Sets of ideal limit points

Adam Kwela

University of Gdańsk

39th International Summer Conference on Real Functions Theory
Rowy, 14–19 September 2025

Preliminaries

$\mathcal{I} \subseteq \mathcal{P}(M)$ is an ideal on M if:

- $A \in \mathcal{I},\ B \subseteq A \implies B \in \mathcal{I}$,
- $A, B \in \mathcal{I} \implies A \cup B \in \mathcal{I}$.

During this talk we also require:

- M is countable,
- all finite subsets of M belong to $\mathcal{I}$,
- $M \notin \mathcal{I}$, i.e., $M \in \mathcal{I}^+ = \{A \subseteq M :\ A \notin \mathcal{I}\}$.

$\mathbf{\Sigma}^0_\alpha$ ($\mathbf{\Pi}^0_\alpha$) denotes the family of all Borel subsets of additive (multiplicative, resp.) class α. In particular, $\mathbf{\Pi}^0_2 = \mathbf{G}_\delta$, $\mathbf{\Sigma}^0_2 = \mathbf{F}_\sigma$ and $\mathbf{\Pi}^0_3 = \mathbf{F}_{\sigma\delta}$.

By identifying subsets of M with their characteristic functions, we equip $\mathcal{P}(M)$ with the topology of 2^M, so that we can assign Borel complexity to ideals on M.

Sierpiński: There are no $\mathbf{\Pi}^0_2$ ideals.

Examples of ideals

- $\mathrm{Fin} = \{A \subseteq \mathbb{N} : A \text{ is finite}\}$ is a $\mathbf{\Sigma^0_2}$ ideal.
- $\mathcal{I}_{1/n} = \{A \subseteq \mathbb{N} : \sum_{n\in A} \frac{1}{n+1} < \infty\}$ is a $\mathbf{\Sigma^0_2}$ ideal.
- $\mathcal{Z} = \{A \subseteq \mathbb{N} : \frac{|A\cap\{0,1,\ldots,n\}|}{n+1} \to 0\}$ is a $\mathbf{\Pi^0_3}$ ideal.
- $\mathrm{nwd} = \{A \subseteq [0,1]\cap\mathbb{Q} : \mathtt{cl}_{[0,1]}(A) \text{ is meager}\}$ is a $\mathbf{\Pi^0_3}$ ideal.
- $\mathrm{null} = \{A \subseteq [0,1]\cap\mathbb{Q} : \mathtt{cl}_{[0,1]}(A) \text{ is of Lebesgue measure } 0\}$ is a $\mathbf{\Pi^0_3}$ ideal.
- $\mathcal{W} = \{A \subseteq \mathbb{N} : \exists_k\ A \text{ does not contain arithmetic progressions of length } k\}$ is a $\mathbf{\Sigma^0_2}$ ideal.
- $\mathcal{H} = \{A \subseteq \mathbb{N} : \neg(\exists_{D\subseteq\mathbb{N}}\ D \text{ is infinite and } \mathrm{FS}(D) \subseteq A)\}$, where $\mathrm{FS}(D) = \{\sum_{n\in F} n : F \subseteq D, F \in \mathrm{Fin}\setminus\{\emptyset\}\}$, is a coanalytic non-Borel ideal.
- $\mathrm{Fin}^2 = \mathrm{Fin}\otimes\mathrm{Fin} = \{A \subseteq \mathbb{N}^2 : \exists_k\ \forall_{n>k}\ A_{(k)} \in \mathrm{Fin}\}$, where $A_{(k)} = \{i : (k,i) \in A\}$, is a $\mathbf{\Sigma^0_4}$ ideal.
- $\mathrm{Fin}^3 = \mathrm{Fin}\otimes\mathrm{Fin}^2 = \{A \subseteq \mathbb{N}^3 : \exists_n\ \forall_{k>n}\ A_{(k)} \in \mathrm{Fin}^2\}$ is a $\mathbf{\Sigma^0_6}$ ideal.

Bolzano-Weierstrass theorem

Bolzano and Weierstrass: The space $[0,1]$ is sequentially compact, i.e., $\forall_{(x_n)\subseteq[0,1]}\ \exists_{A\in\mathrm{Fin}^+}\ (x_n)_{n\in A}$ is convergent.

Definition

FinBW($\mathcal{I}$) is the class of all Hausdorff spaces X such that $\forall_{(x_n)\subseteq X}\ \exists_{A\in\mathcal{I}^+}\ (x_n)_{n\in A}$ is convergent.

FinBW(Fin) = sequentially compact spaces.
Bolzano and Weierstrass: $[0,1]\in$ FinBW(Fin).

$[0,1]\notin$ FinBW(conv), where conv is the ideal on $[0,1]\cap\mathbb{Q}$ consisting of all $A\subseteq[0,1]\cap\mathbb{Q}$ that can be covered by finitely many convergent sequences.

For which ideals $[0,1] \in \mathrm{FinBW}(\mathcal{I})$?

Definition (Katětov order)

Let $\mathcal{I}$ and $\mathcal{J}$ be ideals on M and L, respectively. Write $\mathcal{I} \leq_K \mathcal{J}$ if there is $f : L \to M$ such that $f^{-1}[A] \in \mathcal{J}$ for all $A \in \mathcal{I}$.

Theorem (Filipów, Mrożek, Recław, Szuca, Meza-Alcántara)

$[0,1] \in \mathrm{FinBW}(\mathcal{I}) \iff \mathrm{conv} \not\leq_K \mathcal{I}$.

$\mathrm{conv} \leq_K \mathcal{I}$ for $\mathcal{I} = \mathrm{nwd}, \mathrm{null}, \mathcal{H}, \mathrm{Fin}^2, \mathrm{Fin}^3$.
$\mathrm{conv} \not\leq_K \mathcal{I}$ for $\mathcal{I} = \mathrm{Fin}, \mathcal{I}_{1/n}, \mathcal{W}$ and all other $\mathbf{\Sigma^0_2}$ ideals.

Question (Hrušák)

Let $\mathcal{I}$ be a Borel ideal. Is it true that:

$$\mathrm{conv} \not\leq_K \mathcal{I} \iff \mathcal{I} \text{ is contained in some } \mathbf{\Sigma^0_2} \text{ ideal.}$$

Theorem (K)

No! The counterexample is $\mathbf{\Sigma^0_6}$.

Metric spaces

Theorem (Mazurkiewicz, Sierpiński)

If X is a metric compact and countable space, then it is homeomorphic to $n \cdot \omega^\alpha + 1$, for some $n \in \mathbb{N}$ and ordinal $\alpha < \omega_1$.

Theorem (Filipów, Kowalczuk, K)

Let X be a metric non-discrete space and $\mathcal{I}$ be an ideal.

- *X is not compact* $\implies X \notin \mathrm{FinBW}(\mathcal{I})$.
- *X is compact uncountable* $\implies$ $(X \in \mathrm{FinBW}(\mathcal{I}) \iff \mathrm{conv} \not\leq_K \mathcal{I})$.
- *X is compact countable* $\implies \exists_n \exists_{\alpha<\omega_1} X \cong n \cdot \omega^\alpha + 1 \implies$

$$\exists_{\alpha<\omega_1} (X \in \mathrm{FinBW}(\mathcal{I}) \iff \mathrm{conv}_{1+\alpha} \not\leq_K \mathcal{I}).$$

Theorem (Filipów, Kowalczuk, K)

If $\mathcal{I}$ is a $\mathbf{\Pi^0_4}$ ideal, then $\mathrm{conv}_{1+\alpha} \not\leq_K \mathcal{I}$ for all $\alpha < \omega_1$ (but it may happen that $\mathrm{conv} \leq_K \mathcal{I}$).

Distinguishing ideals

Theorem (K)

Assume CH. The following are equivalent for all $\mathbf{\Pi^0_4}$ *ideals* $\mathcal{I}, \mathcal{J}$*:*

(a) $\mathcal{I} \leq_K \mathcal{J}$,

(b) $\mathrm{FinBW}(\mathcal{J}) \subseteq \mathrm{FinBW}(\mathcal{I})$.

(a) $\Longrightarrow$ (b) is true in ZFC.
(b) $\Longrightarrow$ (a) requires CH and produces $X \in \mathrm{FinBW}(\mathcal{J}) \setminus \mathrm{FinBW}(\mathcal{I})$, which is uncountable, compact, separable, but non-metrizable.

The assumption $\mathcal{I}, \mathcal{J} \in \mathbf{\Pi^0_4}$ cannot be omitted:

Theorem (K)

Let $\mathcal{I}$ *be an ideal. Then:*
$\mathrm{Fin}^2 \leq_K \mathcal{I} \iff \mathrm{FinBW}(\mathcal{I})$ *is the class of finite spaces.*

$\mathrm{Fin}^3 \not\leq_K \mathrm{Fin}^2$, but $\mathrm{FinBW}(\mathrm{Fin}^2) = \mathrm{FinBW}(\mathrm{Fin}^3) =$ finite spaces (as $\mathrm{Fin}^2 \leq_K \mathrm{Fin}^3$).

Sets of ideal limit points

Do far, we have been asking if $X \notin \mathrm{FinBW}(\mathcal{I})$, i.e., if there is a sequence (x_n) in X such that $\{p \in X : \exists_{A \in \mathcal{I}^+} (x_n)_{n \in A} \to p\} = \emptyset$.

Definition

For (x_n) in X, denote $\Lambda(\mathcal{I}) = \{p \in X : \exists_{A \in \mathcal{I}^+} (x_n)_{n \in A} \to p\}$.

$$\Lambda_X(\mathcal{I}) = \{\Lambda_{(x_n)}(\mathcal{I}) : (x_n) \text{ is a sequence in } X\}.$$

~~$\Lambda_X(\mathcal{I}) = \{\Lambda_{(x_n)}(\mathcal{I}) : (x_n)$ is a sequence in $X\}$.~~

Thus, $X \notin \mathrm{FinBW}(\mathcal{I}) \iff \emptyset \in \Lambda_X(\mathcal{I})$. ~~Thus,~~
~~$X \notin \mathrm{FinBW}(\mathcal{I}) \iff \emptyset \in \Lambda_X(\mathcal{I})$.~~

Definition

$$\Lambda_X(\mathcal{I}) = \{\Lambda_{(x_n)}(\mathcal{I}) : (x_n) \text{ is a sequence in } X\} \cup \{\emptyset\}.$$

From now on, X is an uncountable Polish space.

Some examples

Example

- $\Lambda_X(\mathrm{Fin}) = \Lambda_X(\mathcal{I}_{1/n}) = \Lambda_X(\mathcal{W}) = \mathbf{\Pi^0_1}(X)$;
- $\Lambda_X(\mathcal{I}_d) = \Lambda_X(\mathrm{nwd}) = \Lambda_X(\mathrm{null}) = \mathbf{\Sigma^0_2}(X)$;
- $\Lambda_X(\mathrm{Fin}^2) = \mathbf{\Pi^0_3}(X)$.

Theorem (Balcerzak, Głąb, Leonetti)

$\Lambda_X(\mathrm{Fin}^\alpha) = \mathbf{\Pi^0_{2\alpha-1}}(X)$ *and* $\Lambda_X(\emptyset \otimes \mathrm{Fin}^\alpha) = \mathbf{\Sigma^0_{2\alpha}}(X)$.

Theorem (Filipów, K, Leonetti)

$\Lambda_X(\mathcal{H}) = \mathbf{\Sigma^1_1}(X)$.

More examples

For every $\mathcal{I}$ and X, we have $\{\{x\} : x \in X\} \subseteq \Lambda_X(\mathcal{I})$.
Therefore, there are NO ideals $\mathcal{I}$ such that $\Lambda_X(\mathcal{I}) = \mathbf{\Sigma^0_1}(X)$.

Example

If $\mathcal{I}$ is a maximal ideal (i.e., there is no ideal $\mathcal{J}$ such that $\mathcal{I} \subsetneq \mathcal{J}$), then:

$$\begin{aligned}
\Lambda_X(\mathcal{I}) &= \{\{x\} : x \in X\} \\
\Lambda_X(\emptyset \otimes \mathcal{I}) &= \{B \subseteq X : B \text{ is countable and } \neq \emptyset\} \\
\Lambda_X(\mathrm{Fin} \oplus (\emptyset \otimes \mathcal{I})) &= \{A \cup B : A \in \mathbf{\Pi^0_1}(X), B \text{ is countable and } \neq \emptyset\}
\end{aligned}$$

Some combinatorial properties

Definition (Filipów, K, Leonetti)

A family $\{A_s : s \in 2^{<\mathbb{N}}\} \subseteq \mathcal{I}^+$ is an $\mathcal{I}$-scheme if for all $s \in 2^{<\mathbb{N}}$:

1. $A_{s^\frown 0} \cap A_{s^\frown 1} = \emptyset$,
2. $A_{s^\frown 0} \cup A_{s^\frown 1} \subseteq A_s$.

Definition (Filipów, K, Leonetti)

For an $\mathcal{I}$-scheme $\mathcal{A} = \{A_s : s \in 2^{<\mathbb{N}}\}$ we define:

$$B_{\mathcal{I}}(\mathcal{A}) = \left\{x \in 2^{\mathbb{N}} : \neg\left(\exists_{C \in \mathcal{I}^+} \forall_{n \in \mathbb{N}}\, C \setminus A_{x\restriction n} \text{ is finite}\right)\right\}.$$

Definition (Filipów, K, Leonetti)

- $\mathcal{I} \in P(\mathbf{\Pi^0_1})$ if there is an $\mathcal{I}$-scheme with $B_{\mathcal{I}}(\mathcal{A}) = \emptyset$;
- $\mathcal{I} \in P(\mathbf{\Sigma^0_2})$ if there is an $\mathcal{I}$-scheme with $B_{\mathcal{I}}(\mathcal{A}) = \{\langle 0, 0, \ldots\rangle\}$;
- $\mathcal{I} \in P(\mathbf{\Pi^0_3})$ if there is an $\mathcal{I}$-scheme with $B_{\mathcal{I}}(\mathcal{A}) = \mathbb{Q}(2^{\mathbb{N}})$.

$$P(\mathbf{\Pi^0_3}) \subsetneq P(\mathbf{\Sigma^0_2}) \subsetneq P(\mathbf{\Pi^0_1}) \subsetneq \{\mathcal{I} : \text{there is some } \mathcal{I}\text{-scheme}\}$$

Closed sets

Theorem (Filipów, K, Leonetti)

The following are equivalent for every $\mathcal{I}$ and X:

(a) $\mathcal{I} \in P(\mathbf{\Pi^0_1})$;

(b) $\mathbf{\Pi^0_1}(X) \subseteq \Lambda_X(\mathcal{I})$;

(c) $\Lambda_X(\mathcal{I})$ *contains an uncountable analytic set.*

$P(\mathbf{\Pi^0_1})$ is purely combinatorial and does not depend on the topological space, so:

$$\exists_X \exists_{A \in \Lambda_X(\mathcal{I})} A \text{ is uncountable analytic} \implies \forall_Y \mathbf{\Pi^0_1}(Y) \subseteq \Lambda_Y(\mathcal{I}).$$

$\mathbf{F}_\sigma$ sets

Theorem (Filipów, K, Leonetti)

The following are equivalent for every $\mathcal{I}$ and X:

(a) $\mathcal{I} \in P(\mathbf{\Sigma^0_2})$;

(b) $\mathbf{\Sigma^0_2}(X) \subseteq \Lambda_X(\mathcal{I})$;

(c) $\Lambda_X(\mathcal{I})$ *contains an analytic set which is not the union of a closed set and a countable set.*

Moreover, if $\mathcal{I}$ is Borel, then the above are equivalent to:

(d) $\Lambda_X(\mathcal{I})$ *contains a non-*$\mathbf{\Pi^0_1}$ *analytic set.*

Recall that there are NO ideals $\mathcal{I}$ such that $\Lambda_X(\mathcal{I}) = \mathbf{\Sigma^0_1}(X)$.

Corollary (Filipów, K, Leonetti)

There are NO ideals $\mathcal{I}$ such that $\Lambda_X(\mathcal{I}) = \mathbf{\Pi^0_2}(X)$.

Again, $P(\mathbf{\Sigma^0_2})$ is purely combinatorial and does not depend on the topological space, so ...

$\mathbf{F}_{\sigma\delta}$ sets

Theorem (Filipów, K, Leonetti)

The following are equivalent for every $\mathcal{I}$ and X:

(a) $\mathcal{I} \in P(\mathbf{\Pi^0_3})$;

(b) $\mathbf{\Pi^0_3}(X) \subseteq \Lambda_X(\mathcal{I})$;

(c) $\Lambda_X(\mathcal{I})$ *contains a non-*$\mathbf{\Sigma^0_2}$ *analytic set.*

Corollary (Filipów, K, Leonetti)

There are NO ideals $\mathcal{I}$ such that $\Lambda_X(\mathcal{I}) = \mathbf{\Sigma^0_3}(X)$.

Again, $P(\mathbf{\Pi^0_3})$ is purely combinatorial and does not depend on the topological space, so ...

A scale for Borel ideals

Theorem (Xi He, Hang Zhang, and Shuguo Zhang)

If $\mathcal{I}$ is Borel, then $\mathbf{\Pi}^0_1(X) \subseteq \Lambda_X(\mathcal{I})$ (so $\mathcal{I} \in P(\mathbf{\Pi}^0_1)$).

Theorem (Balcerzak and Leonetti)

If $\mathcal{I}$ is Borel, then $\Lambda_X(\mathcal{I}) \subseteq \mathbf{\Sigma}^1_1(X)$.

Corollary (Filipów, K, Leonetti)

If $\mathcal{I}$ is a Borel ideal, then:

$$\Lambda_X(\mathcal{I}) = \mathbf{\Pi}^0_1(X) \iff \mathcal{I} \in P(\mathbf{\Pi}^0_1) \setminus P(\mathbf{\Sigma}^0_2)$$
$$\Lambda_X(\mathcal{I}) = \mathbf{\Sigma}^0_2(X) \iff \mathcal{I} \in P(\mathbf{\Sigma}^0_2) \setminus P(\mathbf{\Pi}^0_3)$$
$$\mathbf{\Pi}^0_3(X) \subseteq \Lambda_X(\mathcal{I}) \subseteq \mathbf{\Sigma}^1_1(X) \iff \mathcal{I} \in P(\mathbf{\Pi}^0_3)$$

Connections with descriptive complexity of ideals

Theorem (He, Zhang, Zhang)

There are ideals of arbitrarily high Borel complexity such that $\Lambda_X(\mathcal{I}) = \mathbf{\Pi}^0_1(X)$.

Theorem (Balcerzak, Głąb, Leonetti)

If $\mathcal{I} \in \mathbf{\Sigma}^0_3$, *then* $\Lambda_X(\mathcal{I}) = \mathbf{\Pi}^0_1(X)$.

Question

Let $\mathcal{I}$ *be a* $\mathbf{\Pi}^0_4$ *ideal. Is* $\Lambda_X(\mathcal{I})$ *either* $\mathbf{\Pi}^0_1(X)$ *or* $\mathbf{\Sigma}^0_2(X)$*?*

Recall that Fin^2 is $\mathbf{\Sigma}^0_4$ and $\Lambda_X(\mathrm{Fin}^2) = \mathbf{\Pi}^0_3(X)$.

Connections with descriptive complexity of ideals

Question

Let $\mathcal{I}$ be a $\mathbf{\Pi}^0_4$ ideal. Is $\Lambda_X(\mathcal{I})$ either $\mathbf{\Pi}^0_1(X)$ or $\mathbf{\Sigma}^0_2(X)$?

Theorem (He, Zhang, Zhang)

If $\mathcal{I}$ is a $\mathbf{\Pi}^0_3$ Farah ideal, then $\Lambda_X(\mathcal{I})$ is either $\mathbf{\Pi}^0_1(X)$ or $\mathbf{\Sigma}^0_2(X)$.

$\mathcal{I}$ is a Farah ideal if $\mathcal{I} = \{A \subseteq \mathbb{N} : \forall_{n\in\mathbb{N}} \exists_{m\in\mathbb{N}} A \setminus [0, m) \in C_n\}$ for compact C_n's closed under taking subsets of their elements.

Question (Farah)

Is every $\mathbf{\Pi}^0_3$ ideal a Farah ideal?

Theorem (Filipów, K, Leonetti)

If $\mathcal{I} \in \mathbf{\Pi}^0_4$, then $\Lambda_X(\mathcal{I})$ is either $\mathbf{\Pi}^0_1(X)$ or $\mathbf{\Sigma}^0_2(X)$ or $\mathbf{\Sigma}^1_1(X)$.

But we do NOT have an example of $\mathcal{I} \in \mathbf{\Pi}^0_4$ such that $\Lambda_X(\mathcal{I}) = \mathbf{\Sigma}^1_1(X)$...

References

 Balcerzak, M. and Głąb, S. and Leonetti, P.,
Topological complexity of ideal limit points,
preprint, https://arxiv.org/abs/2407.12160.

 Balcerzak, M. and Leonetti, P.,
On the relationship between ideal cluster points and ideal limit points,
Topology and its Applications 252 (2019), 178–190.

 Filipów, R. and Kowalczuk, M. and Kwela, A.,
Critical ideals for compact spaces,
preprint, https://arxiv.org/abs/2503.12571.

 Filipów, R. and Kwela, A. and Leonetti, P.,
Borel complexity of sets of ideal limit points,
preprint, https://arxiv.org/abs/2411.10866.

References (cont.)

 Filipów, R. and Mrożek, N. and Recław, I. and Szuca, P.,
Ideal convergence of bounded sequences,
Journal of Symbolic Logic 72 (2007), 501–512.

 He, Xi and Zhang, H. and Zhang, S.,
The Borel complexity of ideal limit points,
Topology and its Applications 312 (2022), 4393937.

 He, Xi and Zhang, H. and Zhang, S.,
More on ideal limit points,
Topology and its Applications 322 (2022), 4505549.

 Hrušák, M. and Meza-Alcántara, D.,
Comparison game on Borel ideals,
Commentationes Mathematicae Universitatis Carolinae 52 (2011),
191–204.

References (cont.)

 Kwela A.,
On extendability to F_σ ideals,
Archive for Mathematical Logic 61 (2022), 881–890.

 Kwela A.,
Unboring ideals,
Fundamenta Mathematicae 261 (2023), 235–272.

 Meza-Alcántara, D.,
Ideals and filters on countable set,
PhD thesis, Universidad Nacional Autónoma de México (2009),
https://ru.dgb.unam.mx/handle/DGB_UNAM/TES01000645364.

Thank you for your attention!