SETS OF RAMSEY-LIMIT POINTS AND IP-LIMIT POINTS

RAFAŁ FILIPÓW, ADAM KWELA, AND PAOLO LEONETTI

Abstract. Let X be an uncountable Polish space and let $\mathcal{H}$ be the Hindman ideal, that is, the family of all $S \subseteq \omega$ which are not IP-sets. For each sequence $x = (x_n)_{n\in\omega}$ taking values in X, let $\Lambda_x(\mathsf{FS})$ be the set of IP-limit points of x. Also, let $\Lambda_x(\mathcal{H})$ be the set of $\mathcal{H}$-limit points of x, that is, the set of ordinary limits of subsequences $(x_n)_{n\in S}$ with $S \notin \mathcal{H}$. After proving that these two notions do not coincide in general, we show that both families of nonempty sets of the type $\Lambda_x(\mathsf{FS})$ and of the type $\Lambda_x(\mathcal{H})$ are precisely the class of nonempty analytic subsets of X.

An analogous result holds also for Ramsey convergence. In the proofs, we use the concept of partition regular functions introduced in [6], which provide a unified approach to these types of convergence.

1. Introduction

For each infinite subset D of the nonnegative integers ω, denote by $\mathsf{FS}(D)$ the family of all finite nonempty sums of distinct elements of D. A set $S \subseteq \omega$ is said to be an IP-*set* if it contains $\mathsf{FS}(D)$ for some infinite $D \subseteq \omega$. It is known that IP-sets are examples of so-called Poincaré sequences which play an important role in the study of recurrences in topological dynamics, see e.g. [11, p. 74]. It follows from Hindman's theorem [15] that the family

$$\mathcal{H} = \{S \subseteq \omega : S \text{ is not an } \mathsf{IP}\text{-set}\} \tag{1}$$

is an ideal on ω, that is, a family of subsets of ω closed under taking finite unions and subsets. Hereafter, $\mathcal{H}$ will be referred to as the *Hindman ideal*, see e.g. [9, p. 109].

Given a sequence $x = (x_n)_{n\in\omega}$ taking values in a topological space X, we say that $\eta \in X$ is an IP-*limit point* (or an FS-*limit point*) of x if there exists an infinite $D \subseteq \omega$ such that for every neighborhood U of η there exists $k \in \omega$ such that $x_n \in U$ for all $n \in \mathsf{FS}(D \setminus k)$; equivalently, the subsequence $(x_n)_{n\in\mathsf{FS}(D)}$ is "IP-convergent" to η, see e.g. [12]. The set of such points is denoted by

$$\Lambda_x(\mathsf{FS}) = \{\eta \in X : \eta \text{ is an IP-limit point of } x\}.$$

In another direction, for each infinite set $D \subseteq \omega$, let $\mathsf{r}(D)$ be the family of subsets of D with cardinality 2. It follows by Ramsey's theorem that the family

$$\mathcal{R} = \{S \subseteq \mathsf{r}(\omega) : S \text{ does not contain } \mathsf{r}(D) \text{ for any infinite } D \subseteq \omega\} \tag{2}$$

Date: January 18, 2026.

2020 *Mathematics Subject Classification.* Primary: 03E75, 05D10, 40A05, 54A20. Secondary: 03E05, 11B05, 28A05, 40A35.

Key words and phrases. Ramsey convergence; IP-convergence; analytic sets; ideal limit points; Hindman ideal; partition regular functions.

The second-listed author was supported by the Polish National Science Centre project OPUS No. 2024/53/B/ST1/02494.

is an ideal on the countably infinite set $\mathsf{r}(\omega)$. Hereafter, $\mathcal{R}$ is called the *Ramsey ideal*, see e.g. [16, 21]. As before, if $y = (y_{\{n,m\}})_{\{n,m\}\in\mathsf{r}(\omega)}$ is a sequence taking values in a topological space X, we say that $\eta \in X$ is an r-*limit point* of y if there exists an infinite set $D \subseteq \omega$ such that for every neighborhood U of η there exists $k \in \omega$ such that $y_{\{n,m\}} \in U$ for all $\{n, m\} \in \mathsf{r}(D \setminus k)$; equivalently, the subsequence $(y_{\{n,m\}})_{\{n,m\}\in\mathsf{r}(D)}$ is "r-convergent" to η, see e.g. [3, 4]. The set of such points is denoted by

$$\Lambda_y(\mathsf{r}) = \{\eta \in X : \eta \text{ is an } \mathsf{r}\text{-limit point of } y\}.$$

The main aim of this work is to show that each of the two families of nonempty sets of the type $\Lambda_x(\mathsf{FS})$ and of the type $\Lambda_y(\mathsf{r})$ coincides with the class of nonempty analytic subsets of X, provided that X is an uncountable Polish space (see Theorem 4.2 and Theorem 4.3 below). This continues the line of research in [1, 2, 8, 13, 14] on the study of attainable sets of limit points with respect to ideal convergence. In fact, after recalling the necessary definitions in Section 2, we show that variants of our main results hold also in the context of ideal limit points for the Hindman ideal $\mathcal{H}$ and Ramsey ideal $\mathcal{R}$ defined in (1) and (2), respectively. In addition, in Section 3 we provide several characterizations of P-like properties, and specialize them to the cases of maps FS and r (see Corollary 3.4 below).

Following [6], we use partition regular functions and study some of their properties. This gives us a unified approach to consider our questions for the types of convergence considered here. For additional motivations, we refer the reader to [6].

2. Preliminaries

2.1. **Notations.** The set of all nonnegative integers is denoted by ω, and each nonnegative integer $n \in \omega$ is identified with the set $\{0, 1, \dots, n-1\}$. Given a set X, we denote by X^ω and $X^{<\omega}$ the families of all infinite sequences $x : \omega \to X$ and the families of all finite sequences $x : n \to X$ with $n \in \omega$, respectively. For a finite sequence $s \in \omega^{<\omega}$ we denote its length by $|s|$. The infinite zero sequence $(0, 0, \dots) \in \omega^\omega$ is denoted by 0^∞. Given a set X, we denote by $[X]^\kappa$ and $[X]^{<\kappa}$ the families of all subsets of X of cardinality κ and of cardinality strictly less than κ, respectively. In particular, we write $\mathsf{Fin}(X) = [X]^{<\omega}$ to denote the family of all finite subsets of X.

If X is a metric space with metric d, we denote by $B(p, r) = \{x \in X : d(x, p) < r\}$ and $\overline{B}(p, r) = \{x \in X : d(x, p) \le r\}$ the open ball and closed ball of radius $r > 0$ and centered at $p \in X$, respectively. Lastly, if X is a Polish space (that is, a separable and completely metrizable space), we use the standard notation $\Sigma^1_1(X)$ and $\Pi^1_1(X)$ (or simply Σ^1_1 and Π^1_1 if X is understood) for the families of analytic and coanalytic subsets of X, respectively, see e.g. [17].

2.2. **Partition regular functions.** Let us start by recalling the main definition of partition regular function from [6, Definition 3.1].

Definition 2.1. Let Ω, Ψ be two countably infinite sets and pick a nonempty family $\mathcal{F} \subseteq [\Omega]^\omega$ such that $A \setminus K \in \mathcal{F}$ for all $A \in \mathcal{F}$ and $K \in \mathsf{Fin}(\Omega)$. We say that a function

$$\rho : \mathcal{F} \to [\Psi]^\omega$$

is *partition regular* if it satisfies the following properties:

(M) $\rho(E) \subseteq \rho(F)$ for all $E, F \in \mathcal{F}$ such that $E \subseteq F$;

(R) for every $F \in \mathcal{F}$ and $A, B \subseteq \Psi$ such that $\rho(F) = A \cup B$, there exists $E \in \mathcal{F}$ such that either $\rho(E) \subseteq A$ or $\rho(E) \subseteq B$;
(S) for every $F \in \mathcal{F}$ there exists a subset $E \subseteq F$, $E \in \mathcal{F}$ such that for every $a \in \rho(E)$ there exists $K \in \mathsf{Fin}(\Omega)$ for which $a \notin \rho(E \setminus K)$.

For each partition regular function ρ as in Definition 2.1, we associate the family

$$\mathcal{I}_\rho = \{S \subseteq \Psi : \forall_{F \in \mathcal{F}}\ \rho(F) \not\subseteq S\}\,. \tag{3}$$

It is not difficult to see that $\mathcal{I}_\rho$ is an ideal on Ψ, that is, a proper subset of $\mathcal{P}(\Psi)$ which contains $\mathsf{Fin}(\Psi)$ and is closed under taking finite unions and subsets; vice versa, for each ideal $\mathcal{I}$ on Ψ, we have $\mathcal{I} = \mathcal{I}_{\rho_\mathcal{I}}$ for the partition regular function $\rho_\mathcal{I} : \mathcal{I}^+ \to [\Psi]^\omega$ defined by $\rho_\mathcal{I}(F) = F$ where $\mathcal{I}^+ = \mathcal{P}(\Psi) \setminus \mathcal{I}$; see [6, Proposition 3.3] for details.

As anticipated in Section 1, in this work we are mainly interested in two partition regular functions $\mathsf{r} : [\omega]^\omega \to [[\omega]^2]^\omega$ and $\mathsf{FS} : [\omega]^\omega \to [\omega]^\omega$ defined by

$$\mathsf{r}(D) = [D]^2 \quad \text{and} \quad \mathsf{FS}(D) = \left\{\sum\nolimits_{n\in\alpha} n : \alpha \in [D]^{<\omega} \setminus \{\emptyset\}\right\}$$

for all $D \in [\omega]^\omega$, respectively, see [6, Theorem 3.4 and Theorem 3.6]. Following the notation in (3), the associated ideals

$$\mathcal{R} = \mathcal{I}_\mathsf{r} \quad \text{and} \quad \mathcal{H} = \mathcal{I}_\mathsf{FS}$$

are called *Ramsey ideal* and *Hindman ideal*, respectively; see [9, 16, 21].

2.3. **Families of cluster and limit points.** We now present our main definitions of ρ-cluster points and ρ-limit points. Hereafter, given a sequence $x : \Psi \to X$, we usually write $x_s = x(s)$ for each $s \in \Psi$.

Definition 2.2. Let $x : \Psi \to X$ be a sequence in a topological space X and pick a partition regular function $\rho : \mathcal{F} \to [\Psi]^\omega$. Denote by $\Gamma_x(\rho)$ the set of **ρ-cluster points** of x, that is, the set of all $\eta \in X$ such that, for all neighborhoods U of η,

$$\exists_{F\in\mathcal{F}} \quad \rho(F) \subseteq \{s \in \Psi : x_s \in U\}\,.$$

We also write $\mathscr{C}_X(\rho)$ for the family of sets of ρ-cluster points of sequences with values in X, together with the empty set, that is,

$$\mathscr{C}_X(\rho) = \{A \subseteq X : \exists_{x\in X^\Psi}\ A = \Gamma_x(\rho)\} \cup \{\emptyset\}.$$

If the topological space X is understood, we simply write $\mathscr{C}(\rho)$.

Equivalently, η is a ρ-cluster point of x if $\{s \in \Psi : x_s \in U\} \notin \mathcal{I}_\rho$ for all neighborhoods U of η, where $\mathcal{I}_\rho$ is the ideal generated by ρ as in (3). In particular, the definition of ρ-cluster point depends only on the ideal generated by ρ, hence it makes sense to write $\Gamma_x(\mathcal{I}_\rho)$. In fact, if $\mathcal{I}$ is an ideal on Ψ, elements of $\Gamma_x(\mathcal{I}) = \Gamma_x(\rho_\mathcal{I})$ are usually called $\mathcal{I}$-cluster points, see e.g. [2, 20]. It is worth noting that the case of an empty set of ρ-cluster points is not really interesting: indeed, it is known that, if X is a metric space and $\mathcal{I}$ is an ideal on ω, then there exists a sequence $x : \omega \to X$ such that $\Gamma_x(\mathcal{I}) = \emptyset$ if and only if X is not compact, see [8, Proposition 5.1].

For the next definition, let X be a topological space, $\rho : \mathcal{F} \to [\Psi]^\omega$ be a partition regular function, and pick a sequence $x : \Psi \to X$. A subsequence y of x is said to be a *ρ-subsequence* of x if there exists $F \in \mathcal{F}$ such that $y = (x_s)_{s\in\rho(F)}$. Informally, a ρ-subsequence is a subsequence of x with an index set which is "not small" with respect to ρ. Accordingly, a ρ-subsequence $y = x \restriction \rho(F)$ is said to be *ρ-convergent*

to $\eta \in X$, shortened as $\rho\text{-}\lim y = \eta$, if for every neighborhood U of η there exists $K \in \mathsf{Fin}(\Omega)$ such that $y_s \in U$ for every $s \in \rho(F \setminus K)$. Notice that this convergence depends on the choice of F. In fact, it may happen that $\rho(F) = \rho(G)$ for some $F, G \in \mathcal{F}$, yet $(y_s)_{s\in\rho(F)}$ is ρ-convergent while $(y_s)_{s\in\rho(G)}$ is not, see [6].

Definition 2.3. Let $x : \Psi \to X$ be a sequence in a topological space X and pick a partition regular function $\rho : \mathcal{F} \to [\Psi]^\omega$. Denote by $\Lambda_x(\rho)$ the set of **ρ-limit points** of x, that is, the set of all $\eta \in X$ such that

$$\rho\text{-}\lim y = \eta$$

for some ρ-subsequence y of x. We also write $\mathscr{L}_X(\rho)$ for the family of sets of ρ-limit points of sequences with values in X together with the empty set, that is,

$$\mathscr{L}_X(\rho) = \{A \subseteq X : \exists_{x\in X^\Psi}\, A = \Lambda_x(\rho)\} \cup \{\emptyset\}. \tag{4}$$

If the topological space X is understood, we simply write $\mathscr{L}(\rho)$.

In the case where $\rho = \rho_\mathcal{I}$ for some ideal $\mathcal{I}$ on ω, we define

$$\Lambda_x(\mathcal{I}) = \Lambda_x(\rho_\mathcal{I}) \quad \text{and} \quad \mathscr{L}_X(\mathcal{I}) = \mathscr{L}_X(\rho_\mathcal{I}).$$

Elements of $\Lambda_x(\mathcal{I})$ are called *$\mathcal{I}$-limit points* of x.

More explicitly, η is a ρ-limit point of x if there exists $F \in \mathcal{F}$ such that, for each neighborhood U of η, there is a finite set $K \in \mathsf{Fin}(\Omega)$ such that $x_s \in U$ for all $s \in \rho(F \setminus K)$. In the case where $\rho = \rho_\mathcal{I}$ for some ideal $\mathcal{I}$, it is easy to see that η is an $\mathcal{I}$-limit point if and only if there exists $E \in \mathcal{I}^+$ such that the subsequence $y = x \restriction E$ is convergent to η; hence, this coincides with the classical notion of $\mathcal{I}$-limit point studied e.g. in [1, 2, 8, 19]. Unlike the case of ρ-cluster points, realizing the empty set in (4) as a set of the form $\Lambda_x(\rho)$ is much more difficult to handle. Indeed, if $\mathcal{I}$ is an ideal on ω, several implications on the existence of a sequence x such that $\Lambda_x(\mathcal{I}) = \emptyset$ can be found in [8, Proposition 5.3 and Figure 3]. Recently, significant progress on this topic was made in [5]. It is worth noting that families of $\mathcal{I}$-limit points and $\mathcal{I}$-cluster points for an arbitrary ideal $\mathcal{I}$ were introduced in [18], however they were already considered in [10] in the case of the ideal of all sets of asymptotic density zero.

Proposition 2.4. *Let $x : \Psi \to X$ be a sequence in a topological space X and pick a partition regular function $\rho : \mathcal{F} \to [\Psi]^\omega$. Then*

$$\Lambda_x(\mathcal{I}_\rho) \subseteq \Lambda_x(\rho) \subseteq \Gamma_x(\rho).$$

Proof. First, suppose that $\eta \in \Lambda_x(\mathcal{I}_\rho)$, i.e., there exists $S \in \mathcal{I}_\rho^+$ such that $\lim x \restriction S = \eta$. Then there is $F \in \mathcal{F}$ such that $\rho(F) \subseteq S$, which gives $\lim x \restriction \rho(F) = \eta$. Since ρ is partition regular, there is $E \in \mathcal{F}$ such that $E \subseteq F$ (which implies $\rho(E) \subseteq \rho(F)$) and for all $a \in \rho(E)$ there is a finite $K \subseteq \Omega$ with $a \notin \rho(E \setminus K)$. Then it is easy to check that the ρ-subsequence $x \restriction \rho(E)$ is ρ-convergent to η. Hence, $\eta \in \Lambda_x(\rho)$. This implies that $\Lambda_x(\mathcal{I}_\rho) \subseteq \Lambda_x(\rho)$.

Second, suppose that $\eta \in \Lambda_x(\rho)$, i.e., there exists $F \in \mathcal{F}$ such that the ρ-subsequence $x \restriction \rho(F)$ is ρ-convergent to η. Then for each neighborhood U of η we can find a finite $K \subseteq \Omega$ such that $x_s \in U$ for all $s \in \rho(F \setminus K)$. Since $\rho(F \setminus K) \in \mathcal{I}_\rho^+$, we conclude that $\eta \in \Gamma_x(\rho)$. Therefore $\Lambda_x(\rho) \subseteq \Gamma_x(\rho)$. □

The next example shows that $\Lambda_x(\mathcal{I}_\rho)$ does not coincide, in general, with $\Lambda_x(\rho)$.

Example 2.5. For each nonzero $n \in \omega$, let $\nu_2(n)$ be the 2-adic valuation of n, that is, the maximal $k \in \omega$ such that 2^k divides n, and define $S_m = \{n \in \omega \setminus \{0\} : \nu_2(n) = m\}$ for each $m \in \omega$. Set $\rho = \mathsf{FS}$ and $X = [0,1]$, so that $\mathcal{F} = [\omega]^\omega$ and $\mathcal{H} = \mathcal{I}_\rho$. We claim that there exists a sequence $x \in X^\omega$ which has no $\mathcal{I}_\rho$-limit points and, on the other hand, it admits a ρ-subsequence which is ρ-convergent. In particular, this would imply that

$$\Lambda_x(\mathcal{H}) = \emptyset \quad \text{and} \quad \Lambda_x(\mathsf{FS}) \neq \emptyset.$$

We are going to show that the sequence $x = (x_n)_{n\in\omega}$ defined by $x_0 = 1/3$ and $x_n = 2^{-\nu_2(n)}$ for each nonzero $n \in \omega$ satisfies our claim.

To this end, we show first that x has no $\mathcal{H}$-limit points. Indeed, suppose for the sake of contradiction that there exists an infinite $E \subseteq \omega$ such that $x \restriction \mathsf{FS}(E)$ is convergent (in the ordinary sense). Since the image of x equals $\{1/3\} \cup \{2^{-n} : n \in \omega\}$, there are only the following two possibilities.

(i) The sequence $(x_n)_{n\in\mathsf{FS}(E)}$ is eventually constant. In other words, there exist $k, m \in \omega$ such that $\mathsf{FS}(E) \setminus k \subseteq S_m$. In particular, since $E \subseteq \mathsf{FS}(E)$, we have $E \setminus k \subseteq S_m$. Now, pick distinct $a, b \in E \setminus k$, so that $\nu_2(a) = \nu_2(b) = m$. Then $a + b \in \mathsf{FS}(E) \setminus k \subseteq S_m$ and, on the other hand, $\nu_2(a + b) \geq m + 1$, contradicting $a + b \in S_m$.

(ii) The sequence $(x_n)_{n\in\mathsf{FS}(E)}$ is convergent to zero. Take any $a \in E \setminus \{0\}$ and pick $k \in \omega$ such that $a \in S_k$. Since $(x_n)_{n\in\mathsf{FS}(E)}$ converges to zero, $\mathsf{FS}(E) \cap S_m$ has to be finite for each $m \in \omega$. In particular, since $E \subseteq \mathsf{FS}(E)$, we have that $E \cap S_m$ is finite for each $m \in \omega$. Then $E \setminus \bigcup_{m\leq k} S_m$ is infinite and for each $b \in E \setminus \bigcup_{m\leq k} S_m$ we have $a+b \in \mathsf{FS}(E) \cap S_k$, which contradicts the fact that $\mathsf{FS}(E) \cap S_k$ is finite.

Lastly, we show that, if $F = \{2^n : n \in \omega\}$ (so that $\mathsf{FS}(F) = \omega \setminus \{0\}$), then the FS-subsequence $y = x \restriction \mathsf{FS}(F)$ is FS-convergent to 0. Indeed, for each $k \in \omega$ and each $n \in \mathsf{FS}(F \setminus 2^k)$, we have $\nu_2(n) \geq k$, hence $y_n \leq 2^{-k}$. Therefore $\mathsf{FS}\text{-}\lim y = 0$.

The main goal of this work is to show that, for each nonempty analytic subset C of an uncountable Polish space X, there exist sequences x and y taking values in X which satisfy

$$C = \Lambda_x(\mathsf{FS}) = \Lambda_x(\mathcal{H}) \quad \text{and} \quad C = \Lambda_y(\mathsf{r}) = \Lambda_y(\mathcal{R}).$$

As an application, we obtain

$$\mathscr{L}(\mathsf{FS}) = \mathscr{L}(\mathcal{H}) = \mathscr{L}(\mathsf{r}) = \mathscr{L}(\mathcal{R}) = \Sigma^1_1,$$

see Theorem 4.2 and Theorem 4.3 below.

3. P-like properties

It will be useful to recall the following definitions, see [6, Section 6]. For a partition regular function $\rho : \mathcal{F} \to [\Psi]^\omega$, we write $\rho(F) \subseteq^\rho B$ if there is a finite set $K \subseteq \Omega$ with $\rho(F \setminus K) \subseteq B$. We remark that "$\rho(F) \subseteq^\rho B$" is a relation between F and B, and not between $\rho(F)$ and B (in fact, it is possible that $\rho(F) = \rho(G)$ and $\rho(F) \subseteq^\rho B$ but $\rho(G) \not\subseteq^\rho B$, see [6, Remark after Definition 6.3]).

A partition regular function $\rho : \mathcal{F} \to [\Psi]^\omega$ is

(i) P^+ if for every $\subseteq$-decreasing sequence $(A_n)_{n\in\omega} \in (\mathcal{I}^+_\rho)^\omega$ there exists $F \in \mathcal{F}$ such that $\rho(F) \subseteq^\rho A_n$ for each $n \in \omega$;

(ii) P^- if for every $\subseteq$-decreasing sequence $(A_n)_{n\in\omega} \in (\mathcal{I}_\rho^+)^\omega$ with $A_n \setminus A_{n+1} \in \mathcal{I}_\rho$ for each $n \in \omega$ there exists $F \in \mathcal{F}$ such that $\rho(F) \subseteq^\rho A_n$ for each $n \in \omega$;
(iii) $P^|$ if for every $\subseteq$-decreasing sequence $(A_n)_{n\in\omega} \in (\mathcal{I}_\rho^+)^\omega$ with $A_n \setminus A_{n+1} \in \mathcal{I}_\rho^+$ for each $n \in \omega$ there exists $F \in \mathcal{F}$ such that $\rho(F) \subseteq^\rho A_n$ for each $n \in \omega$.

Proposition 3.1. *Let $\rho : \mathcal{F} \to [\Psi]^\omega$ be a partition regular function. Then ρ is P^+ if and only if it is both $P^|$ and P^-.*

Proof. Only If part. This is obvious.

If part. Suppose that ρ is $P^|$ and P^-. Fix a $\subseteq$-decreasing sequence $(A_n)_{n\in\omega} \in (\mathcal{I}_\rho^+)^\omega$. Define

$$T = \{0\} \cup \{n \in \omega : A_n \setminus A_{n+1} \in \mathcal{I}_\rho^+\}.$$

If T is finite, define $k = \max T$ and $B_n = A_{k+1+n}$ for all $n \in \omega$. Then $(B_n)_{n\in\omega}$ is a $\subseteq$-decreasing sequence with $B_n \setminus B_{n+1} \in \mathcal{I}_\rho$ for each $n \in \omega$. Since ρ is P^-, there exists $F \in \mathcal{F}$ such that $\rho(F) \subseteq^\rho B_n$ for each $n \in \omega$, hence $\rho(F) \subseteq^\rho A_n$ for each $n \in \omega$.

If T is infinite, let $(t_n)_{n\in\omega}$ be the increasing enumeration of T and define $C_n = A_{t_n}$ for all $n \in \omega$. Then $(C_n)_{n\in\omega}$ is a $\subseteq$-decreasing sequence with $C_n \setminus C_{n+1} \supseteq A_{t_n} \setminus A_{t_n+1} \in \mathcal{I}_\rho^+$ for each $n \in \omega$. Applying the fact that ρ is $P^|$, there exists $E \in \mathcal{F}$ such that $\rho(E) \subseteq^\rho C_n$ for each $n \in \omega$. Also in this case $\rho(E) \subseteq^\rho A_n$ for each $n \in \omega$. Therefore ρ is P^+. □

Proposition 3.2. *Let X be a metric space with an accumulation point, and let $\rho : \mathcal{F} \to [\Psi]^\omega$ be a partition regular function. Then the following hold.*

(i) *ρ is P^+ if and only if $\Lambda_x(\rho) = \Gamma_x(\rho)$ for every sequence $x \in X^\Psi$.*
(ii) *ρ is $P^|$ if and only if $\Lambda_x(\rho)$ is closed for every sequence $x \in X^\Psi$.*
(iii) *Suppose that X is locally compact. Then ρ is P^- if and only if each isolated point in $\Gamma_x(\rho)$ belongs to $\Lambda_x(\rho)$ for every sequence $x \in X^\Psi$.*

Proof. In the following proofs, let d be the metric on X.

(i) Only If part. Fix a sequence $x = (x_s)_{s\in\Psi}$ in X. By Proposition 2.4 we have $\Lambda_x(\rho) \subseteq \Gamma_x(\rho)$, so we only need to show that $\Gamma_x(\rho) \subseteq \Lambda_x(\rho)$. If $\Gamma_x(\rho) = \emptyset$, the claim is clear. Otherwise, fix $p \in \Gamma_x(\rho)$. Then for every $n \in \omega$ we have $A_n = \{s \in \Psi : d(x_s, p) < 2^{-n}\} \in \mathcal{I}_\rho^+$. Moreover, $(A_n)_{n\in\omega}$ is $\subseteq$-decreasing. Since ρ is P^+, there is $F \in \mathcal{F}$ such that $\rho(F) \subseteq^\rho A_n$ for all $n \in \omega$. Now, it is enough to show that $(x_s)_{s\in\rho(F)}$ is ρ-convergent to p. To this aim, let U be a neighborhood of p and pick $n \in \omega$ such that $B(p, 2^{-n}) \subseteq U$. Then there is a finite set $K \subseteq \Omega$ such that $\rho(F \setminus K) \subseteq A_n$. It follows that $x_s \in B(p, 2^{-n}) \subseteq U$ for all $s \in \rho(F \setminus K)$, hence $(x_s)_{s\in\rho(F)}$ is ρ-convergent to p.

If part. Let us suppose that ρ is not P^+, and let $(A_n)_{n\in\omega} \in (\mathcal{I}_\rho^+)^\omega$ be a $\subseteq$-decreasing sequence such that for every $F \in \mathcal{F}$ we have $\rho(F) \not\subseteq^\rho A_n$ for some $n \in \omega$. Let also p be an accumulation point of X and pick an injective sequence $(y_n)_{n\in\omega}$ such that the sequence $(\delta_n)_{n\in\omega}$ given by $\delta_n = d(y_n, p)$ is decreasing and tends to zero. Define the sequence $x = (x_s)_{s\in\Psi}$ by

$$x_s = \begin{cases} y_0 & \text{if } s \in \Psi \setminus A_0, \\ p & \text{if } s \in \bigcap_{n\in\omega} A_n, \\ y_n & \text{if } s \in A_n \setminus A_{n+1}. \end{cases}$$

We will show that $p \in \Gamma_x(\rho)$ and that $p \notin \Lambda_x(\rho)$.

In fact, let U be a neighborhood of p and find $n \in \omega$ such that the closed ball $\overline{B}(p, \delta_n)$ is contained in U. Then $x_s \in \overline{B}(p, \delta_n) \subseteq U$ for all $s \in A_n$. Since $A_n \in \mathcal{I}_\rho^+$, we obtain that $p \in \Gamma_x(\rho)$.

Lastly, to prove that $p \notin \Lambda_x(\rho)$, fix $F \in \mathcal{F}$. We are going to show that the subsequence $(x_s)_{s\in\rho(F)}$ does not ρ-converge to p. By the choice of $(A_n)_{n\in\omega}$, there is $n \in \omega$ such that $\rho(F \setminus K) \not\subseteq A_n$ for all finite $K \subseteq \Omega$. Define $U = B(p, \delta_n)$. Then U is a neighborhood of p. Fix a finite set $K \subseteq \Omega$. There is $s \in \rho(F \setminus K) \setminus A_n$. Then $s \in \rho(F \setminus K)$ and $x_s \in \{y_0, y_1, \dots, y_{n-1}\}$, so $x_s \notin U$. By the arbitrariness of K, we conclude that $(x_s)_{s\in\rho(F)}$ does not ρ-converge to p.

(ii) Only If part. Fix a sequence $x = (x_s)_{s\in\Psi}$ in X and let $(y_n)_{n\in\omega}$ be a sequence in $\Lambda_x(\rho)$ converging to some $p \in X$. Then for each $n \in \omega$ there is $F_n \in \mathcal{F}$ such that the ρ-subsequence $(x_s)_{s\in\rho(F_n)}$ is ρ-convergent to y_n. We need to show that $p \in \Lambda_x(\rho)$. This is clear if $p = y_n$ for some $n \in \omega$, so we can assume that $p \neq y_n$ for all $n \in \omega$. Then it is possible to find a decreasing sequence $(\delta_n)_{n\in\omega}$ of positive reals such that $\lim_n \delta_n = 0$ and for each $n \in \omega$ there is $k \in \omega$ such that $d(y_k, p) \in (\delta_{n+1}, \delta_n)$. Define $A_n = \{s \in \Psi : \ d(x_s, p) < \delta_n\}$ for all $n \in \omega$. Then $(A_n)_{n\in\omega}$ is $\subseteq$-decreasing. Moreover, for every $n \in \omega$ we have $A_n \setminus A_{n+1} \in \mathcal{I}_\rho^+$. Indeed, if $n \in \omega$ then there is $k \in \omega$ such that $\delta_{n+1} < d(y_k, p) < \delta_n$, so $U = B(p, \delta_n) \setminus \overline{B}(p, \delta_{n+1}) = \{x \in X : \ \delta_{n+1} < d(x, p) < \delta_n\}$ is a neighborhood of y_k and we can find a finite $K \subseteq \Omega$ such that $x_s \in U$ for all $s \in \rho(F_k \setminus K)$. Thus, $\rho(F_k \setminus K) \subseteq A_n \setminus A_{n+1}$ and consequently $A_n \setminus A_{n+1} \in \mathcal{I}_\rho^+$ (in particular, $A_n \in \mathcal{I}_\rho^+$).

Since ρ is $P^|$, there is $F \in \mathcal{F}$ such that $\rho(F) \subseteq^\rho A_n$ for all $n \in \omega$. We will show that $(x_s)_{s\in\rho(F)}$ is ρ-convergent to p. Let U be a neighborhood of p and find $n \in \omega$ such that the ball $B(p, \delta_n)$ is contained in U. Then there is a finite set $K \subseteq \Omega$ such that $\rho(F \setminus K) \subseteq A_n$. It follows that $x_s \in B(p, \delta_n) \subseteq U$ for all $s \in \rho(F \setminus K)$, hence $(x_s)_{s\in\rho(F)}$ is ρ-convergent to p. Therefore $p \in \Lambda_x(\rho)$.

If part. Assume that ρ is not $P^|$. Hence it is possible to fix a $\subseteq$-decreasing sequence of sets $(A_n)_{n\in\omega} \in (\mathcal{I}_\rho^+)^\omega$ such that

- $A_n \setminus A_{n+1} \in \mathcal{I}_\rho^+$ for all $n \in \omega$,
- for every $F \in \mathcal{F}$ there is $n \in \omega$ such that $\rho(F) \not\subseteq^\rho A_n$.

Let also p be an accumulation point of X and pick an injective sequence $(y_n)_{n\in\omega}$ such that the sequence $(\delta_n)_{n\in\omega}$ defined by $\delta_n = d(y_n, p)$ is strictly decreasing and has limit zero. At this point, define the sequence $x = (x_s)_{s\in\Psi}$ by

$$x_s = \begin{cases} y_0 & \text{if } s \in \Psi \setminus A_0, \\ p & \text{if } s \in \bigcap_{n\in\omega} A_n, \\ y_n & \text{if } s \in A_n \setminus A_{n+1}. \end{cases}$$

To complete the proof, it will be enough to show that

$$\{y_n : \ n \in \omega\} \subseteq \Lambda_x(\rho) \quad \text{and} \quad p \notin \Lambda_x(\rho). \tag{5}$$

To this aim, fix $n \in \omega$. Then, $A_n \setminus A_{n+1} \in \mathcal{I}_\rho^+$, hence there is $F \in \mathcal{F}$ such that $\rho(F) \subseteq A_n \setminus A_{n+1}$. Since $x_s = y_n$ for all $s \in \rho(F)$, the ρ-subsequence $(x_s)_{s\in\rho(F)}$ is ρ-convergent to y_n. This proves the first part of (5). The second part of (5) can be shown similarly as in the proof of the If part of item (i).

(iii) Only If part. Fix a sequence $x = (x_s)_{s\in\Psi}$ in X and let p be an isolated point of $\Gamma_x(\rho)$. Taking into account that X is locally compact, it is possible to fix an integer $n_0 \in \omega$ such that $\overline{B}(p, 2^{-n_0})$ is compact and $\overline{B}(p, 2^{-n_0}) \cap \Gamma_x(\rho) = \{p\}$.

Since $p \in \Gamma_x(\rho)$, for every $n \in \omega$ we have that

$$A_n = \{s \in \Psi : \ x_s \in \overline{B}(p, 2^{-(n_0+n)})\} \in \mathcal{I}_\rho^+.$$

In addition, $(A_n)_{n\in\omega}$ is $\subseteq$-decreasing. We are going to show that $A_n \setminus A_{n+1} \in \mathcal{I}_\rho$ for all $n \in \omega$. Indeed, suppose for the sake of contradiction that there is $n \in \omega$ such that $A_n \setminus A_{n+1} \in \mathcal{I}_\rho^+$. Then $Z = \overline{B}(p, 2^{-(n_0+n)}) \setminus B(p, 2^{-(n_0+n+1)})$ is compact and $x_s \in Z$ for all $s \in A_n \setminus A_{n+1} \in \mathcal{I}^+$. Thanks to [20, Lemma 3.1(vi)], it is possible to fix a point $p' \in \Gamma_x(\rho) \cap Z$. Then $p' \neq p$ and $p' \in Z \subseteq \overline{B}(p, 2^{-(n_0+n)}) \subseteq \overline{B}(p, 2^{-n_0})$. This contradicts the choice of n_0.

Since ρ is P^-, there is $F \in \mathcal{F}$ such that $\rho(F) \subseteq^\rho A_n$ for all $n \in \omega$. Now, we are going to show that $(x_s)_{s\in\rho(F)}$ is ρ-convergent to p. To this aim, let U be a neighborhood of p and find $n \in \omega$ such that the closed ball $\overline{B}(p, 2^{-(n_0+n)})$ is contained in U. There is a finite set $K \subseteq \Omega$ such that $\rho(F \setminus K) \subseteq A_n$. Then $x_s \in \overline{B}(p, 2^{-(n_0+n)}) \subseteq U$ for all $s \in \rho(F \setminus K)$, so $(x_s)_{s\in\rho(F)}$ is ρ-convergent to p.

If part. Assume that ρ is not P^- and let $(A_n)_{n\in\omega} \in (\mathcal{I}_\rho^+)^\omega$ be a $\subseteq$-decreasing sequence such that:

- $A_n \setminus A_{n+1} \in \mathcal{I}_\rho$ for all $n \in \omega$,
- for every $F \in \mathcal{F}$ there is $n \in \omega$ such that $\rho(F) \not\subseteq^\rho A_n$.

Let also p be an accumulation point of X and pick an injective sequence $(y_n)_{n\in\omega}$ such that the sequence $(\delta_n)_{n\in\omega}$ given by $\delta_n = d(y_n, p)$ is decreasing and tends to zero. Define $x = (x_s)_{s\in\Psi}$ by

$$x_s = \begin{cases} y_0 & \text{if } s \in \Psi \setminus A_0, \\ p & \text{if } s \in \bigcap_{n\in\omega} A_n, \\ y_n & \text{if } s \in A_n \setminus A_{n+1}. \end{cases}$$

Note that $\Gamma_x(\rho) \subseteq \{y_n : \ n \in \omega\} \cup \{p\}$. To finish the proof, we will show that $p \in \Gamma_x(\rho)$, $y_n \notin \Gamma_x(\rho)$ for all $n \geq 1$ (so that p is an isolated point of $\Gamma_x(\rho)$) and that $p \notin \Lambda_x(\rho)$.

To this aim, let U be a neighborhood of p and find $n \in \omega$ such that the ball $\overline{B}(p, \delta_n)$ is contained in U. Then $x_s \in \overline{B}(p, \delta_n) \subseteq U$ for all $s \in A_n$. Since $A_n \in \mathcal{I}_\rho^+$, we obtain that $p \in \Gamma_x(\rho)$.

In addition, pick an integer $n \geq 1$ and find a neighborhood U of y_n such that $p \notin U$ and $y_k \notin U$ for every $k \neq n$. Then $\{s \in \Psi : \ x_s \in U\} = A_n \setminus A_{n+1} \in \mathcal{I}_\rho$. Hence $y_n \notin \Gamma_x(\rho)$.

Lastly, the proof that $p \notin \Lambda_x(\rho)$ goes similarly as in the If part of item (i). □

Remark 3.3. The assumption about local compactness in Proposition 3.2(iii) cannot be dropped. Indeed, in [8, Example 3.8] it is shown that there are an ideal $\mathcal{I}$, a space X, and a sequence $x \in X^\omega$ such that

- X is metric with an accumulation point, but it is not locally compact,
- $\mathcal{I}$ is P^-, which is equivalent to $\rho_\mathcal{I}$ being P^- (by [6, Proposition 6.5(2)]),
- there is an isolated point of $\Gamma_x(\mathcal{I})$ which does not belong to $\Lambda_x(\mathcal{I})$.

Since $\Gamma_x(\rho_\mathcal{I}) = \Gamma_x(\mathcal{I}_{\rho_\mathcal{I}}) = \Gamma_x(\mathcal{I})$ and $\Lambda_x(\mathcal{I}) = \Lambda_x(\rho_\mathcal{I})$, we get an example of a metric space X with an accumulation point, a sequence x in X and a P^- partition regular function $\rho_\mathcal{I}$ such that there is an isolated point of $\Gamma_x(\mathcal{I}_{\rho_\mathcal{I}})$ which does not belong to $\Lambda_x(\rho_\mathcal{I})$.

Corollary 3.4. *Let X be a metric space with an accumulation point. Then the following hold.*

(i) $\Lambda_x(\mathsf{r}) \neq \Gamma_x(\mathsf{r})$ *for some sequence* $x \in X^{[\omega]^2}$.
(ii) $\Lambda_x(\mathsf{r})$ *is not closed for some sequence* $x \in X^{[\omega]^2}$.
(iii) *If X is locally compact, then each isolated point in $\Gamma_x(\mathsf{r})$ belongs to $\Lambda_x(\mathsf{r})$ for every* $x \in X^{[\omega]^2}$.

Similarly, the following hold.

(i′) $\Lambda_x(\mathsf{FS}) \neq \Gamma_x(\mathsf{FS})$ *for some sequence* $x \in X^{\omega}$.
(ii′) $\Lambda_x(\mathsf{FS})$ *is not closed for some sequence* $x \in X^{\omega}$.
(iii′) *If X is locally compact, then each isolated point in $\Gamma_x(\mathsf{FS})$ belongs to $\Lambda_x(\mathsf{FS})$ for every* $x \in X^{\omega}$.

Proof. Thanks to [6, Proposition 6.7], both partition regular functions r and FS are P^-, but not P^+. Then r and FS are not $P^|$ by Proposition 3.1. To conclude, all claims follow by Proposition 3.2. □

4. Main results

In this section, we identify a set $A \subseteq \Omega$ with its characteristic function $\mathbf{1}_A$, and view 2^Ω as the product space $\{0,1\}^\Omega$ with the discrete topology on $\{0,1\}$. Then 2^Ω is a Polish space. The set $[\Omega]^\omega$ is a G_δ subset of 2^Ω, hence it is a Polish space. In the same manner, we think of 2^Ψ and $[\Psi]^\omega$.

Proposition 4.1. *Let X be a Polish space and $\rho : \mathcal{F} \to [\Psi]^\omega$ be a partition regular function. If $\mathcal{F}$ is an analytic subset of $[\Omega]^\omega$ and ρ is a Borel function, then*

$$\mathscr{L}(\rho) \subseteq \Sigma^1_1.$$

Proof. Let d be a compatible complete metric on X. Fix a sequence $x = (x_s)_{s\in\Psi} \in X^\Psi$. We need to show that $\Lambda_x(\rho)$ is an analytic subset of X. Notice that $\Lambda_x(\rho)$ is the projection onto the first coordinate of the set

$$A = \left\{(p,F) \in X \times \mathcal{F} : \forall_{n\in\omega}\, \exists_{K\in[\Omega]^{<\omega}}\, \forall_{s\in\Psi}\, (s \in \rho(F\setminus K) \implies d(p,x_s) < 2^{-n})\right\}.$$

Since A is a subset of the Polish space $X \times [\Omega]^\omega$, the proof will be finished once we show that A is an analytic subset of $X \times [\Omega]^\omega$. Observe that

$$A = \bigcap_{n\in\omega} \bigcup_{K\in[\Omega]^{<\omega}} \bigcap_{s\in\Psi} (B_{n,K,s} \cup C_{n,K,s}),$$

where

$$B_{n,K,s} = \{(p,F) \in X \times \mathcal{F} : s \notin \rho(F\setminus K)\} \quad \text{and}$$
$$C_{n,K,s} = \{(p,F) \in X \times \mathcal{F} : d(p,x_s) < 2^{-n}\}.$$

It is enough to show that the sets $B_{n,K,s}$ and $C_{n,K,s}$ are analytic.

First, we show that $B_{n,K,s}$ is analytic. Since $B_{n,K,s} = X \times D_{K,s}$, where $D_{K,s} = \{F \in \mathcal{F} : s \notin \rho(F\setminus K)\}$, the proof will be finished once we show that $D_{K,s}$ is analytic. Notice that

$$D_{K,s} = \mathcal{F} \cap \phi_K^{-1}\left[\rho^{-1}\left[[\Psi\setminus\{s\}]^\omega\right]\right],$$

where $\phi_K : [\Omega]^\omega \to [\Omega]^\omega$ is given by $\phi_K(A) = A\setminus K$. Since $\mathcal{F}$ is an analytic set, ϕ_K is a continuous function, ρ is a Borel function, and $[\Psi\setminus\{s\}]^\omega$ is an open subset of $[\Psi]^\omega$, we obtain that $D_{K,s}$ is analytic.

Finally, we show that $C_{n,K,s}$ is analytic. Notice that $C_{n,K,s} = B(x_s, 2^{-n}) \times \mathcal{F}$. Since $B(x_s, 2^{-n})$ is open and $\mathcal{F}$ is analytic, we obtain that $C_{n,K,s}$ is analytic. $\square$

It is worth noting that the special case of Proposition 4.1 where $\rho = \rho_{\mathcal{I}}$, for some coanalytic ideal $\mathcal{I}$ on ω, has been already shown in [2, Proposition 4.1], cf. also [14, Theorem 4.1] and [1, Theorem 3.3].

We are finally ready to prove our main results.

Theorem 4.2. *Let X be an uncountable Polish space. Then*

$$\mathscr{L}(\mathsf{r}) = \mathscr{L}(\mathcal{R}) = \Sigma^1_1.$$

Moreover, for every nonempty analytic set $C \subseteq X$, there exists a sequence $y \in X^{[\omega]^2}$ such that $C = \Lambda_y(\mathsf{r}) = \Lambda_y(\mathcal{R})$.

Proof. The inclusion $\mathscr{L}(\mathsf{r}) \subseteq \Sigma^1_1$ follows from [6, Proposition 5.2(2)] and Proposition 4.1. The inclusion $\mathscr{L}(\mathcal{R}) \subseteq \Sigma^1_1$ follows from [2, Proposition 4.1] (or Proposition 4.1 again and [6, Proposition 5.2(5)]). We will show the "moreover" part, from which the converse inclusions $\Sigma^1_1 \subseteq \mathscr{L}(\mathsf{r})$ and $\Sigma^1_1 \subseteq \mathscr{L}(\mathcal{R})$ follow (recall that $\emptyset \in \mathscr{L}(\mathsf{r})$ and $\emptyset \in \mathscr{L}(\mathcal{R})$ by definition).

To this aim, fix a nonempty analytic set $C \subseteq X$. Thanks to [17, Theorem 25.7], there exists a Souslin scheme $\{C_s : s \in \omega^{<\omega}\}$ of nonempty closed sets such that

$$C = \bigcup_{x \in \omega^\omega} \bigcap_{n \in \omega} C_{x\restriction n}. \tag{6}$$

In addition, it can be assumed that $C_s \supseteq C_t$ whenever $s \subseteq t$ for all $t, s \in \omega^{<\omega}$ and $\lim_n \operatorname{diam}(C_{x\restriction n}) = 0$ for all $x \in \omega^\omega$. Note that, since X is complete, for each $x \in \omega^\omega$ there exists $p_x \in X$ such that $\bigcap_n C_{x\restriction n} = \{p_x\}$. Hence $C = \{p_x : x \in \omega^\omega\}$.

Fix any bijection $f : \omega \to \omega^{<\omega}$ such that if $s \subseteq t$ then $f^{-1}(s) \leq f^{-1}(t)$. Define

$$A_s = \big\{\{i,j\} \in [\omega]^2 : i < j \text{ and } s \subseteq f(i) \subseteq f(j)\big\}$$

for each $s \in \omega^{<\omega}$. Observe that $s \subseteq t$ implies $A_t \subseteq A_s$. Moreover, if $n, k \in \omega$ are distinct, then $A_{s^\frown n} \cap A_{s^\frown k} = \emptyset$.

At this point, define the sequence $y = (y_{\{i,j\}})_{\{i,j\} \in [\omega]^2}$ by

$$y_{\{i,j\}} = \begin{cases} p_{0^\infty} & \text{if } \{i,j\} \notin A_\emptyset, \\ p_{f(\max\{i,j\})^\frown 0^\infty} & \text{if } \{i,j\} \in A_\emptyset. \end{cases}$$

Claim 1. $C \subseteq \Lambda_y(\mathcal{R})$.

Proof. Pick $c \in C$. Then $c = p_x$ for some $x \in \omega^\omega$. For each $n \in \omega$, fix $i_n \in \omega$ such that $f(i_n) = x \restriction n$ and define $F = \{i_n : n \in \omega\}$. Note that the properties of f guarantee that F is infinite and that the sequence $(i_n)_{n \in \omega}$ is increasing. We claim that for every neighborhood U of p_x there is a finite set $G \subseteq [\omega]^2$ such that $y_{\{i,j\}} \in U$ for all $\{i,j\} \in [F]^2 \setminus G$.

In fact, fix a neighborhood U of p_x, and pick a sufficiently large $n \in \omega$ such that $C_{x\restriction n} \subseteq U$. Let $G = \{\{i_m, i_k\} \in [F]^2 : \max\{m,k\} \leq n\}$. Then G is finite. Moreover, if $\{i,j\} \in [F]^2 \setminus G$, then $\{i,j\} = \{i_m, i_k\}$ for some $m < k$ with $k > n$. Note that $\{i,j\} \in A_\emptyset$. Then

$$y_{\{i,j\}} = p_{f(\max\{i,j\})^\frown 0^\infty} = p_{f(i_k)^\frown 0^\infty} = p_{(x\restriction k)^\frown 0^\infty} \in C_{x\restriction k} \subseteq C_{x\restriction n} \subseteq U.$$

Therefore $c = p_x \in \Lambda_y(\mathcal{R})$. $\square$

Claim 2. For each $s \in \omega^{<\omega}$ and $F \in [\omega]^\omega$ such that $[F]^2 \subseteq A_s$, there exist $n \in \omega$ and $K \in [\omega]^{<\omega}$ for which $[F \setminus K]^2 \subseteq A_{s^\frown n}$.

Proof. Fix $s \in \omega^{<\omega}$ and $F \in [\omega]^\omega$ such that $[F]^2 \subseteq A_s$. Define $K = \{f^{-1}(s)\}$ and pick $m \in F \setminus K$ such that $|f(m)| = \min\{|f(i)| : i \in F \setminus K\}$. Also, let $n \in \omega$ be such that $s^\frown n \subseteq f(m)$ (which is possible since $[F \setminus K]^2 \subseteq [F]^2 \subseteq A_s$ implies $s \subseteq f(m)$). Now, we are going to show that $[F \setminus K]^2 \subseteq A_{s^\frown n}$.

For, pick $\{i, j\} \in [F \setminus K]^2$ with $i < j$. Since $\{i, j\} \in [F \setminus K]^2 \subseteq [F]^2 \subseteq A_s$, we get $f(i) \subseteq f(j)$. Observe that $s^\frown n \subseteq f(m) \subseteq f(i)$. Indeed, either $m = i$ (which is trivial) or the choice of m together with $\{m, i\} \in [F \setminus K]^2 \subseteq A_s$ give us $f(m) \subseteq f(i)$. Therefore $s^\frown n \subseteq f(m) \subseteq f(i) \subseteq f(j)$, which means that $\{i, j\} \in A_{s^\frown n}$. □

Claim 3. $\Lambda_y(\mathsf{r}) \subseteq C$.

Proof. The claim is trivial if $\Lambda_y(\mathsf{r}) = \emptyset$. Otherwise, fix $z \in \Lambda_y(\mathsf{r})$ and pick $F \in [\omega]^\omega$ such that $(y_{\{i,j\}})_{\{i,j\} \in [F]^2}$ is r-convergent to z.

First, suppose that $[F \setminus K]^2 \setminus A_\emptyset \neq \emptyset$ for all $K \in [\omega]^{<\omega}$. Then $z = p_{0^\infty} \in C$. Indeed, otherwise, we could find a neighborhood U of z such that $p_{0^\infty} \notin U$. Then there is a finite $K \in [\omega]^{<\omega}$ such that $y_{\{i,j\}} \in U$ for all $\{i, j\} \in [F \setminus K]^2$. However, by our assumption, there is some $\{i, j\} \in [F \setminus K]^2 \setminus A_\emptyset$, so for that $\{i, j\}$ we have $y_{\{i,j\}} = p_{0^\infty} \notin U$, which gives a contradiction.

Henceforth, assume that $[F \setminus K]^2 \subseteq A_\emptyset$ for some $K \in [\omega]^{<\omega}$. Then applying Claim 2 we can recursively define $x \in \omega^\omega$ such that for each $n \in \omega$ there is K_n with $[F \setminus K_n]^2 \subseteq A_{x\restriction n}$. We will show that $z = p_x \in C$. Indeed, suppose for the sake of contradiction that $z \neq p_x$. Then there is $n \in \omega$ such that $z \notin C_{x\restriction n}$. Then $p_x \in C_{x\restriction n}$ and, since $C_{x\restriction n}$ is closed, $X \setminus C_{x\restriction n}$ is an open neighborhood of z. Hence, there exists $K \in [\omega]^{<\omega}$ such that $y_{\{i,j\}} \notin C_{x\restriction n}$ for all $\{i, j\} \in [F \setminus K]^2$. However, $F \setminus (K \cup K_n) \in [\omega]^\omega$ and for each $i < j$ with $\{i, j\} \in [F \setminus (K \cup K_n)]^2 \subseteq A_{x\restriction n}$ we have $y_{\{i,j\}} = p_{f(j)^\frown 0^\infty} \in C_{f(j)} \subseteq C_{x\restriction n}$. This is a contradiction. □

Putting together Claim 1 and Claim 3, and using $\Lambda_y(\mathcal{R}) \subseteq \Lambda_y(\mathsf{r})$ (by Proposition 2.4), we conclude that $C \subseteq \Lambda_y(\mathcal{R}) \subseteq \Lambda_y(\mathsf{r}) \subseteq C$. Therefore they coincide. □

Theorem 4.3. *Let X be an uncountable Polish space. Then*

$$\mathscr{L}(\mathsf{FS}) = \mathscr{L}(\mathcal{H}) = \Sigma^1_1.$$

Moreover, for every nonempty analytic set $C \subseteq X$, there exists a sequence $y \in X^\omega$ such that $C = \Lambda_y(\mathsf{FS}) = \Lambda_y(\mathcal{H})$.

Proof. The first part proceeds verbatim as in the proof of Theorem 4.2, replacing r and $\mathcal{R}$ with FS and $\mathcal{H}$, respectively. Fix a nonempty analytic set $C \subseteq X$ and let $\{C_s : s \in \omega^{<\omega}\}$ be a Souslin scheme as in (6), with the same properties. Similarly, we need to show that there exists $y \in X^\omega$ such that $C = \Lambda_y(\mathsf{FS}) = \Lambda_y(\mathcal{H})$.

Thanks to [7, Lemma 2.2], it is possible to fix an infinite set $D \subseteq \omega$ which is "very sparse", that is

(a) for each $a \in \mathsf{FS}(D)$ there is a unique nonempty finite set $\alpha_D(a) \in [D]^{<\omega}$ such that $a = \sum_{i \in \alpha_D(a)} i$;
(b) if $G, H \in [D]^{<\omega}$ are nonempty and $G \cap H \neq \emptyset$, then $\sum_{i \in G} i + \sum_{i \in H} i \notin \mathsf{FS}(D)$.

Fix a bijection $f : D \to \omega^{<\omega}$ such that if $s \subseteq t$ then $f^{-1}(s) \leq f^{-1}(t)$, and set

$$A_s = \bigcup_{m\in\omega} \left\{ \sum_{i=0}^{m} k_i : k_0, \dots, k_m \in D,\, k_0 < \dots < k_m \text{ and } s \subseteq f(k_0) \subseteq \dots \subseteq f(k_m) \right\}$$

for each $s \in \omega^{<\omega}$. Observe that $A_s \subseteq \mathsf{FS}(D)$, and that $s \subseteq t$ implies $A_t \subseteq A_s$. Moreover, since D is very sparse, we have $A_{s^\frown n} \cap A_{s^\frown k} = \emptyset$ for all distinct $n, k \in \omega$.

At this point, define a sequence $y = (y_i)_{i\in\omega}$ by

$$y_i = \begin{cases} p_{0^\infty} & \text{if } i \notin A_\emptyset, \\ p_{f(\max \alpha_D(i))^\frown 0^\infty} & \text{if } i \in A_\emptyset. \end{cases}$$

Claim 1. $C \subseteq \Lambda_y(\mathcal{H})$.

Proof. Pick $c \in C$. Then $c = p_x$ for some $x \in \omega^\omega$. For each $n \in \omega$, fix $i_n \in D$ such that $f(i_n) = x \restriction n$ and define $F = \{i_n : n \in \omega\}$. Note that the properties of f guarantee that F is infinite and $(i_n)_{n\in\omega}$ is increasing. We claim that for every neighborhood U of p_x there is $G \in [\omega]^{<\omega}$ such that $y_i \in U$ for all $i \in \mathsf{FS}(F) \setminus G$.

In fact, fix a neighborhood U of p_x, and pick a sufficiently large $n \in \omega$ such that $C_{x\restriction n} \subseteq U$. Let $G = \{i \in \mathsf{FS}(D) : \max \alpha_D(i) \leq i_n\}$. Then G is finite. Moreover, if $i \in \mathsf{FS}(F) \setminus G \subseteq \mathsf{FS}(D) \setminus G$, then $i_k = \max \alpha_D(i) > i_n$ for some $k \in \omega$, $k > n$ as $(i_j)_{j\in\omega}$ is increasing and D is very sparse. Note that $i \in A_\emptyset$. Hence

$$y_i = p_{f(\max \alpha_D(i))^\frown 0^\infty} = p_{f(i_k)^\frown 0^\infty} = p_{(x\restriction k)^\frown 0^\infty} \in C_{x\restriction k} \subseteq C_{x\restriction n} \subseteq U.$$

Therefore $c = p_x \in \Lambda_y(\mathcal{H})$. □

Claim 2. For each $s \in \omega^{<\omega}$ and $F \in [\omega]^\omega$ such that $\mathsf{FS}(F) \subseteq A_s$, there exist $n \in \omega$ and $K \in [\omega]^{<\omega}$ for which $\mathsf{FS}(F \setminus K) \subseteq A_{s^\frown n}$.

Proof. Fix $s \in \omega^{<\omega}$ and $F \in [\omega]^\omega$ such that $\mathsf{FS}(F) \subseteq A_s$. Observe that $F \subseteq \mathsf{FS}(F) \subseteq A_s \subseteq \mathsf{FS}(D)$. Moreover, if $a, b \in F$ are distinct, then $\alpha_D(a) \cap \alpha_D(b) = \emptyset$. Indeed, otherwise, $a + b \in \mathsf{FS}(F) \subseteq A_s \subseteq \mathsf{FS}(D)$, which would contradict the fact that D is very sparse.

Hence, by the above observation, $K = \{a \in F : f^{-1}(s) \in \alpha_D(a)\}$ has at most one element. In particular, K is finite. Pick $e \in \bigcup_{a\in F\setminus K} \alpha_D(a)$ such that $|f(e)| = \min\{|f(i)| : i \in \bigcup_{a\in F\setminus K} \alpha_D(a)\}$. Let $c \in F \setminus K$ be such that $e \in \alpha_D(c)$ (note that such c is unique) and $n \in \omega$ be such that $s^\frown n \subseteq f(e)$ (such n exists by the definition of K and an observation that $e \in \alpha_D(c)$ implies $s \subseteq f(e)$, as $c \in \mathsf{FS}(F) \subseteq A_s$ and D is very sparse). Thus, we are left to show that $\mathsf{FS}(F \setminus K) \subseteq A_{s^\frown n}$.

Fix distinct $a_0, a_1, \dots, a_m \in F \setminus (K \cup \{c\})$. To complete the proof, we need to prove that $c \in A_{s^\frown n}$, $\sum_{i\leq m} a_i \in A_{s^\frown n}$, and $c + \sum_{i\leq m} a_i \in A_{s^\frown n}$. Let $(d_i)_{i\leq k} \in D^{k+1}$ be an increasing enumeration of the set $\alpha_D(c) \cup \bigcup_{i\leq m} \alpha_D(a_i)$. Since $\alpha_D(a) \cap \alpha_D(b) = \emptyset$ for all distinct $a, b \in F$, we have

$$\sum_{i\leq k} d_i = c + \sum_{i\leq m} a_i \in \mathsf{FS}(F \setminus K) \subseteq \mathsf{FS}(F) \subseteq A_s.$$

Since D is very sparse and $d_0 < \dots < d_k$ in D, we get $s \subseteq f(d_0) \subseteq f(d_1) \subseteq \dots \subseteq f(d_k)$. In particular, $f(d_0) \subseteq f(e)$ (because $e = d_j$ for some j). By the choice of e, we have also $|f(e)| \leq |f(d_0)|$, hence $|f(e)| = |f(d_0)|$ and thus $f(e) = f(d_0)$. Since f is injective, we obtain $e = d_0$. Hence also $s^\frown n \subseteq f(e) = f(d_0) \subseteq f(d_1) \subseteq \dots \subseteq f(d_k)$. This shows that $c + \sum_{i\leq m} a_i \in A_{s^\frown n}$. However, since the increasing

enumerations of $\alpha_D(c)$ and of $\bigcup_{i\leq m}\alpha_D(a_i)$ are some subsequences of $(d_i)_{i\leq k}$, we can also conclude that $c\in A_{s^\frown n}$ and $\sum_{i\leq m}a_i\in A_{s^\frown n}$. □

Claim 3. $\Lambda_y(\mathsf{FS})\subseteq C$.

Proof. The claim is trivial if $\Lambda_y(\mathsf{FS})=\emptyset$. Otherwise, fix $z\in\Lambda_y(\mathsf{FS})$ and pick $F\in[\omega]^\omega$ such that $(y_i)_{i\in\mathsf{FS}(F)}$ is FS-convergent to z.

First, suppose that $\mathsf{FS}(F\setminus K)\setminus A_\emptyset\neq\emptyset$ for all $K\in[\omega]^{<\omega}$. Then $z=p_{0^\infty}$. Indeed, in the opposite, we could find a neighborhood U of z such that $p_{0^\infty}\notin U$. Then there is a finite $K\in[\omega]^{<\omega}$ such that $y_i\in U$ for all $i\in\mathsf{FS}(F\setminus K)$. However, by our assumption, there is some $i\in\mathsf{FS}(F\setminus K)\setminus A_\emptyset$, so for that i we have $y_i=p_{0^\infty}\notin U$, which gives a contradiction.

Hence, assume hereafter that $\mathsf{FS}(F\setminus K)\subseteq A_\emptyset$ for some $K\in[\omega]^{<\omega}$. Then applying Claim 2 we can recursively define $x\in\omega^\omega$ such that for each $n\in\omega$ there is K_n with $\mathsf{FS}(F\setminus K_n)\subseteq A_{x\restriction n}$. We will show that $z=p_x\in C$. Indeed, suppose for the sake of contradiction that $z\neq p_x$. Then there is $n\in\omega$ such that $z\notin C_{x\restriction n}$. Then $p_x\in C_{x\restriction n}$ and, since $C_{x\restriction n}$ is closed, $X\setminus C_{x\restriction n}$ is an open neighborhood of z. Hence, there exists $K\in[\omega]^{<\omega}$ such that $y_i\notin C_{x\restriction n}$ for all $i\in\mathsf{FS}(F\setminus K)$. However, $F\setminus(K\cup K_n)\in[\omega]^\omega$ and each $i\in\mathsf{FS}(F\setminus(K\cup K_n))\subseteq A_{x\restriction n}$ is of the form $\sum_{j=0}^m k_j$ for some increasing finite sequence $(k_j)_{j\leq m}\in D^{m+1}$ such that $x\restriction n\subseteq f(k_0)\subseteq f(k_1)\subseteq\cdots\subseteq f(k_m)$. Thus, $y_i=p_{f(k_m)^\frown 0^\infty}\in C_{f(k_m)}\subseteq C_{x\restriction n}$. This is the claimed contradiction. □

The conclusion follows from putting together Claim 1, Claim 3, and the inclusion $\Lambda_y(\mathcal{H})\subseteq\Lambda_y(\mathsf{FS})$ (by Proposition 2.4), as in the proof of Theorem 4.2. □

References

1. M. Balcerzak, S. Głąb, and P. Leonetti, *Topological complexity of ideal limit points*, `https://arxiv.org/abs/2407.12160` (2024), 1–23.
2. M. Balcerzak and P. Leonetti, *On the relationship between ideal cluster points and ideal limit points*, Topology Appl. **252** (2019), 178–190. MR 3883171
3. V. Bergelson and R. Zelada, *Strongly mixing systems are almost strongly mixing of all orders*, Ergodic Theory Dynam. Systems **44** (2024), no. 6, 1489–1530. MR 4741322
4. M. Bojańczyk, E. Kopczyński, and S. Toruńczyk, *Ramsey's theorem for colors from a metric space*, Semigroup Forum **85** (2012), no. 1, 182–184. MR 2948679
5. R. Filipów, M. Kowalczuk, and A. Kwela, *Critical ideals for countable compact spaces*, `https://arxiv.org/abs/2503.12571` (2025), 1–19.
6. R. Filipów, K. Kowitz, and A. Kwela, *A unified approach to Hindman, Ramsey and van der Waerden spaces*, to appear in J. Symb. Log. (`http://doi.org/10.1017/jsl.2024.8`) (2024), 1–53.
7. R. Filipów, K. Kowitz, A. Kwela, and J. Tryba, *New Hindman spaces*, Proc. Amer. Math. Soc. **150** (2022), no. 2, 891–902. MR 4356195
8. R. Filipów, A. Kwela, and P. Leonetti, *Borel complexity of sets of ideal limit points*, `https://arxiv.org/abs/2411.10866` (2025), 1–44.
9. J. Flašková, *Ideals and sequentially compact spaces*, Topology Proc. **33** (2009), 107–121. MR 2471564
10. J.A. Fridy, *Statistical limit points*, Proc. Amer. Math. Soc. **118** (1993), no. 4, 1187–1192. MR 1181163
11. H. Furstenberg, *Recurrence in ergodic theory and combinatorial number theory*, Princeton University Press, Princeton, NJ, 1981, M. B. Porter Lectures. MR 603625
12. H. Furstenberg and B. Weiss, *Topological dynamics and combinatorial number theory*, J. Analyse Math. **34** (1978), 61–85. MR 531271
13. Xi He, H. Zhang, and S. Zhang, *The Borel complexity of ideal limit points*, Topology Appl. **312** (2022), Paper No. 108061, 12. MR 4393937

14. ———, *More on ideal limit points*, Topology Appl. **322** (2022), Paper No. 108324, 9. MR 4505549
15. N. Hindman, *Finite sums from sequences within cells of a partition of N*, J. Combinatorial Theory Ser. A **17** (1974), 1–11. MR 349574
16. M. Hrušák and D. Meza-Alcántara, *Katětov order, Fubini property and Hausdorff ultrafilters*, Rend. Istit. Mat. Univ. Trieste **44** (2012), 503–511. MR 3019575
17. A. S. Kechris, *Classical descriptive set theory*, Graduate Texts in Mathematics, vol. 156, Springer-Verlag, New York, 1995. MR 1321597
18. P. Kostyrko, T. Šalát, and W. Wilczyński, *$\mathcal{I}$-convergence*, Real Anal. Exchange **26** (2000/01), no. 2, 669–685. MR 1844385
19. P. Leonetti, *Invariance of ideal limit points*, Topology Appl. **252** (2019), 169–177. MR 3883170
20. P. Leonetti and F. Maccheroni, *Characterizations of ideal cluster points*, Analysis (Berlin) **39** (2019), no. 1, 19–26. MR 3920799
21. D. Meza-Alcántara, *Ideals and filters on countable sets*, Ph.D. thesis, Universidad Nacional Autónoma de México, 2009, (`https://ru.dgb.unam.mx/handle/DGB_UNAM/TES01000645364`).

(Rafał Filipów) Institute of Mathematics, Faculty of Mathematics, Physics and Informatics, University of Gdańsk, ul. Wita Stwosza 57, 80-308 Gdańsk, Poland
Email address: `Rafal.Filipow@ug.edu.pl`
URL: `http://mat.ug.edu.pl/~rfilipow`

(Adam Kwela) Institute of Mathematics, Faculty of Mathematics, Physics and Informatics, University of Gdańsk, ul. Wita Stwosza 57, 80-308 Gdańsk, Poland
Email address: `Adam.Kwela@ug.edu.pl`
URL: `https://mat.ug.edu.pl/~akwela`

(Paolo Leonetti) Department of Economics, Universitá degli Studi dell'Insubria, via Monte Generoso 71, Varese 21100, Italy
Email address: `leonetti.paolo@gmail.com`
URL: `https://sites.google.com/site/leonettipaolo`