Hantzsche-Wendt flat manifolds

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Lausanne
December 4, 2008

Crystallographic groups

Let $\mathbb{R}^n$ be n-dimensional Euclidean space, with isometry group $E(n) = O(n) \ltimes \mathbb{R}^n$.

Definition

Γ is a crystallographic group of rank n iff it is a discrete and cocompact subgroup of $E(n)$.

A Bieberbach group is a torsion free crystallographic group.

Basic properties

Theorem

(Bieberbach, 1910)

- *If Γ is a crystallographic group of dimension n, then the set of all translations of Γ is a maximal abelian subgroup of a finite index.*
- *There is only a finite number of isomorphic classes of crystallographic groups of dimension n.*
- *Two crystallographic groups of dimension n are isomorphic if and only if there are conjugate in the group affine transformations $A(n) = GL(n, \mathbb{R}) \ltimes \mathbb{R}^n$.*

Pure abstract point of view

Theorem

(Zassenhaus, 1947) A group Γ is a crystallographic group of dimension n if and only if, it has a normal maximal abelian subgroup $\mathbb{Z}^n$ of a finite index.

Holonomy representation

Definition

Let Γ be a crystallographic group of dimension n with translations subgroup $A \simeq \mathbb{Z}^n$. A finite group $\Gamma/A = G$ we shall call a holonomy group of Γ.

Let $(A, a) \in E(n)$ and $x \in \mathbb{R}^n$. Γ acts on $\mathbb{R}^n$ in the following way:

$$(A, a)(x) = Ax + a.$$

Definition

Let Γ be n-dimensional Bieberbach group. We have the following short exact sequence of groups.

$$0 \to \mathbb{Z}^n \to \Gamma \xrightarrow{p} \Gamma/\mathbb{Z}^n = H \to 0.$$

Let us define a homomorphism $h_\Gamma : H \to GL(n, \mathbb{Z})$. Put

$$\forall h \in H, h_\Gamma(h)(e_i) = \bar{h}^{-1} e_i \bar{h},$$

where $p(\bar{h}) = h$ and $e_i \in \mathbb{Z}^n$ is a standard basis. h_Γ is called a holonomy representation of a group Γ.

Flat manifold

Let $\Gamma \subset E(n)$ be a torsion free crystallographic group. Since Γ is cocompact and discrete subgroup, then the orbit space $\mathbb{R}^n/\Gamma$ is a manifold. If Γ is not torsion free then the orbit space $\mathbb{R}^n/\Gamma$ is an orbifold.

Definition

The above manifolds (orbifolds) we shall call "flat".

From elementary covering theory any compact Riemannian manifold (orbifold) with sectional curvature equal to zero is flat.

Example

Flat surfaces:

- torus $S^1 \times S^1$,
- Klein bottle $S^1 \times S^1/\mathbb{Z}_2$

We shall see that many properties of the Bieberbach Groups correspond to properties of flat manifolds.

Definition

Definition

Any Bieberbach group of dimension n with holonomy group $(\mathbb{Z}_2)^{n-1}$ we shall call Generalized Hantzsche-Wendt (GHW for short) Bieberbach group of rank n. An oriented GHW we shall call Hantzsche-Wendt group (HW for short).

A flat manifold $\mathbb{R}^n/\Gamma$ is GHW (HW) if and only if the Bieberbach group Γ is GHW (HW).

Exercise: Let M^n be a flat Hantzsche-Wendt manifold of dimension n. Shaw that n is an odd natural number.

Example

Klein Bottle is GHW, 3-dimensional flat oriented manifold M^3 with non-cyclic holonomy $\mathbb{Z}_2 \times \mathbb{Z}_2$ is HW manifold. J.Conway called it "didicosm".

It can be proved that $\pi_1(M^3)$ is a Fibonacci group $F(2,6)$, where

$$F(2,6) = \{x_1, ..., x_6 \mid x_1x_2 = x_3, x_2x_3 = x_4, ..., x_6x_1 = x_2\}.$$

It is clear, that any n-dimensional HW group is related to an element of the second cohomology group $H^2(\mathbb{Z}_2^{n-1}, \mathbb{Z}^n)$. Hence the number of non-isomorphic GHW groups of given dimension growth exponentialy.

Theorem

Let Γ be a n-dimensional GHW group. Then

$$h_\Gamma((\mathbb{Z}_2)^{n-1}) \subset GL(n,\mathbb{Z})$$

is a set of the diagonal matrices with ± 1 on diagonal.

The proof is by induction and used the following lemmas.

Lemma

Let $\rho : \mathbb{Z}_2^{n-1} \to GL(n,\mathbb{Z})$ be a diagonal faithful integral representation with -Id $\notin Im(\rho)$. Then there is $g \in \mathbb{Z}_2^{n-1}$ such that $\rho(g) =$ diag(-1,-1,...,-1,1,-1,...,-1).

Moreover, if furthermore $Im(\rho) \not\subset SL(n,\mathbb{Z})$, then there is $g \in \mathbb{Z}_2^{n-1}$ such that $\rho(g) =$ diag(1,...,1,-1,1,...,1).

Lemma

Let Γ be a n-rank Bieberbach group with translation lattice Λ. Suppose that $(B, b) \in \Gamma$ and B has eigenvalues 1 and -1, with corresponding eigenspaces V^+ and V^- of dimension 1 and $n-1$ respectively.

Then $\Lambda = (\Lambda \cap V^+) \oplus (\Lambda \cap V^-)$, and the orthogonal projection of b onto V^+ lies in $1/2(\Lambda \cap V^+) \setminus (\Lambda \cap V^+)$.

Rational homology sphere

Theorem

If $M = \mathbb{R}^n/\Gamma$, where Γ is HW, then M is a rational homology sphere.

We calculate rational homology of M from the following formula

$$H_i(M, \mathbb{Q}) = (\Lambda^i(\mathbb{Z}^n))^{\mathbb{Z}_2^{n-1}}.$$

From definition of the holonomy representation they are zero for all $i \neq 0, n$. Since HW is connected and oriented, then M is a rational homology sphere.

Anosov relation

Let M be a flat manifold and $f : M \to M$ be a continous map. Define the Lefschetz number by

$$L(f) = \Sigma_{i \geqslant 0}(-1)^i \mathrm{Tr}(f_* \mid H_i(M, \mathbb{Q})).$$

Moreover, let $N(f)$ be the Nielsen number of a map f.

Theorem

(Anosov, 1985) If $f : M \to M$ is a continous map of nilmanifolds, then

$$N(f) = \mid L(f) \mid .$$

It is known, that for any non-oriented flat manifold the above theorem is not true.

Theorem

(Bram DE ROCK, 2006) Let M be any HW flat manifold, then for any continous map $f : M \to M$

$$N(f) = \mid L(f) \mid .$$

Abelianization

Theorem

(B.Putrycz, 2006) Let M be any HW flat manifold of dimension $n > 3$, then

$$H_1(M, \mathbb{Z}) = (\mathbb{Z}_2)^{n-1}.$$

In the proof are used methods proposed by R. Miatello and J. P. Rossetti.

Homology

In 2008 K. Dekimpe and N. Petrosyan calculate, by constructing a free resolution, homology of low dimensional oriented GHW groups/manifolds. In dimension five there are this same (for 2 groups). But in dimension 7 there are four classes on 62 groups. They find an algorithm.

Let us present the table for dimension five (2 groups):

$H_k(\Gamma)$	k = 0	k = 1	k = 2	k = 3	k = 4	k = 5
	$\mathbb{Z}$	$\mathbb{Z}_2^4$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^4$	0	$\mathbb{Z}$

,

for dimension seven (62 groups):

$H_k(\Gamma)$	k=0	k=1	k=2	k=3	k=4	k=5	k=6	k=7
I	$\mathbb{Z}$	$\mathbb{Z}_2^6$	$\mathbb{Z}_2^8$	$\mathbb{Z}_4^6$	$\mathbb{Z}_2^8$	$\mathbb{Z}_2^6$	0	$\mathbb{Z}$
II	$\mathbb{Z}$	$\mathbb{Z}_2^6$	$\mathbb{Z}_2^9$	$\mathbb{Z}_2^{10} \oplus \mathbb{Z}_4^2$	$\mathbb{Z}_2^9$	$\mathbb{Z}_2^6$	0	$\mathbb{Z}$
III	$\mathbb{Z}$	$\mathbb{Z}_2^6$	$\mathbb{Z}_2^8$	$\mathbb{Z}_2^4 \oplus \mathbb{Z}_4^4$	$\mathbb{Z}_2^8$	$\mathbb{Z}_2^6$	0	$\mathbb{Z}$
IV	$\mathbb{Z}$	$\mathbb{Z}_2^6$	$\mathbb{Z}_2^8$	$\mathbb{Z}_2^8 \oplus \mathbb{Z}_4^2$	$\mathbb{Z}_2^8$	$\mathbb{Z}_2^6$	0	$\mathbb{Z}$

.

Theorem

For any generalized Hantzsche-Wendt group Γ of dimension $n \geqslant 3$ and any subgroup A of index two of the maximal abelian subgroup $\mathbb{Z}^n$ we can associate the quadratic function

$$Q_A^\Gamma : (\mathbb{Z}_2)^{n-1} \to \mathbb{Z}_2$$

and its associated alternating, bilinear quadratic form

$$B_A^\Gamma : (\mathbb{Z}_2)^{n-1} \times (\mathbb{Z}_2)^{n-1} \to \mathbb{Z}_2.$$

Moreover, the function Q_A^Γ and form B_A^Γ corresponds to the following short exact sequence of finite groups

$$0 \to \mathbb{Z}_2 = \mathbb{Z}^n/A \to \Gamma/A \to (\mathbb{Z}_2)^{n-1} \to 0.$$

The proof follows from the lemma.

Lemma

Any subgroup $A \subset \mathbb{Z}^n$ of index two of maximal abelian subgroup of Γ is a normal subgroup of Γ.

Proof: We have $\Gamma \subset E(n) = O(n) \ltimes \mathbb{R}^n$. For any $a \in A$ and $(G, g) \in \Gamma$ we have $(G, g)(I, a)(G, g)^{-1} = (I, G(a))$. Since $(G - I)x \in 2(\mathbb{Z}^n)$, for any $x \in \mathbb{Z}^n$, then $G(a) \in A$.

We define the form B in language of crystallographic group Γ. Let $(X,x),(Y,y) \in \Gamma$ are mapped by p to $X,Y \in V = (\mathbb{Z}_2)^{n-1}$. We have

$$B(X,Y) = (X-I)y - (Y-I)x \in \mathbb{Z}^n$$

and

$$Q(X) = (X,x)^2 = (X+I)x \in \mathbb{Z}^n.$$

It is easy to see that B and Q are well defined. It means does not depend from the choice of an element $(X,x) \in \Gamma$. By this same argument the bilinear form B is alternating. However, it depends from an index 2 subgroup $A \subset \mathbb{Z}^n$.

We shall present an example. Let $D \subset \mathbb{Z}^n$ be generated by the following elements:

$$2e_1, e_1 - e_2, e_2 - e_3, ..., e_{n-1} - e_n,$$

where $e_i, i = 1, 2, ..., n$ is a standard basis of $\mathbb{Z}^n$. D is a subgroup of index two in $\mathbb{Z}^n$. For $n \geqslant 2$, let Γ_n be the subgroup of $E(n)$ generated by the set $\{(B_i, s(B_i)) \mid 1 \leqslant i \leqslant n-1\}$. Recall, $B_i's$ are the $n \times n$ diagonal matrices:

$$B_i = \text{diag}(-1, \ldots, -1, \underbrace{1}_{i}, -1, \ldots, -1)$$

and

$$s(B_i) = e_i/2 + e_{i+1}/2 \text{ for } 1 \leqslant i \leqslant n-1.$$

From definition we have

$$B_D^{\Gamma_n}(B_i, Bj) = \begin{cases} 0 & \text{if } i = j \\ 1 & \text{if } i = j+1 \\ 0 & \text{if } i \geqslant j+2 \end{cases}$$

and a matrix

$$X = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 1 & 0 & 1 & 0\dots & 0 \\ \dots & \dots & \dots & \dots & \\ 0 & \dots 0 & 1 & 0 & 1 \\ 0 & \dots & 0 & 1 & 0 \end{bmatrix}.$$

Moreover for any $B_i \in (Z_2)^{n-1}$, where $1 \leqslant i \leqslant n-1$

$$Q_D^{\Gamma_n}(B_i) = (B_i + I)(e_i/2 + e_{i+1}/2) = e_i \notin D.$$

We want to calculate the Arf invariant $\mathbf{c}$ of $Q_D^{\Gamma_n}$. For this, we have to transform X to a symplectic matrix. Let us introduce a new basis $f_1, f_2, \ldots, f_k, f_{k+1}, \ldots, f_{2k}$:

$$f_i = B_{2i-1}, 1 \leqslant i \leqslant k$$

and

$$f_{k+i} = B_{2i} + B_{2i+2} + B_{2i+4} + \cdots + B_{2k}, 1 \leqslant i \leqslant k.$$

It is easy to see that the matrix X with respect to the new basis is a symplectic,

$$\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}.$$

Finally we have

Proposition

For any odd number $n = 2k + 1$ the group Γ_n/D is an extraspecial group with quadratic form of the quaternion type, (for $k = 2l - 1, 2l$ and l-even) or the real type (for $k = 2l - 1, 2l$ and l-odd).

Let $M = \mathbb{R}^n/\Gamma$ be a flat manifold.

Definition

A holonomy representation $\Psi_\Gamma : H \to GL(n, \mathbb{Z})$ is essentially complex if there exists a matrix $A \in GL(n, \mathbb{R})$, such that,

$$\forall h \in H, A\Psi_\Gamma(h)A^{-1} \in GL(\frac{1}{2}n, \mathbb{C}).$$

Theorem

(F.E.A. Johnson, E. Rees, 1991) The following conditions are equivalent:

- M *is a flat Kähler manifold,*
- Ψ_Γ *is essentially complex,*
- Γ *is a discrete cocompact torsion-free subgroup of* $U(\frac{1}{2}n) \ltimes \mathbb{C}^{\frac{1}{2}n}$.

In the same paper is given the following characterization of an essentially complex representation.

$$\Psi_\Gamma : H \to GL(n, \mathbb{Z})$$

is essentially complex if and only if n is an even number and each $\mathbb{R}$-irreducible summand of Ψ_Γ which is also $\mathbb{C}$-irreducible occurs with even multiplicity.

Definition

A flat manifold has a $\mathbb{C}$omplex structure if and only if their holonomy representation is essentially complex.

In Algebraic geometry the flat Kähler manifolds are called hyperelliptic varieties.

2-dimensional hyperelliptic varieties :

holonomy	CARAT notations
1	15.1.1
$\mathbb{Z}_2$	18.1.1; 18.1.2
$\mathbb{Z}_3$	35.1.1; 35.1.2
$\mathbb{Z}_4$	25.1.2; 27.1.1
$\mathbb{Z}_6$	70.1.1

There is also a list of all 3-dimensional complex flat manifolds. There are 174 such objects. If a holonomy group of a flat manifold M is a subgroup of $SU(n)$, then M is called Calabi-Yau manifold.

Theorem

(Hodge, 1941) Let M be n-dimensional complex Kähler manifold. We have:

- $H^r(M,\mathbb{C}) = \Sigma_{p+q=r} H^{p,q}(M)$,
- *if* $h^{p,q}(M) = \dim_{\mathbb{C}} H^{p,q}(M)$, *then* $h^{p,q}(M) = h^{q,p}(M)$,
- *number* $b_r(M) = \Sigma_{p+q=r} h^{p,q}(M)$ *is even if* r *is odd.*

The table of numbers $\{h^{p,q}(M), 0 \leqslant p,q \leqslant n\}$ is called the Hodge diamond of M.

Definition

A flat Kähler manifold of complex - dimension n with $\mathbb{Z}_2^{n-1}$ holonomy group is called a complex Hantzsche-Wendt manifold.

They are Calabi-Yau manifolds. Here we present their Hodge diamond:

$$
\begin{array}{c}
\binom{n}{0} \\
0 \qquad\qquad 0 \\
0 \qquad\qquad \binom{n}{1} \qquad\qquad 0 \\
0 \qquad\qquad 0 \qquad\qquad 0 \qquad\qquad 0 \\
0 \qquad\qquad 0 \qquad\qquad \binom{n}{2} \qquad\qquad 0 \qquad\qquad 0 \\
\cdots \qquad\qquad \cdots \qquad\qquad \cdots \qquad\qquad \cdots \qquad\qquad \cdots \qquad\qquad \cdots \\
\binom{n}{0} \qquad \binom{n}{1} \qquad \binom{n}{2} \qquad \cdots \qquad \cdots \qquad \binom{n}{n-1} \qquad \binom{n}{n} \\
\cdots \qquad\qquad \cdots \qquad\qquad \cdots \qquad\qquad \cdots \qquad\qquad \cdots \qquad\qquad \cdots \\
0 \qquad\qquad 0 \qquad\qquad \binom{n}{n-2} \qquad\qquad 0 \qquad\qquad 0 \\
0 \qquad\qquad 0 \qquad\qquad 0 \qquad\qquad 0 \\
0 \qquad\qquad \binom{n}{n-1} \qquad\qquad 0 \\
0 \qquad\qquad 0 \\
\binom{n}{0}
\end{array}
$$

Thank You.