ON EXTENDABILITY TO F_σ IDEALS

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ABSTRACT. Answering in negative a question of M. Hrušák, we construct a Borel ideal not extendable to any F_σ ideal and such that it is not Katětov above the ideal conv.

1. Introduction

We use standard set-theoretic notation. In particular, a collection $\mathcal{I}$ of subsets of a set X is called an ideal on X if it is closed under subsets and finite unions of its elements. We assume additionally that $\mathcal{P}(X)$ (i.e., the power set of X) is not an ideal, and that every ideal contains all finite subsets of X (hence, $X = \bigcup\mathcal{I}$). All ideals considered in this paper are defined on infinite countable sets.

We treat the power set $\mathcal{P}(X)$ as the space 2^X of all functions $f : X \to 2$ (equipped with the product topology, where each space $2 = \{0, 1\}$ carries the discrete topology) by identifying subsets of X with their characteristic functions. Thus, we can talk about descriptive complexity of subsets of $\mathcal{P}(X)$ (in particular, of ideals on X).

This article is motivated by a problem of M. Hrušák concerning characterization of Borel ideals that can be extended to a $\mathbf{\Sigma^0_2}$ ideal (i.e., such Borel ideals $\mathcal{I}$ that there is a $\mathbf{\Sigma^0_2}$ ideal $\mathcal{J}$ with $\mathcal{I} \subseteq \mathcal{J}$). In order to formulate this question in a precise way, we need to recall two definitions:

- if $\mathcal{I}$ and $\mathcal{J}$ are ideals then we say that $\mathcal{I}$ is below $\mathcal{J}$ in the Katětov preorder (and write $\mathcal{I} \leq_K \mathcal{J}$), if there is $f : \bigcup\mathcal{J} \to \bigcup\mathcal{I}$ such that $f^{-1}[A] \in \mathcal{J}$ for each $A \in \mathcal{I}$ (cf. [5, Subsection 1.3] or [11, Subsection 1.5]);
- by conv we denote the ideal on $\mathbb{Q} \cap [0, 1]$ generated by sequences in $\mathbb{Q} \cap [0, 1]$ that are convergent in $[0, 1]$, i.e., $A \subseteq \mathbb{Q} \cap [0, 1]$ belongs to the ideal conv if it can be covered by finitely many such sequences (cf. [5, Subsection 3.4] or [11, Subsection 1.6]).

A property of ideals can often be expressed by finding a critical ideal (in sense of the Katětov preorder) with respect to this property. This approach proved to be especially effective in many papers including [5], [6], [8], [9] and [11]. The above mentioned question of M. Hrušák is the following:

Question 1. Is it true that, if $\mathcal{I}$ is a Borel ideal then either conv $\leq_K \mathcal{I}$ or there is a $\mathbf{\Sigma^0_2}$ ideal containing $\mathcal{I}$?

This problem has been asked in [5, Question 5.16] and repeated in [6, Question 5.8]. It is known that conv is a $\mathbf{\Sigma^0_4}$ ideal that cannot be extended to any $\mathbf{\Sigma^0_2}$ ideal (by [3, Propositions 3.4 and 4.1] and [11, Subsection 2.7]). Thus, a positive answer

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to M. Hrušák's question would establish a combinatorial characterization of Borel ideals extendable to $\mathbf{\Sigma^0_2}$ ideals. However, in this paper we answer it in the negative.

A very similar problem to the above one has been posed by D. Meza-Alcántara in [11, Question 4.4.6]: Is it true that, if $\mathcal{I}$ is a Borel ideal then either $\mathrm{conv} \leq_K \mathcal{I}|A$ (here $\mathcal{I}|A = \{B \subseteq A : B \in \mathcal{I}\}$) for some $A \notin \mathcal{I}$ or there is a $\mathbf{\Sigma^0_2}$ ideal containing $\mathcal{I}$? This question remains open.

$\mathbf{\Sigma^0_2}$ ideals are closely related to the notion of P^+-ideals. We say that an ideal $\mathcal{I}$ on X is a P^+-ideal if for each decreasing sequence (A_n) of sets not belonging to $\mathcal{I}$ one can find $B \notin \mathcal{J}$ such that $B \setminus A_n$ is finite for all $n \in \omega$ ([11, Definition 2.2.3]). This notion has been studied by for instance in [5, Subsection 1.1]. By [11, Theorem 3.2.7], a Borel ideal is extendable to a $\mathbf{\Sigma^0_2}$ ideal if and only if it is extendable to a P^+-ideal. Therefore, original question of M. Hrušák can be reformulated in the following way: Is is true that a Borel ideal $\mathcal{I}$ is extendable to a P^+-ideal if and only if $\mathrm{conv} \not\leq_K \mathcal{I}$?

It is worth mentioning that M. Hrušák's conjecture holds for all analytic P-ideals (an ideal $\mathcal{I}$ is called a P-ideal if for every $(A_n) \subseteq \mathcal{I}$ there is $A \in \mathcal{I}$ with $A \setminus A_n$ finite for all n), i.e., an analytic P-ideal $\mathcal{I}$ is extendable to a $\mathbf{\Sigma^0_2}$ ideal if and only if $\mathrm{conv} \not\leq_K \mathcal{I}$ (cf. [3, Theorem 4.2] and [11, Subsection 2.7]). Moreover, by [11, Theorem 3.2.14], if $\mathcal{I}$ is a Borel ideal such that the forcing $\mathcal{P}(\omega)/\mathcal{I}$ is proper then either it is extendable to a $\mathbf{\Sigma^0_2}$ ideal or there is $A \notin \mathcal{I}$ with $\mathrm{conv} \leq_K \mathcal{I}|A$.

In Section 2 we introduce some necessary notions. Section 3 contains the solution of M. Hrušák's question. Section 4 is devoted to some concluding remarks.

2. Preliminaries

If $\mathcal{I}$ and $\mathcal{J}$ are ideals on X and Y, respectively, then the ideal:

$$\mathcal{I} \oplus \mathcal{J} = \{A \subseteq (\{0\} \times X) \cup (\{1\} \times Y) :$$
$$\{x \in X : (0,x) \in A\} \in \mathcal{I} \text{ and } \{y \in Y : (1,y) \in A\} \in \mathcal{J}\}$$

is their disjoint sum (see [1]). Two ideals $\mathcal{I}$ and $\mathcal{J}$ are isomorphic if there is a bijection $f : \bigcup \mathcal{J} \to \bigcup \mathcal{I}$ such that

$$f^{-1}[A] \in \mathcal{J} \iff A \in \mathcal{I}$$

for all $A \subseteq \bigcup \mathcal{I}$. Isomorphisms of ideals have been deeply studied for instance in [10] (see also [1] and [9]).

Recall the definition of the classical ideal of asymptotic density zero sets:

$$\mathcal{I}_d = \left\{ A \subseteq \omega : \lim_{n \to \infty} \frac{|A \cap [0,n]|}{n+1} = 0 \right\},$$

where by $[0,n]$ we denote the set $\{0,1,\ldots,n\}$. This ideal has been deeply investigated in the past in the context of convergence (see e.g. [2], [4], [13] and [12]) as well as from the set-theoretic point of view (see e.g. [1], [7] and [9]).

In our considerations we will need some new notions which we introduce below.

In this paper we denote by $\mathcal{S}$ the family of all sequences in $(0, \frac{1}{2}]$ decreasing to zero, i.e.:

$$\mathcal{S} = \left\{ (\alpha_n) \in \left(0, \tfrac{1}{2}\right]^\omega : \lim_n \alpha_n = 0 \text{ and } \alpha_{n+1} < \alpha_n \text{ for all } n \in \omega \right\}.$$

For $x \in [0,1]$ and $r > 0$ by $B(x,r)$ we denote the ball of radius r centered at x, i.e., $B(x,r) = (x-r, x+r)$. Note that for every $(\alpha_n) \in \mathcal{S}$ and every $x \in [0,1]$ we have $(B(x,\alpha_0) \setminus B(x,\alpha_1)) \cap [0,1] \neq \emptyset$.

Definition 2.1. If $(\alpha_n) \in \mathcal{S}$ and $\mathcal{I}$ is an ideal on ω then we say that a sequence $(x_k) \in [0,1]^\omega$ converges $\mathcal{I}$-quickly with respect to (α_n) if there is $x \in [0,1]$ such that $\lim_k x_k = x$ and:

$$\{n \in \omega :\ \{x_k :\ k \in \omega\} \cap (B(x,\alpha_n) \setminus B(x,\alpha_{n+1})) \neq \emptyset\} \in \mathcal{I}.$$

Definition 2.2. If $(\alpha_n) \in \mathcal{S}$ and $\mathcal{I}$ is an ideal on ω then $\mathrm{conv}(\mathcal{I},(\alpha_n))$ is the ideal on $[0,1] \cap \mathbb{Q}$ generated by all sequences converging $\mathcal{I}$-quickly with respect to (α_n), i.e., $A \subseteq \mathbb{Q} \cap [0,1]$ belongs to $\mathrm{conv}(\mathcal{I},(\alpha_n))$ if it can be covered by finitely many sequences converging $\mathcal{I}$-quickly with respect to (α_n).

Our counterexample will be of the form $\mathrm{conv}(\mathcal{I},(\alpha_n))$. We start with a simple general result concerning such ideals.

Proposition 2.3. *If $\mathcal{I}$ is an analytic ideal on ω and $(\alpha_n) \in \mathcal{S}$ then $\mathrm{conv}(\mathcal{I},(\alpha_n))$ is also analytic.*

Proof. Fix an ideal $\mathcal{I}$ on ω and $(\alpha_n) \in \mathcal{S}$. Observe that:

$$\mathrm{conv}(\mathcal{I},(\alpha_n)) = \Big\{A \subseteq [0,1] \cap \mathbb{Q} :\ \exists_{k\in\omega}\ \exists_{x_0,\ldots,x_{k-1}\in[0,1]}\ \left(\forall_{i<k}\ A \in f_{x_i}^{-1}[\mathcal{I}]\right)\ \wedge \left(\forall_{m\in\omega}\ \exists_{F\in[\mathbb{Q}\cap[0,1]]^{<\omega}}\ A \subseteq F \cup \bigcup_{i<k} B\left(x_i, \tfrac{1}{m+1}\right)\right)\Big\},$$

where $f_x : \mathcal{P}([0,1]\cap\mathbb{Q}) \to \mathcal{P}(\omega)$, for each $x \in [0,1]$, is given by:

$$f_x(A) = \{n \in \omega :\ A \cap (B(x,\alpha_n) \setminus B(x,\alpha_{n+1})) \neq \emptyset\},$$

for all $A \subseteq [0,1] \cap \mathbb{Q}$.

Notice that if $x \in [0,1]$ and $l \in \omega$ then $f_x^{-1}[\{A \subseteq \omega :\ l \in A\}]$ is open and $f_x^{-1}[\{A \subseteq \omega :\ l \notin A\}]$ is closed. Hence, f_x is of Baire class 1, for every $x \in [0,1]$, and $f_x^{-1}[\mathcal{I}]$ is analytic, for each analytic ideal $\mathcal{I}$. It follows that $\mathrm{conv}(\mathcal{I},(\alpha_n))$ is analytic for every analytic ideal $\mathcal{I}$. ☐

3. The counterexample

The following sequence of lemmas will lead us to the solution of M. Hrušák's problem.

Lemma 3.1. *The following are equivalent for every ideal $\mathcal{I}$ on ω and every sequence $(\alpha_n) \in \mathcal{S}$:*

(a) *$\mathcal{I}$ can be extended to a P^+-ideal;*
(b) *$\mathrm{conv}(\mathcal{I},(\alpha_n))$ can be extended to a P^+-ideal.*

Proof. (b) $\implies$ (a): Suppose that $\mathrm{conv}(\mathcal{I},(\alpha_n)) \subseteq \mathcal{J}$ for some P^+-ideal $\mathcal{J}$ (on $[0,1]\cap\mathbb{Q}$). If each convergent sequence is in $\mathcal{J}$, then $\mathrm{conv} \subseteq \mathcal{J}$, which contradicts the choice of $\mathcal{J}$ (as conv cannot be extended to any P^+-ideal – see the Introduction for details). Thus, there is a convergent sequence $A \in \mathrm{conv} \setminus \mathcal{J}$ (so also $A \notin \mathrm{conv}(\mathcal{I},(\alpha_n))$). Let $x \in [0,1]$ be the limit of A. Without loss of generality we may assume that $A \subseteq B(x,\alpha_0)$. Note that $\mathcal{I} \leq_K \mathrm{conv}(\mathcal{I},(\alpha_n))|A \subseteq \mathcal{J}|A$ as witnessed by the function $f : A \to \omega$ given by:

$$f(i) = n \ \Leftrightarrow\ i \in B(x,\alpha_n) \setminus B(x,\alpha_{n+1})$$

for all $i \in A$.

Observe that $\mathcal{I}$ is a subset of $\mathcal{J}' = \{C \subseteq \omega :\ f^{-1}[C] \in \mathcal{J}|A\}$ as for each $C \in \mathcal{I}$ we have $f^{-1}[C] \in \mathcal{J}|A$. To finish the proof, we will show that $\mathcal{J}'$ is a P^+-ideal.

It is easy to check that $\mathcal{J}'$ is indeed an ideal, so let (C_n) be a decreasing sequence of sets not belonging to $\mathcal{J}'$. Define $B_n = f^{-1}[C_n] \subseteq A$ for all $n \in \omega$. Then each B_n does not belong to $\mathcal{J}|A$ (as $C_n \notin \mathcal{J}'$) and (B_n) is decreasing. Since $\mathcal{J}$ is a P^+-ideal, so is $\mathcal{J}|A$. Hence, there is $X \subseteq A$, $X \notin \mathcal{J}|A$ such that $X \setminus B_n$ is finite for each n. Then $f[X] \setminus C_n \in \mathrm{Fin}$ and $f[X] \notin \mathcal{J}'$ (as $f^{-1}[f[X]] \supseteq X \notin \mathcal{J}|A$). Thus, $\mathcal{J}'$ is a P^+-ideal.

(a) $\implies$ (b): Suppose that $\mathcal{J}$ is a P^+-ideal such that $\mathcal{I} \subseteq \mathcal{J}$. Consider the sequence $X = \{\alpha_n : n \in \omega\}$. Clearly, $\lim X = 0$ and $X \notin \mathrm{conv}(\mathcal{I}, (\alpha_n))$. Let $f : \omega \to X$ be given by $f(n) = \alpha_n$ for all n. Observe that f witnesses that $\mathcal{I}$ and $\mathrm{conv}(\mathcal{I}, (\alpha_n))|X$ are isomorphic. Consider the ideal $\mathcal{J}' = \{A \subseteq \mathbb{Q} \cap [0,1] : f^{-1}[A \cap X] \in \mathcal{J}\}$ (which is isomorphic with $\mathcal{J} \oplus \mathcal{P}(\omega)$). Note that $\mathrm{conv}(\mathcal{I}, (\alpha_n)) \subseteq \mathcal{J}'$. Moreover, similarly as above it can be shown that $\mathcal{J}'$ is a P^+-ideal. This ends the proof. □

Lemma 3.2. *We have* $\mathrm{conv} \not\leq_K \mathrm{conv}(\mathcal{I}, (\alpha_n))$*, for every* $(\alpha_n) \in \mathcal{S}$ *and every ideal* $\mathcal{I}$ *on* ω*.*

Proof. We will use the following characterization: $\mathrm{conv} \leq_K \mathcal{J}$ if and only if there is a countable family $\{X_n : n \in \omega\} \subseteq [\bigcup \mathcal{J}]^\omega$ such that for every $A \notin \mathcal{J}$ there is $n \in \omega$ such that both $A \cap X_n$ and $A \setminus X_n$ are infinite (cf. [11, Theorem 2.4.3]).

Let $\{X_n : n \in \omega\} \subseteq [[0,1] \cap \mathbb{Q}]^\omega$. We will recursively define $(i_n) \in 2^\omega$, $(Y_n) \subseteq [[0,1] \cap \mathbb{Q}]^\omega$ and a sequence (I_n) of closed subintervals of $[0,1]$ such that:

(a) $I_{n+1} \subseteq \mathrm{int}(I_n)$ and the length of I_n is at most $\frac{1}{2^{n+1}}$, for all $n \in \omega$;
(b) $Y_{n+1} \subseteq Y_n$ and $\overline{Y_n} = I_n$, for all $n \in \omega$;
(c) if $i_n = 0$ then $Y_n \cap X_n = \emptyset$ and if $i_n = 1$ then $Y_n \subseteq X_n$.

First, since $[0,1] = \overline{[0,1] \cap \mathbb{Q}} = \overline{Y_0^0} \cup \overline{Y_0^1}$, where $Y_0^0 = ([0,1] \cap \mathbb{Q}) \setminus X_0$ and $Y_0^1 = X_0$, there is $i_0 \in \{0,1\}$ such that $Y_0^{i_0}$ is dense in some open subinterval of $[0,1]$. Find a closed interval $I_0 \subseteq [0,1]$ of length at most $\frac{1}{2}$ such that $Y_0^{i_0}$ is dense in I_0 and define $Y_0 = I_0 \cap Y_0^{i_0}$. Note that $\overline{Y_0} = I_0$. At step $n+1$, if all i_j, I_j and Y_j for $j \leq n$ are already defined, observe that $I_n = \overline{Y_n} = \overline{Y_{n+1}^0} \cup \overline{Y_{n+1}^1}$, where $Y_{n+1}^0 = Y_n \setminus X_{n+1}$ and $Y_{n+1}^1 = Y_n \cap X_{n+1}$. Then there is $i_{n+1} \in \{0,1\}$ such that $Y_{n+1}^{i_{n+1}}$ is dense in some open subinterval of I_n. Find a closed interval $I_{n+1} \subseteq \mathrm{int}(I_n)$ of length at most $\frac{1}{2^{n+2}}$ such that $Y_{n+1}^{i_{n+1}}$ is dense in I_{n+1} and define $Y_{n+1} = I_{n+1} \cap Y_{n+1}^{i_{n+1}}$. Then $\overline{Y_{n+1}} = I_{n+1}$ and $Y_{n+1} \subseteq Y_n$.

Once the recursion is completed, let $x \in [0,1]$ be the unique point such that $\{x\} = \bigcap_n I_n$. Denote:

$$k_0 = \min\{k \in \omega : \mathrm{int}(I_0) \cap (B(x, \alpha_k) \setminus B(x, \alpha_{k+1})) \neq \emptyset\}.$$

Pick x_k, for each $k \geq k_0$, such that $x_k \in Y_{m_k} \cap (B(x, \alpha_k) \setminus B(x, \alpha_{k+1}))$, where:

$$m_k = \max\{n \in \omega : \mathrm{int}(I_n) \cap (B(x, \alpha_k) \setminus B(x, \alpha_{k+1})) \neq \emptyset\}$$

(item (a) guarantees that each m_k is well-defined). This is possible as each Y_n is dense in I_n (by (b)). Note that $\lim_k m_k = \infty$ (by (a) and the choice of x).

Observe that $X = \{x_k : k \in \omega\} \notin \mathrm{conv}(\mathcal{I}, (\alpha_n))$. Indeed, let $B_0, \ldots, B_m$ be sequences converging $\mathcal{I}$-quickly with respect to (α_n) and assume to the contrary that $X \subseteq \bigcup_{i \leq m} B_i$. Since $\mathcal{I}$ is an ideal, without loss of generality we may assume that $\lim B_i \neq \lim B_j$ whenever $i, j \leq m$ are distinct (as a union of finitely many sequences converging $\mathcal{I}$-quickly with respect to (α_n) to the same limit is a sequence converging $\mathcal{I}$-quickly with respect to (α_n)). Since $\lim X = x$, there is $i_0 \leq m$

such that $x = \lim B_{i_0}$. Then $X \cap \bigcup_{i \leq m, i \neq i_0} B_i$ is finite and $X \setminus \bigcup_{i \leq m, i \neq i_0} B_i$ does not converge $\mathcal{I}$-quickly with respect to (α_n) (as it intersects almost all $B(x, \alpha_k) \setminus B(x, \alpha_{k+1})$). Hence, $X \setminus \bigcup_{i \leq m, i \neq i_0} B_i$ cannot be covered by B_{i_0}. This shows that $X \notin \mathrm{conv}(\mathcal{I}, (\alpha_n))$.

We claim that for each $n \in \omega$ either $X \cap X_n$ or $X \setminus X_n$ is finite. Indeed, fix any $n \in \omega$. If $i_n = 0$ then $Y_j \cap X_n = \emptyset$ for all $j \geq n$ (by (b) and (c)). Thus, by (b) and $\lim_k m_k = \infty$ we get $X \cap X_n \subseteq X \setminus Y_n \in [\mathbb{Q} \cap [0,1]]^{<\omega}$. On the other hand, if $i_n = 1$ then (b) and (c) give us $Y_j \subseteq X_n$ for all $j \geq n$. Therefore, similarly as before, $X \setminus X_n \subseteq X \setminus Y_n \in [\mathbb{Q} \cap [0,1]]^{<\omega}$. This finishes the proof. □

Lemma 3.3. *The ideal* $\mathrm{conv}(\mathcal{I}_d, (\frac{1}{2^{n+1}}))$ *is* $\mathbf{\Sigma^0_6}$.

Proof. Recall that:

$$\mathcal{I}_d = \mathrm{Exh}(\phi) = \left\{ A \subseteq \omega : \lim_n \phi(A \setminus [0,n]) = 0 \right\},$$

where $\phi : \mathcal{P}(\omega) \to [0,1]$ given by:

$$\phi(A) = \sup_{n \in \omega} \frac{|A \cap [0,n]|}{n+1},$$

for all $A \subseteq \omega$, is a lower semicontinuous submeasure, i.e., it satisfies $\phi(\emptyset) = 0$, $\phi(A) \leq \phi(A \cup B) \leq \phi(A) + \phi(B)$ and $\phi(A) = \lim_n \phi(A \cap [0,n])$ (lower semicontinuity) for all $A, B \subseteq \omega$ (see [1, Example 1.2.3.(d)]).

We will need the following observation: if $G \in [\omega]^{<\omega}$ and $\min G > 0$ then $\phi((G-1) \cup G \cup (G+1)) \leq 4\phi(G)$ (here $B + d = \{b + d : b \in B\}$ for $d \in \mathbb{Z}$ and $B \subseteq \omega$). Indeed, it is obvious that $\phi(G+1) \leq \phi(G)$. Moreover, since G is finite, there is $d \in \omega$ such that $\phi(G-1) = \frac{|(G-1) \cap [0,d]|}{d+1}$. If $|(G-1) \cap [0,d]| = 1$ then $\phi(G-1) = \frac{1}{d+1} \leq \frac{2}{d+2} = 2\phi(G)$. If $|(G-1) \cap [0,d]| \neq 1$ then we have:

$$\phi(G-1) = \frac{|(G-1) \cap [0,d]|}{|(G-1) \cap [0,d]| - 1} \cdot \frac{|(G-1) \cap [0,d]| - 1}{d+1} \leq$$

$$2 \cdot \frac{|(G-1) \cap [0,d]| - 1}{d+1} \leq 2\phi(G).$$

Thus, $\phi((G-1) \cup G \cup (G+1)) \leq \phi(G-1) + \phi(G) + \phi(G+1) \leq 4\phi(G)$.

Denote $\alpha_n = \frac{1}{2^{n+1}}$ for all n. We will show that $A \in \mathrm{conv}(\mathcal{I}, (\alpha_n))$ is equivalent to:

$$\exists_{k \in \omega} \ \forall_{m \in \omega} \ \exists_{n \in \omega} \ \exists_{H \in [\mathbb{Q} \cap [0,1]]^{<\omega}} \ \forall_{l \in \omega} \ \exists_{x_0, \ldots, x_k \in \mathbb{Q} \cap [0,1]} \ \exists_{G_0, \ldots, G_k \subseteq [n,l]}$$

$$\left(\left(\forall_{i \leq k} \ \phi(G_i) < \tfrac{1}{m+1} \right) \ \wedge \ (\forall_{i,j \leq k, i \neq j} \ B(x_i, \alpha_n) \cap B(x_j, \alpha_n) = \emptyset) \ \wedge \right.$$

$$A \subseteq H \cup \bigcup_{i \leq k} \left(B(x_i, \alpha_{l+1}) \cup \bigcup_{j \in G_i} (B(x_i, \alpha_j) \setminus B(x_i, \alpha_{j+1})) \right) \ \wedge$$

$$\left. (\forall_{i \leq k} \ \forall_{j \in G_i} \ A \cap (B(x_i, \alpha_j) \setminus B(x_i, \alpha_{j+1})) \text{ is finite}) \right).$$

This will finish the proof as the right-hand side condition is clearly $\mathbf{\Sigma^0_6}$.

($\Rightarrow$): Let $A \in \mathrm{conv}(\mathcal{I}, (\alpha_n))$. Then there is $k \in \omega$ and sequences $A_0, \ldots, A_k$ converging $\mathcal{I}$-quickly with respect to $(\frac{1}{2^{n+1}})$ such that $A \subseteq A_0 \cup \ldots \cup A_k$. Let $m \in \omega$ be arbitrary. For each $i \leq k$ there is $n_i \in \omega$ such that

$$\phi\left(\{j \in \omega \setminus [0, n_i) : \ A_i \cap (B(\lim A_i, \alpha_j) \setminus B(\lim A_i, \alpha_{j+1})) \neq \emptyset\}\right) < \tfrac{1}{4(m+1)}.$$

There is also $n' \in \omega$ such that $|\lim A_i - \lim A_j| > 2\alpha_{n'}$ for all $i, j \leq k$, $i \neq j$. Let $n = \max(\{n_i : i \leq k\} \cup \{1, n'\})$ and $H = A \setminus \bigcup_{i\leq k} B(\lim A_i, \alpha_{n+1}) \in [\mathbb{Q} \cap [0,1]]^{<\omega}$. Let $l \in \omega$ be arbitrary and put $G_i = ((G'_i - 1) \cup G'_i \cup (G'_i + 1)) \cap [n, l]$ for all $i \leq k$, where

$$G'_i = \{j \in [n, l+1] : \ A_i \cap (B(\lim A_i, \alpha_j) \setminus B(\lim A_i, \alpha_{j+1})) \neq \emptyset\}.$$

Note that $\phi(G_i) < \frac{1}{m+1}$ (by the observation from the first paragraph of this proof). For each $i \leq k$ find $x_i \in [0,1] \cap \mathbb{Q}$ close to $\lim A_i$ such that:

- $\forall_{i,j\leq k, i\neq j}\ B(x_i, \alpha_n) \cap B(x_j, \alpha_n) = \emptyset$ (this is possible as $|\lim A_i - \lim A_j| > 2\alpha_{n'}$);
- $\forall_{i\leq k}\ B(\lim A_i, \alpha_{l+2}) \subseteq B(x_i, \alpha_{l+1})$;
- if $i \leq k$ and $n+1 \leq j \leq l+1$ then

$$B(\lim A_i, \alpha_j) \setminus B(\lim A_i, \alpha_{j+1}) \subseteq \bigcup_{j-1\leq p\leq j+1} B(x_i, \alpha_p) \setminus B(x_i, \alpha_{p+1}).$$

Then the first two items above guarantee that $A \cap (B(x_i, \alpha_j) \setminus B(x_i, \alpha_{j+1}))$ is finite for each $i \leq k$ and $j \in G_i \subseteq [0, l]$. Moreover, we have:

$$A \subseteq H \cup \bigcup_{i\leq k} \left(B(x_i, \alpha_{l+1}) \cup \bigcup_{j\in G_i} (B(x_i, \alpha_j) \setminus B(x_i, \alpha_{j+1})) \right).$$

Hence, A satisfies the right-hand side condition.

($\Leftarrow$): First, observe that if A satisfies the right-hand side condition then there is $k \in \omega$ such that for each $l \in \omega$ there are $x_0, \ldots, x_k \in [0,1] \cap \mathbb{Q}$ such that $A \setminus \bigcup_{i\leq k} B(x_i, \alpha_{l+1})$ is finite. Thus, $A \in \mathrm{conv}$ (see [11, Subsection 1.6]). We need to show that each $A \in \mathrm{conv} \setminus \mathrm{conv}(\mathcal{I}, (\alpha_n))$ does not satisfy the right-hand side condition.

Notice that each $A \in \mathrm{conv}$ is contained in finitely many convergent sequences and if $A \notin \mathrm{conv}(\mathcal{I}, (\alpha_n))$ then at least one of those sequences does not belong to $\mathrm{conv}(\mathcal{I}, (\alpha_n))$. Hence, as the right-hand side condition is closed under subsets, it suffices to show that each convergent sequence $A \notin \mathrm{conv}(\mathcal{I}, (\alpha_n))$ does not satisfy the right-hand side condition.

Suppose that A is a convergent sequence not belonging to $\mathrm{conv}(\mathcal{I}, (\alpha_n))$. Let $k \in \omega$ be arbitrary and using $A \notin \mathrm{conv}(\mathcal{I}, (\alpha_n))$ find $m \in \omega$ such that for every $n \in \omega$ we have:

$$\phi\left(\{j \in \omega \setminus [0, n] : \ A \cap (B(\lim A, \alpha_j) \setminus B(\lim A, \alpha_{j+1})) \neq \emptyset\}\right) > \tfrac{4}{m+1}.$$

Fix arbitrary $n \in \omega$ and $H \in [\mathbb{Q} \cap [0,1]]^{<\omega}$. Using lower semicontinuity of ϕ, pick $l \in \omega$ such that:

$$\phi\left(\{j \in (n, l-1) : \ (A \setminus H) \cap (B(\lim A, \alpha_j) \setminus B(\lim A, \alpha_{j+1})) \neq \emptyset\}\right) > \tfrac{4}{m+1}.$$

Observe that the above implies that $l > n + 1$.

Let $x_0, \ldots, x_k \in [0,1] \cap \mathbb{Q}$ and $G_0, \ldots, G_k \subseteq [n, l]$ be arbitrary such that:

(i) $\phi(G_i) < \frac{1}{m+1}$ for all $i \leq k$;
(ii) $B(x_i, \alpha_n) \cap B(x_j, \alpha_n) = \emptyset$ for all $i, j \leq k$, $i \neq j$;
(iii) $A \cap B(x_i, \alpha_j) \setminus B(x_i, \alpha_{j+1})$ is finite for all $i \leq k$ and $j \in G_i$.

Assume to the contrary that

$$A \subseteq H \cup \bigcup_{i\leq k}\left(B(x_i,\alpha_{l+1}) \cup \bigcup_{j\in G_i}(B(x_i,\alpha_j)\setminus B(x_i,\alpha_{j+1}))\right).$$

By (ii), (iii) and the fact that A is convergent, there is $i_0 \leq k$ such that $A \setminus B(x_{i_0},\alpha_{l+1})$ is finite. Then $|x_{i_0} - \lim A| \leq \alpha_{l+1} = \frac{1}{2^{l+2}} < \frac{1}{2^{n+2}}$ (by $l > n+1$) and using (ii) we get:

$$\begin{gathered}(A\setminus H)\cap B(\lim A,\alpha_{n+1}) \subseteq (A\setminus H)\cap B(x_{i_0},\alpha_n) \subseteq \\ B(x_{i_0},\alpha_{l+1}) \cup \bigcup_{j\in G_{i_0}}(B(x_{i_0},\alpha_j)\setminus B(x_{i_0},\alpha_{j+1})) \subseteq \\ B(x_{i_0},\alpha_{l}) \cup \bigcup_{j\in G_{i_0}\setminus\{l\}}(B(x_{i_0},\alpha_j)\setminus B(x_{i_0},\alpha_{j+1})) \subseteq \\ B(\lim A,\alpha_{l-1}) \cup \bigcup_{j\in G_{i_0}\setminus\{l\}}(B(x_{i_0},\alpha_j)\setminus B(x_{i_0},\alpha_{j+1})).\end{gathered}$$

Note that if $x \in B(x_{i_0},\alpha_j)\setminus B(x_{i_0},\alpha_{j+1})$ for some $n \leq j < l$ then:

$$x \in \bigcup_{p\in\{j-1,j,j+1\}}(B(\lim A,\alpha_p)\setminus B(\lim A,\alpha_{p+1})) = B(\lim A,\alpha_{j-1})\setminus B(\lim A,\alpha_{j+2}).$$

Indeed, $|x - x_{i_0}| < \alpha_j = \frac{1}{2^{j+1}}$ implies:

$$|x-\lim A| \leq |x-x_{i_0}| + |x_{i_0}-\lim A| < \frac{1}{2^{j+1}} + \frac{1}{2^{l+2}} \leq \frac{1}{2^{j+1}} + \frac{1}{2^{j+3}} < \frac{1}{2^j} = \alpha_{j-1}.$$

Analogously, $|x - x_{i_0}| \geq \alpha_{j+1} = \frac{1}{2^{j+2}}$ implies:

$$|x-\lim A| \geq |x-x_{i_0}| - |x_{i_0}-\lim A| > \frac{1}{2^{j+2}} - \frac{1}{2^{l+2}} \geq \frac{1}{2^{j+2}} - \frac{1}{2^{j+3}} = \frac{1}{2^{j+3}} = \alpha_{j+2}.$$

Therefore, we obtain:

$$\begin{gathered}(A\setminus H)\cap B(\lim A,\alpha_{n+1}) \subseteq B(\lim A,\alpha_{l-1})\cup \\ \bigcup_{j\in(G_{i_0}-1)\cup G_{i_0}\cup(G_{i_0}+1)} B(\lim A,\alpha_j)\setminus B(\lim A,\alpha_{j+1})\end{gathered}$$

However,

$$\phi(((G_{i_0}-1)\cup G_{i_0}\cup(G_{i_0}+1))\cap(n,l-1)) < 4\phi(G_{i_0}) < \tfrac{4}{m+1}$$

(by the observation from the first paragraph of this proof). This contradicts the choice of l and finishes the proof. □

We are ready to answer the M. Hrušák's question formulated in the Introduction.

Theorem 3.4. *There is a $\mathbf{\Sigma^0_6}$ ideal $\mathcal{I}$ not extendable to a P^+-ideal and such that* conv $\not\leq_K \mathcal{I}$.

Proof. Consider the ideal $\mathrm{conv}(\mathcal{I}_d,(\frac{1}{2^{n+1}}))$. By Lemmas 3.2 and 3.3, it is $\mathbf{\Sigma^0_6}$ and such that conv $\not\leq_K \mathrm{conv}(\mathcal{I}_d,(\frac{1}{2^{n+1}}))$. Moreover, $\mathrm{conv}(\mathcal{I}_d,(\frac{1}{2^{n+1}}))$ cannot be extended to a P^+-ideal by Lemma 3.1, since $\mathcal{I}_d$ is not extendable to a P^+-ideal (cf. [3, Theorem 4.2 and the discussion below Proposition 3.3]). □

4. Concluding remarks

Proposition 2.3 shows that for every Borel ideal $\mathcal{I}$ (and every $(\alpha_n) \in \mathcal{S}$) the family $\text{conv}(\mathcal{I}, (\alpha_n))$ is analytic, however in the general case we were not able to show that it is Borel. In Lemma 3.3 we have done it only in one special case. Thus, we have produced only one counterexample for the M. Hrušák's question. It should be expected that Lemma 3.3 can be generalized for a broader class of ideals.

It seems that our method cannot give any counterexample for the M. Hrušák's question of Borel class lower than $\mathbf{\Sigma^0_6}$. Indeed, if we want to apply Lemma 3.1, we cannot take $\mathcal{I}$ of class lower than $\mathbf{\Pi^0_3}$ (as there are no $\mathbf{\Pi^0_2}$ ideals, all $\mathbf{\Sigma^0_2}$ ideals are clearly extendable to a $\mathbf{\Sigma^0_2}$ ideal and there are no examples of $\mathbf{\Sigma^0_3}$ ideals not extendable to a $\mathbf{\Sigma^0_2}$ ideal). Hence, there is still a chance that the original M. Hrušák's conjecture works for instance in the case of all $\mathbf{\Sigma^0_4}$ ideals.

Finally, it is also worth noticing that the proof of Lemma 3.1 heavily uses the fact that conv cannot be extended to a $\mathbf{\Sigma^0_2}$ ideal. This suggests that conv is indeed a critical ideal for the property of extendability to a $\mathbf{\Sigma^0_2}$ ideal. However, the potential characterization of this property with the use of conv has to be more complicated than just the condition "conv $\not\leq_K \mathcal{I}$".

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