let U_x be a neighborhood of x. We have:

$$\begin{aligned}\{n \in \omega : y_n \notin U_x\} &= \left\{n \in B : x_{g(n)} \notin U_x\right\}\\ &= \{n \in B : g(n) \in \{m \in \omega : x_m \notin U_x\} \cap A\}\\ &= g^{-1}\left[\{m \in \omega : x_m \notin U_x\} \cap A\right] \in \mathcal{J}|B\end{aligned}$$

as $\{m \in \omega : x_m \notin U_x\} \in \mathcal{I}$.

On the other hand:

$$\begin{aligned}\{n \in \omega : f(y_n) \notin U_{f(x)}\} &= \{n \in B : f(x_{g(n)}) \notin U_{f(x)}\}\\ &= \{n \in B : x_{g(n)} \in f^{-1}[Y \setminus U_{f(x)}]\}\\ &= \{n \in B : g(n) \in A\} = g^{-1}[A] = B \in \mathcal{J}^+,\end{aligned}$$

a contradiction with the fact that f preserves $\mathcal{J}$-convergence.

(2) $\implies$ (1): Suppose that there exists $A \in \mathcal{I}^+$ such that $\mathcal{I}|A \not\leq_K \mathcal{J}|B$ for all $B \in \mathcal{J}^+$. Let $f : X(\mathcal{I}|A) \longrightarrow X(\mathcal{J})$ be such that

$$f(x) = \begin{cases} 1, & \text{if } x \in A,\\ \omega, & \text{if } x = \omega.\end{cases}$$

Consider the sequence $(y_n)_{n\in\omega}$ given by $y_n = n$ if $n \in A$ and $y_n = \omega$ if $n \notin A$. Note that it is $\mathcal{I}$-convergent to ω in $X(\mathcal{I}|A)$. On the other hand, $(f(y_n))_{n\in\omega}$ is not $\mathcal{I}$-convergent to $f(\omega) = \omega$ in $X(\mathcal{J})$. Hence, f does not preserve $\mathcal{I}$-convergence.

Now, it remains to show that f preserves $\mathcal{J}$-convergence. Let $(x_n)_{n\in\omega}$ be a $\mathcal{J}$-convergent sequence in $X(\mathcal{I}|A)$. In the case $x_n \xrightarrow{\mathcal{J}} x \neq \omega$, it is easy to prove that $f(x_n) \xrightarrow{\mathcal{J}} f(x)$ in $X(\mathcal{J})$. Therefore, assume that $x_n \xrightarrow{\mathcal{J}} \omega$. Let $B = \{n \in \omega : x_n \neq \omega\}$. Suppose that $B \notin \mathcal{J}$. Then $\mathcal{I}|A \not\leq_K \mathcal{J}|B$, so there exists $C \in \mathcal{I}|A$ such that $\{n \in B : x_n \in C\} \notin \mathcal{J}|B$. But $\{n \in \omega : x_n \in C\} \in \mathcal{J}$ by $\mathcal{J}$-convergence of $(x_n)_{n\in\omega}$ to ω in $X(\mathcal{I}|A)$ – a contradiction. Thus, if a sequence $(x_n)_{n\in\omega}$ is $\mathcal{J}$-convergent to ω in $X(\mathcal{I}|A)$, then $B = \{n \in \omega : x_n \neq \omega\} \in \mathcal{J}$ and consequently $f(x_n) \xrightarrow{\mathcal{J}} \omega = f(\omega)$ in $X(\mathcal{J})$. Hence, f preserves $\mathcal{J}$-convergence.

□

The following are immediate consequences of Theorem 5.1.

Corollary 5.2. *Let $\mathcal{J}$ be an ideal. Then every function that preserves $\mathcal{J}$-convergence also preserves ordinary convergence, i.e., is sequentially continuous.*

Corollary 5.3. *The following conditions are equivalent for an ideal $\mathcal{I}$:*

1. *$\mathcal{I}$ is nowhere tall;*
2. *every sequentially continuous function preserves $\mathcal{I}$-convergence;*
3. *any function is sequentially continuous if and only if it preserves $\mathcal{I}$-convergence.*

6. Functions preserving $\mathcal{I}$-convergence versus $\mathcal{I}$-continuity

We start this section with a result characterizing when every $\mathcal{J}$-continuous function preserves $\mathcal{I}$-convergence. Next, we answer a certain question posed in [10]. At the end of this section we study the reversed implication, i.e., we characterize when preservation of $\mathcal{I}$-convergence implies $\mathcal{J}$-continuity.

Theorem 6.1. *Let $\mathcal{I}$ and $\mathcal{J}$ be ideals. The following conditions are equivalent:*

1. *$\mathcal{I}|A \leq_K \mathcal{J}$ for all $A \in \mathcal{I}^+$;*
2. *every $\mathcal{J}$-continuous function preserves $\mathcal{I}$-convergence.*

Proof. Without loss of generality we may assume that both $\mathcal{I}$ and $\mathcal{J}$ are ideals on ω.

(1) $\implies$ (2): Let $f : X \longrightarrow Y$ be a $\mathcal{J}$-continuous function and $(x_n)_{n\in\omega} \subseteq X$ be such that $x_n \xrightarrow{\mathcal{I}} x \in X$. We will show that $f(x_n) \xrightarrow{\mathcal{I}} f(x)$. Let $U_{f(x)} \subseteq Y$ be a neighborhood of $f(x)$. Then $Y \setminus U_{f(x)}$ is closed, so it is also $\mathcal{J}$-closed (as every closed set is $\mathcal{J}$-closed). Since f is $\mathcal{J}$-continuous, $f^{-1}[Y \setminus U_{f(x)}]$ is $\mathcal{J}$-closed.

We will show that

$$A = \{n \in \omega : x_n \in f^{-1}[Y \setminus U_{f(x)}]\} \in \mathcal{I}.$$

Suppose towards a contradiction that $A \notin \mathcal{I}$. By our assumption, $\mathcal{I}|A \leq_K \mathcal{J}$. Hence, there exists $g : \omega \longrightarrow A$ such that $g^{-1}[M] \in \mathcal{J}$ for any $M \in \mathcal{I}|A$. We will show that $x_{g(n)} \xrightarrow{\mathcal{J}} x$. Let U_x be a neighborhood of x. Then we have:

$$\begin{aligned}\{n \in \omega : x_{g(n)} \notin U_x\} &= \{n \in \omega : g(n) \in \{m \in \omega : x_m \notin U_x\} \cap A\} \\ &= g^{-1}[\{m \in \omega : x_m \notin U_x\} \cap A] \in \mathcal{J}\end{aligned}$$

as $\{m \in \omega : x_m \notin U_x\} \in \mathcal{I}$. Thus, $x_{g(n)} \xrightarrow{\mathcal{J}} x$. Since $f^{-1}[Y \setminus U_{f(x)}]$ is $\mathcal{J}$-closed, $(x_{g(n)})_{n\in\omega} \subseteq f^{-1}[Y \setminus U_{f(x)}]$ and $x \notin f^{-1}[Y \setminus U_{f(x)}]$, we obtain a contradiction. Therefore $A \in \mathcal{I}$.

Observe that:

$$\{n \in \omega : f(x_n) \notin U_{f(x)}\} = \{n \in \omega : f(x_n) \in Y \setminus U_{f(x)}\} = A \in \mathcal{I}.$$

Hence, f preserves $\mathcal{I}$-convergence.

(2) $\implies$ (1): Suppose that there is $A \in \mathcal{I}^+$ such that $\mathcal{I}|A \not\leq_K \mathcal{J}$. Define a function $f : X(\mathcal{I}) \longrightarrow X(\mathcal{I})$ by

$$f(x) = \begin{cases} 1, & \text{if } x \in A, \\ \omega, & \text{if } x \notin A. \end{cases}$$

We will show that f is $\mathcal{J}$-continuous but fails to preserve $\mathcal{I}$-convergence.

It is easy to see that the sequence $(x_n)_{n\in\omega}$ given by $x_n = n$ for all $n \in \omega$ is such that $(x_n)_{n\in\omega} \xrightarrow{\mathcal{I}} \omega$ in $X(\mathcal{I})$. On the other hand, the sequence $(f(x_n))_{n\in\omega}$ is not $\mathcal{I}$-convergent to $f(\omega) = \omega$ in $X(\mathcal{I})$, because $A \notin \mathcal{I}$. Therefore, f does not preserve $\mathcal{I}$-convergence.

Let B be a $\mathcal{J}$-closed set in $X(\mathcal{I})$. If $\omega \in f^{-1}[B]$, then $f^{-1}[B]$ is $\mathcal{J}$-closed. In the case $\omega \notin f^{-1}[B]$ we have $\omega \notin B$, so either $f^{-1}[B] = \emptyset$ or $f^{-1}[B] = A$. Since $\mathcal{I}|A \not\leq_K \mathcal{J}$, it follows from Lemma 2.6 that the set A is $\mathcal{J}$-closed in $X(\mathcal{I})$. Obviously, $\emptyset$ is also $\mathcal{J}$-closed. Hence, f is $\mathcal{J}$-continuous.

□

The following are immediate consequences of Theorem 6.1.

Corollary 6.2. *Let $\mathcal{I}$ be an ideal. The following conditions are equivalent:*

1. *$\mathcal{I}$ is K-uniform;*
2. *any $\mathcal{I}$-continuous function preserves $\mathcal{I}$-convergence.*

Corollary 6.3. *Let $\mathcal{I}$ be an ideal. Then every $\mathcal{I}$-continuous function is sequentially continuous.*

Proof. Follows from the above theorem combined with the facts that for any infinite $A \subseteq \omega$ we have $\mathrm{Fin}|A \leq_K \mathcal{I}$ and that a function is sequentially continuous if and only if it preserves Fin-convergence. □

Now we give a positive answer to the following question posed by Hang Zhang and Shuguo Zhang.

Question 6.4. [10, Question 5] *Do there exist tall ideals $\mathcal{I}$ and $\mathcal{J}$ and a function $\phi : \bigcup\mathcal{I} \longrightarrow \bigcup\mathcal{I}$ such that $\tilde{\phi} : X(\mathcal{I}) \longrightarrow X(\mathcal{I})$ given by:*

$$\tilde{\phi} = \begin{cases} \phi(x), & \text{if } x \neq \omega, \\ \omega, & \text{if } x = \omega, \end{cases}$$

is $\mathcal{J}$-continuous but does not preserve $\mathcal{J}$-convergence?

Before answering this question, we need to prove a certain lemma.

Lemma 6.5. *If $\mathcal{I}$ and $\mathcal{J}$ are ideals such that $\mathcal{I} \not\leq_K \mathcal{J}$, $\mathcal{J} \leq_K \mathcal{I}$ and $\mathcal{J}$ is K-uniform, then the following conditions are equivalent for any $A \subseteq \bigcup\mathcal{I} \times \{0\} \cup \bigcup\mathcal{J} \times \{1\}$:*

1. *$\mathcal{J}|A_1 \not\leq_K \mathcal{I} \oplus \mathcal{J}$;*
2. *$A \notin K(\mathcal{I} \oplus \mathcal{J})$;*
3. *$A_1 \in \mathcal{J}$;*

where by A_1 we denote the set $\{n \in \bigcup\mathcal{J} : (n,1) \in A\}$.

Proof. (2) $\implies$ (1): Obvious.

(1) $\implies$ (3): Assume that $A_1 \in \mathcal{J}^+$. Since $\mathcal{J}$ is K-uniform, we get $\mathcal{J}|A_1 \leq_K \mathcal{J} \leq_K \mathcal{I} \oplus \mathcal{J}$ (by Lemma 4.2 and the facts that $\mathcal{J} \leq_K \mathcal{I}$ and $\mathcal{J} \leq_K \mathcal{J}$).

(3) $\implies$ (2): Assume that $A_1 \in \mathcal{J}$. It is enough to prove that $\mathcal{I} \oplus (\mathcal{J}|A_1) \not\leq_K \mathcal{I} \oplus \mathcal{J}$, because it implies $(\mathcal{I} \oplus \mathcal{J})|A \not\leq_K \mathcal{I} \oplus \mathcal{J}$. There are two possible cases:

Case 1: A_1 is finite. Then $\mathcal{I} \oplus (\mathcal{J}|A_1)$ is isomorphic to $\mathcal{I}$.

Case 2: A_1 is infinite. Observe that $\mathcal{I} \oplus (\mathcal{J}|A_1)$ is isomorphic to $\mathcal{I} \oplus \mathcal{P}(\omega)$. Since $\mathcal{I} \not\leq_K \mathcal{J}$, $\mathcal{I}$ has to be tall, so $\mathcal{I} \oplus \mathcal{P}(\omega)$ (and consequently also $\mathcal{I} \oplus (\mathcal{J}|A_1)$) is isomorphic to $\mathcal{I}$.

In both cases, $\mathcal{I} \oplus (\mathcal{J}|A_1)$ is isomorphic to $\mathcal{I}$. Since $\mathcal{I} \not\leq_K \mathcal{I} \oplus \mathcal{J}$ (by $\mathcal{I} \not\leq_K \mathcal{J}$ and Lemma 4.2), we obtain $\mathcal{I} \oplus (\mathcal{J}|A_1) \not\leq_K \mathcal{I} \oplus \mathcal{J}$.

□

Now we are ready to answer Question 6.4.

Proposition 6.6. *There exists a tall ideal $\mathcal{K}$ and a function $\phi : \bigcup \mathcal{K} \longrightarrow \bigcup \mathcal{K}$ such that $\tilde{\phi} : X(\mathcal{K}) \longrightarrow X(\mathcal{K})$ is $\mathcal{K}$-continuous and does not preserve $\mathcal{K}$-convergence.*

Proof. Let $\mathcal{I}$ and $\mathcal{J}$ be tall ideals such that $\mathcal{J} \leq_K \mathcal{I}$, $\mathcal{I} \not\leq_K \mathcal{J}$ and $\mathcal{J}$ is K-uniform (for example, $\mathcal{I} = \mathcal{I}_d$ and $\mathcal{J} = \mathcal{W}$). Define $\mathcal{K} = \mathcal{I} \oplus \mathcal{J}$. Clearly, $\mathcal{K}$ is a tall ideal. Let $\phi : \omega \times \{0,1\} \longrightarrow \omega \times \{0,1\}$ be such that $\phi(n,1) = (n,1)$ and $\phi(n,0) = (0,0)$ for all $n \in \omega$.

We need to show that the function $\tilde{\phi}$ is $\mathcal{K}$-continuous but does not preserve $\mathcal{K}$-convergence. We begin by proving that $\tilde{\phi}$ is $\mathcal{K}$-continuous. Fix a $\mathcal{K}$-closed set $A \subseteq X(\mathcal{K})$. If $\omega \in A$, then we have $\omega \in \tilde{\phi}^{-1}[A]$, so $\tilde{\phi}^{-1}[A]$ is $\mathcal{K}$-closed in $X(\mathcal{K})$. Now we need to consider the case $\omega \notin A$. It follows from Lemma 2.6 that $\mathcal{K}|A \not\leq_K \mathcal{K}$. By Lemma 6.5 and the way the function $\tilde{\phi}$ is defined, we have:

$$\{n \in \omega : (n,1) \in \tilde{\phi}^{-1}[A]\} = \{n \in \omega : (n,1) \in A\} \in \mathcal{J}.$$

Using Lemma 6.5 once again, $\mathcal{K}|\tilde{\phi}^{-1}[A] \not\leq_K \mathcal{K}$. Applying Lemma 2.6 one more time, we obtain that $\tilde{\phi}^{-1}[A]$ is $\mathcal{K}$-closed in $X(\mathcal{K})$.

It remains to show that $\tilde{\phi}$ does not preserve $\mathcal{K}$-convergence. It is clear that the sequence $(x_{n,i})_{(n,i)\in\omega\times\{0,1\}}$ given by $x_{n,i} = (n,i)$ for all $n \in \omega$ and $i \in \{0,1\}$ is $\mathcal{K}$-convergent to ω in $X(\mathcal{K})$. On the other hand, the sequence $(\tilde{\phi}(x_{n,i}))_{(n,i)\in\omega\times\{0,1\}}$ is not $\mathcal{K}$-convergent to $\tilde{\phi}(\omega) = \omega$. Indeed, it follows from the fact that $\{(n,i) \in \omega \times \{0,1\} : \tilde{\phi}(x_{n,i}) = (0,0)\} = \omega \times \{0\} \notin \mathcal{K}$. Hence, $\tilde{\phi}$ fails to preserve $\mathcal{K}$-convergence. □

We end this section with the following characterization.

Theorem 6.7. *Let $\mathcal{I}$ and $\mathcal{J}$ be any ideals. The following conditions are equivalent:*

1. *every function that preserves $\mathcal{J}$-convergence is $\mathcal{I}$-continuous;*
2. *every function defined on $X(\mathcal{I})$ that preserves $\mathcal{J}$-convergence is $\mathcal{I}$-continuous;*
3. *for every ideal $\mathcal{M}$ on $\bigcup \mathcal{I}$ such that $\mathcal{M} \not\leq_K \mathcal{I}$ there exists $B \notin \mathcal{J}$ and a function h that is a witness for $\mathcal{I} \leq_K \mathcal{J}|B$ and is not a witness for $\mathcal{M} \leq_K \mathcal{J}|B$.*

Proof. Without loss of generality, we will assume that both $\mathcal{I}$ and $\mathcal{J}$ are ideals on ω.

(1) $\implies$ (2): Obvious.

(2) $\implies$ (3): Let $\mathcal{M}$ be such an ideal that $\mathcal{M} \not\leq_K \mathcal{I}$ and every function h that is a witness for $\mathcal{I} \leq_K \mathcal{J}|B$, for some $B \notin \mathcal{J}$, is also a witness for $\mathcal{M} \leq_K \mathcal{J}|B$.

Consider the function $f : X(\mathcal{I}) \to X(\mathcal{M})$ given by $f(x) = x$ for all $x \in X(\mathcal{I})$. Since $\mathcal{I} \leq_K \mathcal{I}$ and $\mathcal{M} \not\leq_K \mathcal{I}$, by Lemma 6.5 we have that the set ω is $\mathcal{I}$-closed in $X(\mathcal{M})$ while $f^{-1}[\omega] = \omega$ is not $\mathcal{I}$-closed in $X(\mathcal{I})$. Hence, f is not $\mathcal{I}$-continuous.

On the other hand, take any sequence $(h(n))_{n\in\omega}$ of elements in $X(\mathcal{I})$ that is $\mathcal{J}$-convergent to some $x \in X(\mathcal{I})$. If $\{n \in \omega : h(n) \neq x\} \in \mathcal{J}$, then clearly $f(h(n)) \xrightarrow{\mathcal{J}} f(x)$. Thus, we can assume that $x = \omega$ and $B = \{n \in \omega : h(n) \neq \omega\} \notin \mathcal{J}$. Therefore, we obtain that for every $A \in \mathcal{I}$ we have $h^{-1}[A] = \{n \in \omega : h(n) \in A\} \in$

$\mathcal{J}$. It follows that $h \restriction B$ is a witness for $\mathcal{I} \leq_K \mathcal{J}|B$, thus $h \restriction B$ is also a witness for $\mathcal{M} \leq_K \mathcal{J}|B$. Therefore, for every $A \in \mathcal{M}$ we get

$$\{n \in B : f(h(n)) \in A\} = \{n \in B : h(n) \in A\} = (h \restriction B)^{-1}[A] \in \mathcal{J}|B.$$

Since $f(h(n)) = \omega$ for every $n \notin B$, it follows that $f(h(n)) \xrightarrow{\mathcal{J}} \omega = f(x)$. Thus, f preserves $\mathcal{J}$-convergence.

(3) $\implies$ (1): Let $f : X \to Y$ be a function that preserves $\mathcal{J}$-convergence, but is not $\mathcal{I}$-continuous. Since f is not $\mathcal{I}$-continuous, there exists an $\mathcal{I}$-closed set $A \subseteq Y$ such that $f^{-1}[A]$ is not $\mathcal{I}$-closed. It follows that there exists a sequence $(x_n)_{n\in\omega}$ of elements in $f^{-1}[A]$ that is $\mathcal{I}$-convergent to some $x \notin f^{-1}[A]$.

Let $\mathcal{M}$ be the ideal generated by

$$\mathcal{B} = \{\{n \in \omega : f(x_n) \notin U\} : U \text{ is a neighborhood of } f(x)\}.$$

Family $\mathcal{B}$ is closed under taking finite unions of its elements, so $\mathcal{M} = \{A \subseteq \omega : \exists_{B\in\mathcal{B}}\ A \subseteq B\}$. We will show that $\mathcal{M} \not\leq_K \mathcal{I}$. Fix any $\phi : \omega \to \omega$. Observe that $f(x_{\phi(n)})$ is not $\mathcal{I}$-convergent to $f(x)$. Indeed, otherwise A would not be $\mathcal{I}$-closed as all $f(x_{\phi(n)})$ are elements of A while $f(x) \notin A$. Hence, there exists a neighborhood V of the point $f(x)$ such that $\{n \in \omega : f(x_{\phi(n)}) \notin V\} \notin \mathcal{I}$. Now, note that

$$\{n \in \omega : f(x_{\phi(n)}) \notin V\} = \phi^{-1}[\{n \in \omega : f(x_n) \notin V\}],$$

thus there exists $C = \{n \in \omega : f(x_n) \notin V\} \in \mathcal{M}$ such that $\phi^{-1}[C] \notin \mathcal{I}$. Therefore, $\mathcal{M} \not\leq_K \mathcal{I}$.

Next, let h be a function that is a witness for $\mathcal{I} \leq_K \mathcal{J}|B$, for some $B \notin \mathcal{J}$. Consider the sequence $(y_n)_{n\in\omega}$ given by $y_n = x_{h(n)}$ for $n \in B$ and $y_n = x$ otherwise. Then for any neighborhood U of x we have:

$$\{n \in \omega : y_n \notin U\} = \{n \in B : x_{h(n)} \notin U\} = h^{-1}[\{n \in \omega : x_n \notin U\}].$$

Since $x_n \xrightarrow{\mathcal{I}} x$, $\{n \in \omega : x_n \notin U\} \in \mathcal{I}$. Thus, $\{n \in \omega : y_n \notin U\} \in \mathcal{J}$, as h is a witness for $\mathcal{I} \leq_K \mathcal{J}|B$. Hence, $y_n \xrightarrow{\mathcal{J}} x$. Since f preserves $\mathcal{J}$-convergence, $f(y_n) \xrightarrow{\mathcal{J}} f(x)$. Now, for any $C \in \mathcal{M}$ there exists a neighborhood U of $f(x)$ such that:

$$C \subseteq \{n \in \omega : f(x_n) \notin U\}.$$

Thus,

$$\begin{aligned} h^{-1}[C] &\subseteq h^{-1}[\{n \in \omega : f(x_n) \notin U\}] \\ &= \{n \in B : f(x_{h(n)}) \notin U\} \\ &= \{n \in B : f(y_n) \notin U\} \in \mathcal{J}|B. \end{aligned}$$

Therefore, h is a witness for $\mathcal{M} \leq_K \mathcal{J}|B$. □

Corollary 6.8. *Let $\mathcal{I}$ be an ideal. The following conditions are equivalent:*

1. *any function is sequentially continuous if and only if it is $\mathcal{I}$-continuous;*
2. *every sequentially continuous functions is $\mathcal{I}$-continuous;*
3. *$\mathcal{I}$ is not tall.*

Proof. (1) $\implies$ (2) is obvious, while (2) $\implies$ (1) follows from Corollary 6.3. In the proof of (2) $\iff$ (3) we will use the characterization from Theorem 6.7.

(2) $\implies$ (3): If $\mathcal{I}$ is tall, then $\mathcal{I} \not\leq_K \mathrm{Fin}$. Thus, we can pick as $\mathcal{M}$ any ideal such that $\mathcal{M} \not\leq_K \mathcal{I}$ and for any $B \notin \mathrm{Fin}$ there will be no witness for $\mathcal{I} \leq_K \mathrm{Fin}|B$.

(3) $\implies$ (2): Let $A \subseteq \bigcup \mathcal{I}$ be such that $\mathcal{I}|A \cong \mathrm{Fin}$. Let $\mathcal{M}$ be any ideal such that $\mathcal{M} \not\leq_K \mathcal{I}$. Then $\mathcal{M} \not\leq_K \mathrm{Fin}$. It remains to notice that for $B = \omega \notin \mathrm{Fin}$ any injection $h : \omega \to A$ is a witness for $\mathcal{I} \leq_K \mathrm{Fin}$ that cannot be a witness for $\mathcal{M} \leq_K \mathrm{Fin}$, because $\mathcal{M} \not\leq_K \mathrm{Fin}$. □

7. $\mathcal{I}$-CONTINUITY VERSUS $\mathcal{J}$-CONTINUITY

In this section we study when $\mathcal{J}$-continuity implies $\mathcal{I}$-continuity.

Theorem 7.1. *Let $\mathcal{I}$ and $\mathcal{J}$ be any ideals and Y be a topological space. The following conditions are equivalent:*

(a) *there is a $\mathcal{J}$-continuous function with codomain Y, which is not $\mathcal{I}$-continuous;*
(b) *there is a $\mathcal{J}$-continuous $f : X(\mathcal{I}) \to Y$, which is not $\mathcal{I}$-continuous;*
(c) *there is a $\mathcal{J}$-continuous $f : X(\mathcal{I}) \to Y$ and an $\mathcal{I}$-closed set $A \subseteq Y$ such that $f^{-1}[A] = \bigcup \mathcal{I}$.*

Proof. The implications (c) $\implies$ (b) and (b) $\implies$ (a) are obvious (as $\bigcup \mathcal{I}$ is not $\mathcal{I}$-closed in $X(\mathcal{I})$), so we only need to prove (a) $\implies$ (c). Without loss of generality, we may assume that both $\mathcal{I}$ and $\mathcal{J}$ are ideals on ω.

Suppose that there are a topological space X and a $\mathcal{J}$-continuous function $g : X \to Y$, which is not $\mathcal{I}$-continuous. Then there is an $\mathcal{I}$-closed subset A of Y such that $g^{-1}[A]$ is not $\mathcal{I}$-closed in X. Hence, there is a sequence $h : \omega \to g^{-1}[A]$ which is $\mathcal{I}$-convergent to some $x \in X \setminus g^{-1}[A]$. Define $f : X(\mathcal{I}) \to Y$ by $f(\omega) = g(x)$ and $f(n) = g(h(n))$ for all $n \in \omega$. Then f is not $\mathcal{I}$-continuous as the set A is $\mathcal{I}$-closed in Y, but its preimage $f^{-1}[A] = h^{-1}[g^{-1}[A]] = \omega$ is not $\mathcal{I}$-closed in $X(\mathcal{I})$ (by Lemma 2.6).

To finish the proof, we need to show that f is $\mathcal{J}$-continuous. Fix a $\mathcal{J}$-closed set $B \subseteq Y$. If $g(x) \in B$, then $\omega \in f^{-1}[B]$, so $f^{-1}[B]$ is $\mathcal{J}$-closed in $X(\mathcal{I})$. Hence, suppose that $g(x) \notin B$ (so $\omega \notin f^{-1}[B]$) and fix a sequence $(z_n)_{n\in\omega}$ in $f^{-1}[B] = h^{-1}[g^{-1}[B]]$ which is $\mathcal{J}$-convergent to some $l \in X(\mathcal{I})$. We claim that $l \neq \omega$. This will finish the proof as it means that $\{n \in \omega : z_n \neq l\} \in \mathcal{J}$, so $l \in f^{-1}[B]$.

Assume towards a contradiction that $(z_n)_{n\in\omega}$ is $\mathcal{J}$-convergent to ω. For any neighborhood U of x in X we have $C = \{n \in \omega : h(n) \notin U\} \in \mathcal{I}$ (as h is $\mathcal{I}$-convergent to x), so $\{n \in \omega : h(z_n) \notin U\} = \{n \in \omega : z_n \in C\} \in \mathcal{J}$ (as $X(\mathcal{I}) \setminus C$ is a neighborhood of ω in $X(\mathcal{I})$). Hence, $(h(z_n))_{n\in\omega}$ is a sequence in $g^{-1}[B]$ which $\mathcal{J}$-converges to $x \notin g^{-1}[B]$, which contradicts $\mathcal{J}$-continuity of g. □

Definition 7.2. For a countable set Ω denote $\mathcal{R}^\Omega_0 = \{\{0\}\}$ and

$$\mathcal{R}^\Omega_\alpha = \left\{ \bigcup_{i\in\Omega} \{i\} \times R_i : (R_i)_{i\in\Omega} \text{ is a sequence in } \bigcup_{\beta<\alpha} \mathcal{R}^\Omega_\beta \right\}$$

for all $0 < \alpha < \omega_1$ (in particular, $\mathcal{R}^\Omega_1 = \{\Omega \times \{0\}\}$). For simplicity of notation, we will write $\mathcal{R}_\alpha$ instead of $\mathcal{R}^\omega_\alpha$.

Definition 7.3. Let $\mathcal{J}$ be any ideal on Ω. Inductively define ideals $\mathcal{J}_R$ on R for all $R \in \bigcup_{\alpha<\omega_1} \mathcal{R}^\Omega_\alpha$ in the following way. If $R = \{0\} \in \mathcal{R}^\Omega_0$, then let $\mathcal{J}_R = \{\emptyset\}$ (in this case $\mathcal{J}_R$ does not contain all finite subsets of R). If $R = \bigcup_{i\in\Omega}\{i\} \times R_i \in \mathcal{R}^\Omega_\alpha$ for some $\alpha < \omega_1$ and all $\mathcal{J}_{R'}$ for $R' \in \bigcup_{\beta<\alpha} \mathcal{R}^\Omega_\beta$ are defined, let

$$\mathcal{J}_R = \left\{ A \subseteq R : \left\{ i \in \Omega : A_{(i)} \notin \mathcal{J}_{R_i} \right\} \in \mathcal{J} \right\},$$

where $A_{(i)} = \{r \in R_i : (i, r) \in A\}$.

Definition 7.4. For two ideals $\mathcal{I}$ and $\mathcal{J}$ write $\mathcal{I} \preceq \mathcal{J}$ if $\mathcal{I} \leq_K \mathcal{J}_R$ for some $R \in \bigcup_{\alpha<\omega_1} \mathcal{R}_\alpha^{\bigcup \mathcal{J}}$.

Definition 7.5. Let $\mathcal{I}$ and $\mathcal{J}$ be ideals. We say that the pair $(\mathcal{I}, \mathcal{J})$ satisfies:

- condition (C1) if there is a (possibly improper) ideal $\mathcal{M}$ on $\bigcup \mathcal{I}$ such that $\mathcal{M} \not\leq_K \mathcal{I}$ and for every $B \subseteq \bigcup \mathcal{I}$ with $\mathcal{M}|B \not\leq_K \mathcal{J}$ we have $\mathcal{I}|B \not\leq_K \mathcal{J}$;
- condition (C2) if there is a (possibly improper) ideal $\mathcal{M}$ on $\bigcup \mathcal{I}$ such that $\mathcal{M} \not\leq_K \mathcal{I}$ and for every $B \subseteq \bigcup \mathcal{I}$ with $\mathcal{M}|B \not\preceq \mathcal{J}$ we have $\mathcal{I}|B \not\leq_K \mathcal{J}$.

Remark. Note that the only difference between (C1) and (C2) is the replacement of $\mathcal{M}|B \not\leq_K \mathcal{J}$ with $\mathcal{M}|B \not\preceq \mathcal{J}$. We will see in Theorems 7.7 and 7.9 that:

- if a pair $(\mathcal{I}, \mathcal{J})$ satisfies condition (C1), then there is a $\mathcal{J}$-continuous function, which is not $\mathcal{I}$-continuous;
- if there is a $\mathcal{J}$-continuous function, which is not $\mathcal{I}$-continuous, then the pair $(\mathcal{I}, \mathcal{J})$ satisfies condition (C2).

Lemma 7.6. *Let $\mathcal{K}$ be an ideal (possibly improper), $f : \omega \to \bigcup \mathcal{K}$ and $\mathcal{M}$ be the ideal on ω generated by all sets of the form $f^{-1}[C]$ for $C \in \mathcal{K}$. Then for every $B \subseteq \omega$, ideals $\mathcal{M}|B$ and $\mathcal{K}|f[B]$ are K-equivalent.*

Proof. Family $\{f^{-1}[C] : C \in \mathcal{K}\}$ is closed under taking finite unions of its elements. Hence, $\mathcal{M} = \{A : \exists_{C \in \mathcal{K}}\ A \subseteq f^{-1}[C]\}$. The function $f \restriction B$ witnesses that $\mathcal{K}|f[B] \leq_K \mathcal{M}|B$. Thus, we only need to show $\mathcal{M}|B \leq_K \mathcal{K}|f[B]$. Let $g : f[B] \to B$ be any function such that $g(i) \in f^{-1}[\{i\}] \cap B$ for all $i \in f[B]$. Observe that $f(g(i)) = i$ for all $i \in f[B]$. We will show that g is a witness for $\mathcal{M}|B \leq_K \mathcal{K}|f[B]$. It suffices to check that the preimages (under g) of sets of the form $f^{-1}[C] \cap B$, for $C \in \mathcal{K}$, belong to $\mathcal{K}$. Actually, we will show that $g^{-1}[f^{-1}[C] \cap B] \subseteq C \in \mathcal{K}$. Fix $i \in g^{-1}[f^{-1}[C] \cap B]$. Then $g(i) \in f^{-1}[C] \cap B$, so using the above observation we get $i = f(g(i)) \in C$. □

Theorem 7.7. *Let $\mathcal{I}$ and $\mathcal{J}$ be any ideals. The following conditions are equivalent:*

(a) *there are ideals $\mathcal{K}$ and $\mathcal{K}'$ and a function $f : X(\mathcal{K}') \to X(\mathcal{K})$, which is $\mathcal{J}$-continuous, but not $\mathcal{I}$-continuous;*
(b) *the pair $(\mathcal{I}, \mathcal{J})$ satisfies condition (C1).*

Proof. Without loss of generality, we may assume that all considered ideals are on ω. In this proof we will often use Lemma 2.6 and the remark below it without any reference.

(b) $\Longrightarrow$ (a): Assume that the pair $(\mathcal{I}, \mathcal{J})$ satisfies condition (C1). Let $\mathcal{K}' = \mathcal{I}$, $\mathcal{K} = \mathcal{M}$ and $f : X(\mathcal{I}) \to X(\mathcal{M})$ be the identity function. Since $\mathcal{M} \not\leq_K \mathcal{I}$, the set ω is $\mathcal{I}$-closed in $X(\mathcal{M})$. However, its preimage $f^{-1}[\omega] = \omega$ is not $\mathcal{I}$-closed in $X(\mathcal{I})$ (as $\mathcal{I} \leq_K \mathcal{I}$). Hence, f is not $\mathcal{I}$-continuous. Now we show that it is $\mathcal{J}$-continuous. Let $B \subseteq X(\mathcal{M})$ be $\mathcal{J}$-closed in $X(\mathcal{M})$. If $\omega \in B$, then also $\omega \in f^{-1}[B]$, so $f^{-1}[B]$ is $\mathcal{J}$-closed in $X(\mathcal{I})$. On the other hand, if $\omega \notin B$, then from $\mathcal{J}$-closedness of B we get that $\mathcal{M}|B \not\leq_K \mathcal{J}$, so our assumption gives us $\mathcal{I}|f^{-1}[B] = \mathcal{I}|B \not\leq_K \mathcal{J}$. Hence, $f^{-1}[B]$ is $\mathcal{J}$-closed in $X(\mathcal{I})$.

(a) $\Longrightarrow$ (b): Observe that if $\mathcal{I} \not\leq_K \mathcal{J}$, we put as $\mathcal{M}$ any ideal such that $\mathcal{M} \not\leq_K \mathcal{I}$ and then for any $B \subseteq \omega$ we have $\mathcal{I}|B \not\leq_K \mathcal{J}$ (as otherwise we would have $\mathcal{I} \leq_K \mathcal{I}|B \leq_K \mathcal{J}$, which contradicts $\mathcal{I} \not\leq_K \mathcal{J}$). Thus, we can assume that $\mathcal{I} \leq_K \mathcal{J}$.

By Theorem 7.1, without loss of generality we may assume that $f : X(\mathcal{I}) \to X(\mathcal{K})$ and there is an $\mathcal{I}$-closed set $A \subseteq X(\mathcal{K})$ such that $f^{-1}[A] = \omega$. Observe that

$\omega \notin A$ as otherwise A would be $\mathcal{J}$-closed in $X(\mathcal{K})$, but its preimage $f^{-1}[A] = \omega$ would not be $\mathcal{J}$-closed in $X(\mathcal{I})$ (by $\mathcal{I} \leq_K \mathcal{J}$), contradicting $\mathcal{J}$-continuity of f.

Let $\mathcal{M}$ be the ideal generated by all sets of the form $f^{-1}[C]$ for $C \in \mathcal{K}$. Observe that $\mathcal{K}|A \not\leq_K \mathcal{I}$ (as A is $\mathcal{I}$-closed in $X(\mathcal{K})$ and $\omega \notin A$). Then also $\mathcal{K}|f[\omega] \not\leq_K \mathcal{I}$ (as $f[\omega] \subseteq A$). Therefore, $\mathcal{M} = \mathcal{M}|\omega \not\leq_K \mathcal{I}$ by Lemma 7.6. Fix now any $B \subseteq \omega$ such that $\mathcal{M}|B \not\leq_K \mathcal{J}$. By Lemma 7.6 we get that $\mathcal{K}|f[B] \not\leq_K \mathcal{J}$, so $f[B]$ is $\mathcal{J}$-closed in $X(\mathcal{K})$. Since f is $\mathcal{J}$-continuous, $f^{-1}[f[B]]$ has to be $\mathcal{J}$-closed in $X(\mathcal{I})$. Hence, $\mathcal{I}|f^{-1}[f[B]] \not\leq_K \mathcal{J}$ (recall that $f[B] \subseteq f[\omega] \subseteq A$, so $f^{-1}[f[B]] \subseteq f^{-1}[A] = \omega$). Since $B \subseteq f^{-1}[f[B]]$, we obtain $\mathcal{I}|B \not\leq_K \mathcal{J}$. □

Lemma 7.8. *Let Y be a topological space, $D \subseteq Y$ be a countable set and $\mathcal{J}$ be an ideal. Then for every $y \in Y$, if y belongs to the $\mathcal{J}$-closure of D, then $\mathcal{L}_{y,D} \preceq \mathcal{J}$, where $\mathcal{L}_{y,D}$ is the (possibly not containing all finite sets) ideal on D generated by $D \setminus U$ for open neighborhoods U of y.*

Proof. Without loss of generality, we may assume that $\mathcal{J}$ is an ideal on ω. By Lemma 3.6, each $y \in cl_{\mathcal{J}}(D)$ belongs to $D^{(\alpha)}$ for some $\alpha < \omega_1$, where $D^{(0)} = D$ and

$$D^{(\alpha)} = \left\{ x \in X : x \text{ is the } \mathcal{J}\text{-limit of some sequence } (x_n)_{n\in\omega} \text{ in } \bigcup_{\beta<\alpha} D^{(\beta)} \right\}.$$

Thus, it is enough to inductively show for every $\alpha < \omega_1$ the following: for each $z \in D^{(\alpha)}$ there is $R \in \mathcal{R}_\alpha$ such that $\mathcal{L}_{z,D} \leq_K \mathcal{J}_R$.

For the case $\alpha = 0$, observe that if $z \in D^{(0)} = D$, then $z \notin C$ for every $C \in \mathcal{L}_{z,D}$, i.e., $\mathcal{L}_{z,D} \subseteq \{C \subseteq D : z \notin C\}$. Hence, the function $f : \{0\} \to D$ given by $f(0) = z$ witnesses that $\mathcal{L}_{z,D} \leq_K \mathcal{J}_{\{0\}} = \{\emptyset\}$, as $f^{-1}[C] = \emptyset$ for all $C \in \mathcal{L}_{z,D}$.

Assume now that $0 < \alpha < \omega_1$ and for every $z \in \bigcup_{\beta<\alpha} D^{(\beta)}$ there is $R \in \bigcup_{\beta<\alpha} \mathcal{R}_\beta$ such that $\mathcal{L}_{z,D} \leq_K \mathcal{J}_R$. Let $z \in D^{(\alpha)}$. Then there is a sequence $(z_n)_{n\in\omega}$ in $\bigcup_{\beta<\alpha} D^{(\beta)}$ that is $\mathcal{J}$-convergent to z. By our assumption, for each $n \in \omega$ there is $R_n \in \bigcup_{\beta<\alpha} \mathcal{R}_\beta$ such that $\mathcal{L}_{z_n,D} \leq_K \mathcal{J}_{R_n}$. Let $f_n : R_n \to D$ be the function witnessing $\mathcal{L}_{z_n,D} \leq_K \mathcal{J}_{R_n}$.

Define $R = \bigcup_{n\in\omega} \{n\} \times R_n \in \mathcal{R}_\alpha$. Let $f : R \to D$ be given by $f(n, r) = f_n(r)$ for all $(n, r) \in R$. We will show that f witnesses $\mathcal{L}_{z,D} \leq_K \mathcal{J}_R$. Let $C \in \mathcal{L}_{z,D}$. Then $C \subseteq D \setminus U$ for some open neighborhood U of z. Since $(z_n)_{n\in\omega}$ is $\mathcal{J}$-convergent to z, we have $S = \{n \in \omega : z_n \notin U\} \in \mathcal{J}$. Then U is an open neighborhood of z_n, for each $n \in \omega \setminus S$, so $C \subseteq D \setminus U \in \mathcal{L}_{z_n,D}$ for all $n \in \omega \setminus S$. By the choice of the functions f_n we get that $f_n^{-1}[C] \in \mathcal{J}_{R_n}$ for every $n \in \omega \setminus S$. Thus,

$$f^{-1}[C] \subseteq \left(\bigcup_{n\in S} \{n\} \times R_n \right) \cup \left(\bigcup_{n\in\omega\setminus S} \{n\} \times f_n^{-1}[C] \right) \in \mathcal{J}_R.$$

□

Theorem 7.9. *Let $\mathcal{I}$ and $\mathcal{J}$ be any ideals. If there is a $\mathcal{J}$-continuous function, which is not $\mathcal{I}$-continuous, then the pair $(\mathcal{I}, \mathcal{J})$ satisfies condition (C2).*

Proof. We can assume that all considered ideals are on ω.

By Theorem 7.1, without loss of generality we may assume that there are a topological space Y, a $\mathcal{J}$-continuous $f : X(\mathcal{I}) \to Y$ and an $\mathcal{I}$-closed set $A \subseteq Y$ such that

$f^{-1}[A] = \omega$. Let $\mathcal{K}$ be the (possibly improper) ideal on $f[\omega]$ generated by $f[\omega] \setminus U$ for open neighborhoods U of $f(\omega)$.

Observe that $\mathcal{K}|f[\omega] = \mathcal{K} \not\leq_K \mathcal{I}$. Indeed, if there would exist $h : \omega \to f[\omega]$ witnessing $\mathcal{K} \leq_K \mathcal{I}$, then $(h(n))_{n\in\omega}$ would be a sequence in $f[\omega] \subseteq A$ that is $\mathcal{I}$-convergent to $f(\omega) \notin A$ (by $f^{-1}[A] = \omega$) – a contradiction with the fact that A is $\mathcal{I}$-closed in Y.

Let $\mathcal{M}$ be the (possibly improper) ideal generated by all sets of the form $f^{-1}[C]$ for $C \in \mathcal{K}$. Then $\mathcal{M} = \mathcal{M}|\omega \not\leq_K \mathcal{I}$ by Lemma 7.6.

Fix now any $B \subseteq \omega$ such that $\mathcal{M}|B \not\leq \mathcal{J}$. Then clearly $\mathcal{K}|f[B] \not\leq \mathcal{J}$ as otherwise there would be $R \in \bigcup_{\alpha<\omega_1} \mathcal{R}_\alpha$ such that $\mathcal{K}|f[B] \leq_K \mathcal{J}_R$ while $\mathcal{M}|B \not\leq_K \mathcal{J}_R$, which contradicts the fact that $\mathcal{M}|B$ and $\mathcal{K}|f[B]$ are K-equivalent by Lemma 7.6.

Now, by applying Lemma 7.8 to $\mathcal{K}|f[B]$ as $\mathcal{L}_{f(\omega),f[B]}$ we obtain that $f(\omega) \notin cl_{\mathcal{J}}(f[B])$, hence $\omega \notin f^{-1}[cl_{\mathcal{J}}(f[B])]$. Since f is $\mathcal{J}$-continuous, $f^{-1}[cl_{\mathcal{J}}(f[B])]$ has to be $\mathcal{J}$-closed in $X(\mathcal{I})$. Hence, $\mathcal{I}|f^{-1}[cl_{\mathcal{J}}(f[B])] \not\leq_K \mathcal{J}$ (by Lemma 2.6). Since $B \subseteq f^{-1}[f[B]] \subseteq f^{-1}[cl_{\mathcal{J}}(f[B])]$, we obtain $\mathcal{I}|B \not\leq_K \mathcal{J}$. □

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