ON A CONJECTURE OF DEBS AND SAINT RAYMOND

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ABSTRACT. Borel separation rank of an analytic ideal $\mathcal{I}$ on ω is the minimal ordinal $\alpha < \omega_1$ such that there is $\mathcal{S} \in \mathbf{\Sigma^0_{1+\alpha}}$ with $\mathcal{I} \subseteq \mathcal{S}$ and $\mathcal{I}^\star \cap \mathcal{S} = \emptyset$, where $\mathcal{I}^\star$ is the filter dual to the ideal $\mathcal{I}$. Answering in negative a question of G. Debs and J. Saint Raymond [Fund. Math. 204 (2009), no. 3], we construct a Borel ideal of rank > 2 which does not contain an isomorphic copy of the ideal Fin^3.

1. Introduction

A collection $\mathcal{I}$ of subsets of a set X is called an *ideal on* X if it is closed under subsets and finite unions of its elements. We assume additionally that $\mathcal{P}(X)$ (i.e., the power set of X) is not an ideal, and that every ideal contains all finite subsets of X (hence, $X = \bigcup \mathcal{I}$). All ideals considered in this paper are defined on infinite countable sets.

We treat the power set $\mathcal{P}(X)$ as the space 2^X of all functions $f : X \to 2$ (equipped with the product topology, where each space $2 = \{0, 1\}$ carries the discrete topology) by identifying subsets of X with their characteristic functions. Thus, we can talk about descriptive complexity of subsets of $\mathcal{P}(X)$ (in particular, of ideals on X).

For $A, B, S \subseteq \mathcal{P}(X)$ we say that S *separates* A *from* B if $A \subseteq S$ and $S \cap B = \emptyset$. Following G. Debs and J. Saint Raymond [2], for an analytic ideal $\mathcal{I}$ we define its *Borel separation rank* by:

$$\mathrm{rk}(\mathcal{I}) = \min\left\{\alpha < \omega_1 : \text{ there is } \mathcal{S} \in \mathbf{\Sigma^0_{1+\alpha}} \text{ which separates } \mathcal{I} \text{ from } \mathcal{I}^\star\right\},$$

where $\mathcal{I}^\star = \{A^c : A \in \mathcal{I}\}$ is the *filter dual to the ideal* $\mathcal{I}$ (actually, authors of [2] use the dual notion of filters instead of ideals). In this paper by *rank of* $\mathcal{I}$ we mean $\mathrm{rk}(\mathcal{I})$.

This article is motivated by a conjecture of G. Debs and J. Saint Raymond from 2009 concerning combinatorial characterization of ideals of a given rank. Before formulating the mentioned conjecture, we need to introduce some tools.

Let $\mathcal{I}$ and $\mathcal{J}$ be ideals. Then:

- $\mathcal{I}$ and $\mathcal{J}$ are *isomorphic* if there is a bijection $f : \bigcup \mathcal{J} \to \bigcup \mathcal{I}$ such that:

$$\forall_{A \subseteq \bigcup \mathcal{I}} \left(A \in \mathcal{I} \iff f^{-1}[A] \in \mathcal{J}\right);$$

- $\mathcal{J}$ *contains an isomorphic copy of* $\mathcal{I}$ ($\mathcal{I} \sqsubseteq \mathcal{J}$) if there is a bijection $f : \bigcup \mathcal{J} \to \bigcup \mathcal{I}$ such that:

$$\forall_{A \subseteq \bigcup \mathcal{I}} \left(A \in \mathcal{I} \implies f^{-1}[A] \in \mathcal{J}\right);$$

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- the *Fubini product of* $\mathcal{I}$ *and* $\mathcal{J}$ is an ideal given by:

$$\mathcal{I}\otimes\mathcal{J}=\left\{A\subseteq\bigcup\mathcal{I}\times\bigcup\mathcal{J}:\ \left\{x\in\bigcup\mathcal{I}:\ A_{(x)}\notin\mathcal{J}\right\}\in\mathcal{I}\right\},$$

  where $A_{(x)}=\{y\in\bigcup\mathcal{J}:\ (x,y)\in A\}$.

Isomorphisms of ideals have been deeply studied for instance in [11] (see also [4]), while the preorder $\sqsubseteq$ is examined in [1], [2] and [12].

Inspired by M. Katětov (cf. [8, p. 240]), G. Debs and J. Saint Raymond defined ideals Fin_α such that $\mathrm{rk}(\mathrm{Fin}_\alpha)=\alpha$, for all $0<\alpha<\omega_1$. In the case of finite ranks we have $\mathrm{Fin}_1=\mathrm{Fin}=[\omega]^{<\omega}$ and $\mathrm{Fin}_{n+1}=\mathrm{Fin}^{n+1}=\mathrm{Fin}\otimes\mathrm{Fin}^n$. The mentioned conjecture of G. Debs and J. Saint Raymond from 2009 is the following:

Conjecture 1.1 ([2, Conjecture 7.8])**.** *Let* $\mathcal{I}$ *be an analytic ideal. Then* $\mathrm{rk}(\mathcal{I})\geq\alpha$ *if and only if* $\mathcal{I}$ *contains an isomorphic copy of* Fin_α.

In [10] we have shown that the above is false for $\alpha=\omega$ and proposed new versions of Fin_α in the case of limit ordinals, for which the conjecture could remain true. Moreover, the implication $\Longleftarrow$ is true in general. In the next section we show that Conjecture 1.1 is false in the case of $\alpha=3$ by constructing a Σ^0_6 ideal of rank >2 which does not contain an isomorphic copy of Fin^3.

Our example is the best possible in the sense that Conjecture 1.1 is true for $\alpha=1$ and $\alpha=2$ as shown in [2, Theorem 7.5] and [12, Theorem 4]. What is interesting, although both results prove almost the same thing, they use completely different methods – the methods of G. Debs and J. Saint Raymond from [2] allowed to extend some of their results for the case of higher ranks and develop a bigger theory, while the method of M. Laczkovich and I. Recław from [12] gives us more information about the separating set. The latter method has been used for instance in [5] in a completely different context associated to some variants of the bounding number.

It is also worth mentioning that a property of ideals can often be expressed by finding a critical ideal in sense of the Katětov preorder $\leq_K$ with respect to this property (recall that $\mathcal{I}\leq_K\mathcal{J}$, if there is $f:\bigcup\mathcal{J}\to\bigcup\mathcal{I}$, not necessary a bijection, such that $f^{-1}[A]\in\mathcal{J}$ for each $A\in\mathcal{I}$). This approach proved to be effective in many papers including [6], [7], [9] and [13]. By [1, Example 4.1], $\mathrm{Fin}^n\sqsubseteq\mathcal{I}$ if and only if $\mathrm{Fin}^n\leq_K\mathcal{I}$, for any $n\in\omega$ and any ideal $\mathcal{I}$. Thus, Conjecture 1.1 can be seen as an attempt to characterize ideals of given rank in the above mentioned way.

Let us point out a connection of the above conjecture to descriptive complexity of ideals. Each ideal Fin_n, for $n\in\omega$, is Σ^0_{2n}. By a result of G. Debs and J. Saint Raymond from [3], any ideal $\mathcal{I}$ containing an isomorphic copy of Fin_n cannot be Π^0_{2n}. Thus, Conjecture 1.1 would lead to an interesting result, by giving lower estimate of Borel complexity of ideals of a given finite rank. Although we show that Conjecture 1.1 is false, the general question about minimal complexity of an ideal of a given rank remains open (see [2, Conjecture 8.5]). It is only known that no Π^0_4 ideal can have rank >1 (cf. [2, Theorem 9.1]).

In [2] and [12] it is shown that ranks of analytic ideals are important for studying ideal pointwise limits of sequences of continuous functions: $\mathrm{rk}(\mathcal{I})=\alpha$ if and only if the family of all $\mathcal{I}$-pointwise limits of continuous real-valued functions defined on a given zero-dimensional Polish space is equal to the family of all functions of Borel class α (the definition of $\mathcal{I}$-pointwise limit can be found in [2] or [12]). Conjecture 1.1 would imply that we can skip the assumption about zero-dimensionality in this result (cf. [2, Corollary 7.6]). In fact, studies of $\mathcal{I}$-pointwise limits of continuous functions were the main motivation for introducing ranks of ideals (however, earlier S. Solecki in [14] studied this notion in a different context).

2. The counterexample

We start this section by introducing the ideal which will be a counterexample for Conjecture 1.1.

Definition 2.1. The *Counterexample ideal* $\mathcal{CEI}$ is a collection of subsets of ω^4 consisting of all $A \subseteq \omega^4$ such that there is $n \in \omega$ satisfying:

- $\forall_{i<n}\ A_{(i)} \in \mathrm{Fin}^3$;
- $\forall_{i\geq n}\ \exists_{k\in\omega}\ \forall_{j\geq k}\ \exists_{m\in\omega}\ A \cap \left(\bigcup_{l\geq m}\{(i,j,l)\} \times \omega\right)$ is finite.

Proposition 2.2. *$\mathcal{CEI}$ is a Borel ideal of class Σ^0_6.*

Proof. It is easy to verify that $\mathcal{CEI}$ is indeed an ideal (while showing closedness under finite unions one needs to note that $A \in \mathcal{CEI}$ implies $A_{(i)} \in \mathrm{Fin}^3$ for all $i \in \omega$).

The Borel complexity of $\mathcal{CEI}$ follows from the facts that Fin^3 is Σ^0_6 (see [2, Proposition 6.4]) and that for every $(i,j,m) \in \omega^3$ the map $f_{i,j,m} : \mathcal{P}(\omega^4) \to \mathcal{P}(\omega^4)$ given by $f_{i,j,m}(A) = A \cap \left(\bigcup_{l\geq m}\{(i,j,l) \times \omega\}\right)$ is continuous, so the set:

$$\left\{A \subseteq \omega^4 :\ A \cap \left(\bigcup_{l\geq m}\{(i,j,l) \times \omega\}\right) \text{ is finite}\right\} = f^{-1}_{i,j,l}[[\omega^4]^{<\omega}]$$

is Σ^0_2. □

Lemma 2.3. $\mathrm{Fin}^3 \not\sqsubseteq \mathcal{CEI}$.

Proof. Assume to the contrary that there is a bijection $f : \omega^3 \to \omega^4$ witnessing $\mathrm{Fin}^3 \not\sqsubseteq \mathcal{CEI}$. Denote:

$$T = \big\{(i,j,k) \in \omega^3 :\ (\{(i,j,k)\} \times \omega) \cap f[\{(n,m)\} \times \omega] \neq \emptyset \text{ for infinitely many } (n,m) \in \omega^2\big\}$$

and $Y = T \times \omega$.

Claim. $Y \in \mathcal{CEI}$.

Proof. Assume to the contrary that $Y \notin \mathcal{CEI}$. Then $|T| = \omega$. Fix any bijections $g : \omega \to \omega^2$ and $h : \omega \to T$ and for each $n \in \omega$ pick a finite set $P_n \in [\omega^3]^{<\omega}$ such that:

- $P_n \subseteq \{g(n)\} \times \omega$;
- if $n' \leq n$ and $(\{h(n')\}\times\omega)\cap f[\{g(n)\}\times\omega] \neq \emptyset$ then $f[P_n]\cap(\{h(n')\}\times\omega) \neq \emptyset$.

Define $P = \bigcup_{n\in\omega} P_n$. Clearly, $P \in \mathrm{Fin}^3$. However, $f[P] \notin \mathcal{CEI}$ as for every $(i,j,k) \in T$ the set $f[P] \cap (\{(i,j,k)\} \times \omega)$ is infinite and $Y = T \times \omega \notin \mathcal{CEI}$. This contradicts the choice of f. □

Define now:

$$S = \big\{(i,j) \in \omega^2 :\ \left(\{(i,j)\} \times \omega^2\right) \setminus Y \text{ is covered by finitely many sets of the form } f[\{(n,m)\} \times \omega]\big\}$$

and $Z = S \times \omega^2$.

Claim. $Z \in \mathcal{CEI}$.

Proof. Assume to the contrary that $Z \notin \mathcal{CEI}$. Then also $Z \setminus Y \notin \mathcal{CEI}$ (as $Y \in \mathcal{CEI}$). Consider the set:

$$S' = \left\{(i,j,k) \in \omega^3 :\ (i,j,k) \notin T \text{ and } (i,j) \in S\right\}.$$

Then $S' \times \omega = Z \setminus Y \notin \mathcal{CEI}$, so there has to be $i_0 \in \omega$ such that $S_{(i_0)} \notin \mathrm{Fin}^2$ (as for each $A \subseteq \omega^3$ with $A_{(n)} \in \mathrm{Fin}^2$ for every n we have $A \times \omega \in \mathcal{CEI}$). Hence, $J = \{j \in \omega :\ (i_0, j, k) \in S' \text{ for infinitely many } k\}$ is infinite. Thus, for each $j \in J$ we can find $(n_j, m_j) \in \omega^2$ such that:

$$\{k \in \omega :\ (i_0, j, k) \in S' \text{ and } |(\{(i_0,j,k)\} \times \omega) \cap f[\{(n_j,m_j)\} \times \omega]| = \omega\}$$

is infinite (as $(\{(i_0,j)\} \times \omega^2) \setminus Y$ is covered by finitely many sets of the form $f[\{(n,m)\} \times \omega]$).

Recall that the ideal Fin^2 is tall (see [13, page 24]), i.e., for each infinite $B \subseteq \omega^2$ there is an infinite $C \subseteq B$ with $C \in \mathrm{Fin}^2$. Therefore, there is some $R \in \mathrm{Fin}^2$ with $(n_j, m_j) \in R$ for infinitely many $j \in J$. Then $R \times \omega \in \mathrm{Fin}^3$, but $(f[R \times \omega])_{(i_0)} \notin \mathrm{Fin}^3$ and consequently $f[R \times \omega] \notin \mathcal{CEI}$. Again we obtain a contradiction with the choice of f. □

Since $Z \in \mathcal{CEI}$, the set $\omega^2 \setminus S$ is infinite. Let $g : \omega \to \omega^2$ and $h : \omega \to \omega^2 \setminus S$ be fixed bijections. For every $n \in \omega$ pick a finite set $G_n \in [\omega^3]^{<\omega}$ such that:

- $G_n \subseteq \{g(n)\} \times \omega$;
- if $n' \leq n$ and $((\{h(n')\} \times \omega^2) \setminus Y) \cap f[\{g(n)\} \times \omega] \neq \emptyset$ then $f[G_n] \cap ((\{h(n')\} \times \omega^2) \setminus Y) \neq \emptyset$.

Define $G = \bigcup_{n\in\omega} G_n$. Obviously, $G \in \mathrm{Fin}^3$. However, $f[G] \notin \mathcal{CEI}$ as $((\omega^2 \setminus S) \times \omega^2) \setminus Y \notin \mathcal{CEI}$ and for every $(i,j) \notin S$ and $k \in \omega$ we have:

$$\left| f[G] \cap \left(\bigcup_{l \geq k} (\{(i,j,l)\} \times \omega) \setminus Y \right) \right| = \omega$$

(since $\bigcup_{l<k} (\{(i,j,l)\} \times \omega) \setminus Y$ is covered by finitely many sets of the form $f[\{(n,m)\} \times \omega]$ while $(\{(i,j)\} \times \omega^2) \setminus Y$ is not). This contradicts the choice of f and finishes the entire proof. □

Lemma 2.4. $\mathrm{rk}(\mathcal{CEI}) > 2$.

Proof. Suppose that $S \in \Sigma^0_3$ is arbitrary such that $\mathcal{CEI} \subseteq S$. Then there are finite sets $F_{i,j,k}, G_{i,j,k} \subseteq \omega^4$, for $i,j,k \in \omega$, such that:

$$S = \left\{A \subseteq \omega^4 :\ \exists_{i\in\omega}\ \forall_{j\in\omega}\ \exists_{k\in\omega}\ F_{i,j,k} \cap A = \emptyset \text{ and } G_{i,j,k} \subseteq A\right\}.$$

We will construct a set $X \in \mathcal{CEI}$ such that:

$$\exists_{i\in\omega}\ \forall_{j\in\omega}\ \exists_{k\in\omega}\ F_{i,j,k} \subseteq X \text{ and } G_{i,j,k} \cap X = \emptyset.$$

This will finish the proof, since the above implies $\omega^4 \setminus X \in \mathcal{CEI}^\star \cap S$ and consequently $S \cap \mathcal{CEI}^\star \neq \emptyset$.

We need to introduce some notation. For each $A \in S$ there is $I(A) \subseteq \omega$, $I(A) \neq \emptyset$ such that for each $i \in I(A)$ we have:

$$\forall_{j\in\omega}\ \exists_{k\in\omega}\ F_{i,j,k} \cap A = \emptyset \text{ and } G_{i,j,k} \subseteq A.$$

Moreover, for each $A \in S$, $i \in I(A)$ and $j \in \omega$ let $k(A,i,j) \in \omega$ be such that $F_{i,j,k(A,i,j)} \cap A = \emptyset$ and $G_{i,j,k(A,i,j)} \subseteq A$.

For $s = (s_0, s_1, \ldots) \in \omega^\omega$ denote:

$$A(s) = \bigcup_{m \in \omega} \left(\left((\omega \setminus m) \times s_m \times \omega^2 \right) \cup (\omega \times (\omega \setminus m) \times s_m \times \omega) \right)$$

(here and in the rest of this proof we use standard set-theoretic notation and identify $n \in \omega$ with the set $\{0, 1, \ldots, n-1\}$). Observe that $A(s) \in \mathcal{CEI}$. Moreover, if $s, t \in \omega^\omega$ are such that $s_n \leq t_n$ for all n then $A(s) \subseteq A(t)$ (we will use this observation without any reference). Put also $A((s_0, \ldots, s_m)) = A((s_0, \ldots, s_m, 0, 0, \ldots))$ for each $s = (s_0, \ldots, s_m) \in \omega^{<\omega}$. For $i \in \omega$ denote by $0_i \in \omega^i$ the sequence consisting of i zeros and define $\mathrm{sq}(i) = i \times i \times \omega^2$. Finally, for each $A \in S$, $i \in I(A)$ and $j \in \omega$ put:

$$p(A,i,j) = \min \left\{ p \in \omega : \ G_{i,j,k(A,i,j)} \subseteq \mathrm{sq}(i) \cup A(0_i^\frown(p)) \right\}.$$

Note that the above is well defined as the set $G_{i,j,k(A,i,j)}$ is finite and:

$$\bigcup_{p \in \omega} (\mathrm{sq}(i) \cup A(0_i^\frown(p))) = \omega^4.$$

Claim. There are $i \in \omega$, $T \subseteq \mathrm{sq}(i)$, $t \in \omega^i$ and $B \in [\omega^4]^{<\omega}$ such that for each finite sequence $C_0, C_1, \ldots, C_n \in \mathcal{CEI}$, if for each $m \leq n$ we have:

- $i \in I(C_m)$;
- $C_m \cap \mathrm{sq}(i) = T$;
- $C_m \supseteq A(t^\frown(p_m)) \setminus \left(B \cup \bigcup_{j<m} F_{i,j,k(C_j,i,j)} \right)$, where:
$$p_m = \max(\{p(C_j,i,j) : \ j < m\} \cup \{m\});$$
- $C_m \cap \left(B \cup \bigcup_{j<m} F_{i,j,k(C_j,i,j)} \right) = \emptyset$;

then one can find $C_{n+1} \in \mathcal{CEI}$ such that:

- $i \in I(C_{n+1})$;
- $C_{n+1} \cap \mathrm{sq}(i) = T$;
- $C_{n+1} \supseteq A(t^\frown(p_{n+1})) \setminus \left(B \cup \bigcup_{j \leq n} F_{i,j,k(C_j,i,j)} \right)$, where:
$$p_{n+1} = \max(\{p(C_j,i,j) : \ j \leq n\} \cup \{n+1\});$$
- $C_{n+1} \cap \left(B \cup \bigcup_{j \leq n} F_{i,j,k(C_j,i,j)} \right) = \emptyset$.

(In other words, each such finite sequence $C_0, C_1, \ldots, C_n \in \mathcal{CEI}$ can be extended by one more element.)

Proof. Suppose otherwise towards contradiction. Then for each $i \in \omega$, $T \subseteq \mathrm{sq}(i)$, $t \in \omega^i$ and $B \in [\omega^4]^{<\omega}$ one can find a finite sequence in $\mathcal{CEI}$ with the required properties that cannot be extended.

For each $n \in \omega$ we will inductively pick $b_n, s_n \in \omega$, $(A_n^m)_{m<b_n} \subseteq \mathcal{CEI}$, $B_n \in [\omega^4]^{<\omega}$ and $A_n \in \mathcal{CEI}$ such that:

(a) for each $m < b_n$ we have:
   - $n \in I(A_n^m)$;
   - $A_n^m \cap \mathrm{sq}(n) = A_{n-1} \cap \mathrm{sq}(n)$;
   - $A_n^m \supseteq A((s_0, \ldots, s_{n-1}, p_n^m)) \setminus \left(B_{n-1} \cup \bigcup_{j<m} F_{n,j,k(A_n^j,n,j)} \right)$, where we put $p_n^m = \max(\{p(A_n^j, n, j) : \ j < m\} \cup \{m\})$;

  - $A_n^m \cap \left(B_{n-1} \cup \bigcup_{j<m} F_{n,j,k(A_n^j,n,j)}\right) = \emptyset$;

(b) $s_n = \max(\{p(A_n^j, n, j) : j < b_n\} \cup \{n\})$;

(c) $B_n = B_{n-1} \cup \bigcup_{j<b_n} F_{n,j,k(A_n^j,n,j)}$;

(d) $A_n = A((s_0, \ldots, s_{n-1}, s_n)) \setminus B_n$;

(e) there is no $A \in \mathcal{CEI}$ such that:
  - $n \in I(A)$;
  - $A \cap \mathrm{sq}(n) = A_{n-1} \cap \mathrm{sq}(n)$;
  - $A \supseteq A_n$;
  - $A \cap B_n = \emptyset$;

(in the above we put $A_{-1} = B_{-1} = \emptyset$).

In the first induction step it suffices to apply our assumption to $i = 0$, $T = \emptyset$, $t = \emptyset$ and $B = \emptyset$ in order to obtain $b_0 \in \omega$ and the required sequence $(A_0^m)_{m<b_0} \subseteq \mathcal{CEI}$ and then define s_0, B_0 and A_0 according to items (b), (c) and (d).

In the nth induction step, if b_t, s_t, $(A_t^m)_{m<b_t}$, B_t and A_t, for all $t < n$, are already defined, using our assumption applied to $i = n$, $t = (s_0, \ldots, s_{n-1})$, $T = A_{n-1} \cap \mathrm{sq}(n)$ and $B = B_{n-1}$ once again we get $b_n \in \omega$ and a finite sequence $(A_n^m)_{m<b_n} \subseteq \mathcal{CEI}$ with the required properties and we can put s_n, B_n and A_n according to items (b), (c) and (d). This finishes the inductive construction.

Notice that the sequence (B_n) is non-decreasing. The rest of this proof strongly relies on the observation that $A_k = A(s_0, \ldots, s_k) \setminus \bigcup_{n\in\omega} B_n$, for every $k \in \omega$. Indeed, $A_k \supseteq A(s_0, \ldots, s_k) \setminus \bigcup_{n\in\omega} B_n$ is obvious, so we only need to prove $A_k \subseteq A(s_0, \ldots, s_k) \setminus \bigcup_{n\in\omega} B_n$. Fix $x \in A_k = A(s_0, \ldots, s_k) \setminus B_k$ and suppose to the contrary that $x \notin A(s_0, \ldots, s_k) \setminus \bigcup_{n\in\omega} B_n$, i.e., $x \in B_n \setminus B_{n-1} = \bigcup_{j<b_n} F_{n,j,k(A_n^j,n,j)}$ for some $n > k$. Let $m < b_n$ be minimal such that $x \in F_{n,m,k(A_n^m,n,m)}$. Then:

$$x \in A((s_0, \ldots, s_k)) \setminus \left(B_{n-1} \cup \bigcup_{j<m} F_{n,j,k(A_n^j,n,j)}\right) \subseteq$$
$$A((s_0, \ldots, s_{n-1})) \setminus \left(B_{n-1} \cup \bigcup_{j<m} F_{n,j,k(A_n^j,n,j)}\right) \subseteq A_n^m,$$

which contradicts $F_{n,m,k(A_n^m,n,m)} \cap A_n^m = \emptyset$.

Consider the set $D = A(s_0, s_1, \ldots) \setminus \bigcup_{n\in\omega} B_n \in \mathcal{CEI}$. We will show that $I(D) = \emptyset$, which will contradict $\mathcal{CEI} \subseteq S$ and finish the proof.

Fix any $k \in \omega$. Clearly, $D \cap B_k = \emptyset$. Moreover, $D \supseteq A_k$, as:

$$D = A(s_0, s_1, \ldots) \setminus \bigcup_{n\in\omega} B_n \supseteq A(s_0, \ldots, s_k) \setminus \bigcup_{n\in\omega} B_n = A_k.$$

Finally, $D \cap \mathrm{sq}(k) = A_{k-1} \cap \mathrm{sq}(k)$ since:

$$D \cap \mathrm{sq}(k) = \left(A(s_0, s_1, \ldots) \setminus \bigcup_{n\in\omega} B_n\right) \cap \mathrm{sq}(k) =$$
$$\left(A(s_0, \ldots, s_{k-1}) \setminus \bigcup_{n\in\omega} B_n\right) \cap \mathrm{sq}(k) = A_{k-1} \cap \mathrm{sq}(k).$$

(by $A(s_0, s_1, \ldots) \cap \mathrm{sq}(k) = A(s_0, \ldots, s_{k-1}) \cap \mathrm{sq}(k)$). Thus, $k \notin I(D)$ by item (e). Since k was arbitrary, we obtain $I(D) = \emptyset$. This finishes the claim. $\square$

Using the above Claim inductively we get $i \in \omega$, $T \subseteq \mathrm{sq}(i)$, $t \in \omega^i$, $B \in [\omega^4]^{<\omega}$ and an infinite sequence $(C_m) \in \mathcal{CEI}$ such that for each $m \in \omega$ we have:

(a) $i \in I(C_m)$;
(b) $C_m \cap \mathrm{sq}(i) = T$;
(c) $C_m \supseteq A(t^\frown(p_m)) \setminus \left(B \cup \bigcup_{j<m} F_{i,j,k(C_j,i,j)}\right)$, where:
$$p_m = \max(\{p(C_j, i, j) : \ j < m\} \cup \{m\});$$
(d) $C_m \cap \left(B \cup \bigcup_{j<m} F_{i,j,k(C_j,i,j)}\right) = \emptyset$.

Define $X = \bigcup_{m\in\omega} F_{i,m,k(C_m,i,m)}$. Observe that $X \cap G_{i,m,k(C_m,i,m)} = \emptyset$ for all m. Indeed, fix $x \in X$ and suppose towards contradiction that $x \in G_{i,n,k(C_n,i,n)}$ for some $n \in \omega$. Denote by m the minimal integer such that $x \in F_{i,m,k(C_m,i,m)}$. There are three possibilities:

- The case $m = n$ cannot happen as $G_{i,n,k(C_n,i,n)} \subseteq C_n$ and $F_{i,n,k(C_n,i,n)} \cap C_n = \emptyset$.
- The case $m < n$ is impossible since $F_{i,m,k(C_m,i,m)} \cap C_n = \emptyset$ (by item (d)), while $G_{i,n,k(C_n,i,n)} \subseteq C_n$.
- If $m > n$ then note that (b) and the definition of $p(C_n, i, n)$ imply:
$$\begin{aligned} x \in G_{i,n,k(C_n,i,n)} &\subseteq C_n \cap (\mathrm{sq}(i) \cup A(0_i^\frown(p(C_n, i, n)))) \subseteq \\ &T \cup A(0_i^\frown(p(C_n, i, n))). \end{aligned}$$
Moreover, $x \notin B \cup \bigcup_{j<m} F_{i,j,k(C_j,i,j)}$ by $x \in G_{i,n,k(C_n,i,n)} \subseteq C_n$, $C_n \cap B = \emptyset$ (by (d)) and the choice of m. This in turn means that:
$$x \in \ (T \cup A(0_i^\frown(p(C_n, i, n)))) \setminus \left(B \cup \bigcup_{j<m} F_{i,j,k(C_j,i,j)}\right) \subseteq$$
$$T \cup \left(A(0_i^\frown(p(C_n, i, n))) \setminus \left(B \cup \bigcup_{j<m} F_{i,j,k(C_j,i,j)}\right)\right) \subseteq C_m$$
(by (b) and (c)). The latter contradicts $F_{i,m,k(C_m,i,m)} \cap C_m = \emptyset$. Hence, $m > n$ neither can hold.

Therefore, $i \in \omega$ is such that for each $m \in \omega$ there is $k = k(C_m, i, m)$ with $F_{i,m,k} \subseteq X$ and $G_{i,m,k} \cap X = \emptyset$. To finish the proof we need to show that $X \in \mathcal{CEI}$.

We will show that $X \cap A(0_i^\frown(m)) \subseteq B \cup \bigcup_{j<m} F_{i,j,k(C_j,i,j)}$, for each m. Assume to the contrary that for some $m \in \omega$ there is $x \in X \cap A(0_i^\frown(m))$ such that $x \notin B \cup \bigcup_{j<m} F_{i,j,k(C_j,i,j)}$ and let $m' \geq m$ be minimal such that $x \in F_{i,m',k(C_{m'},i,m')}$. Then:
$$x \in A(0_i^\frown(m')) \setminus \left(B \cup \bigcup_{j<m'} F_{i,j,k(C_j,i,j)}\right) \subseteq C_{m'}.$$
(by item (c) and the fact that $m \leq m'$ implies $A(0_i^\frown(m)) \subseteq A(0_i^\frown(m')))$. On the other hand, $F_{i,m',k(C_{m'},i,m')} \cap C_{m'} = \emptyset$. This contradiction proves that $X \cap A(0_i^\frown(m)) \subseteq B \cup \bigcup_{j<m} F_{i,j,k(C_j,i,j)}$.

Since $B \cup \bigcup_{j<m} F_{i,j,k(C_j,i,j)}$ is finite, for each m, the rest of the proof follows from the previous paragraph and an easy observation that each set $Y \subseteq \omega^4$ with $Y \cap A(0_i^\frown(m))$ finite for all m, belongs to the ideal $\mathcal{CEI}$. Indeed, for each such set Y we have:

- if $i' \geq i$ and $j \in \omega$ then:
$$Y \cap \left(\{(i', j)\} \times \omega^2\right) \subseteq Y \cap A(0_i^\frown(i'+1)) \in [\omega^4]^{<\omega};$$
- if $i' < i$, $j \geq i$ and $k \in \omega$ then:
$$Y \cap (\{(i', j, k)\} \times \omega) \subseteq Y \cap A(0_i^\frown(k+1)) \in [\omega^4]^{<\omega};$$
- $Y \cap \mathrm{sq}(i) \subseteq \mathrm{sq}(i) \in \mathcal{CEI}$.

□

Theorem 2.5. *There is a Σ^0_6 ideal of rank > 2 not containing an isomorphic copy of* Fin^3.

Proof. $\mathcal{CEI}$ is a good example as shown in Proposition 2.2 and Lemmas 2.3 and 2.4. □

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