Hantzsche-Wendt flat manifolds

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Definitions and examples

Hantzsche-Wendt groups

Some properties

Relations with quadratic forms over $\mathbb{Z}_2$

Complex GHW
Crystallographic groups

Let $\mathbb{R}^n$ be $n$-dimensional Euclidean space, with isometry group $E(n) = O(n) \rtimes \mathbb{R}^n$.

**Definition**

$\Gamma$ is a crystallographic group of rank $n$ iff it is a discrete and cocompact subgroup of $E(n)$.

A Bieberbach group is a torsion free crystallographic group.
Basic properties

Theorem
(Bieberbach, 1910)

- If $\Gamma$ is a crystallographic group of dimension $n$, then the set of all translations of $\Gamma$ is a maximal abelian subgroup of a finite index.

- There is only a finite number of isomorphic classes of crystallographic groups of dimension $n$.

- Two crystallographic groups of dimension $n$ are isomorphic if and only if there are conjugate in the group affine transformations $A(n) = GL(n, \mathbb{R}) \rtimes \mathbb{R}^n$. 

Hantzsche-Wendt flat manifolds

A. Szczepański
Pure abstract point of view

**Theorem**  
(Zassenhaus, 1947) A group $\Gamma$ is a crystallographic group of dimension $n$ if and only if, it has a normal maximal abelian subgroup $\mathbb{Z}^n$ of a finite index.
Holonomy representation

**Definition**
Let $\Gamma$ be a crystallographic group of dimension $n$ with translations subgroup $A \cong \mathbb{Z}^n$. A finite group $\Gamma/A = G$ we shall call a holonomy group of $\Gamma$.

Let $(A, a) \in E(n)$ and $x \in \mathbb{R}^n$. $\Gamma$ acts on $\mathbb{R}^n$ in the following way:

$$(A, a)(x) = Ax + a.$$
Definition

Let $\Gamma$ be $n$-dimensional Bieberbach group. We have the following short exact sequence of groups.

$$0 \rightarrow \mathbb{Z}^n \rightarrow \Gamma \xrightarrow{p} \Gamma/\mathbb{Z}^n = H \rightarrow 0.$$ 

Let us define a homomorphism $h_{\Gamma} : H \rightarrow GL(n, \mathbb{Z})$. Put

$$\forall h \in H, h_{\Gamma}(h)(e_i) = \bar{h}^{-1}e_i\bar{h},$$

where $p(\bar{h}) = h$ and $e_i \in \mathbb{Z}^n$ is a standard basis. $h_{\Gamma}$ is called a holonomy representation of a group $\Gamma$. 

A. Szczepański
Flat manifold

Let $\Gamma \subset E(n)$ be a torsion free crystallographic group. Since $\Gamma$ is cocompact and discrete subgroup, then the orbit space $\mathbb{R}^n / \Gamma$ is a manifold. If $\Gamma$ is not torsion free then the orbit space $\mathbb{R}^n / \Gamma$ is an orbifold.

Definition
The above manifolds (orbifolds) we shall call "flat".

From elementary covering theory any compact Riemannian manifold (orbifold) with sectional curvature equal to zero is flat.
Example

Flat surfaces:

- torus $S^1 \times S^1$,
- Klein bottle $S^1 \times S^1 / \mathbb{Z}_2$

We shall see that many properties of the Bieberbach Groups correspond to properties of flat manifolds.
Definition

Any Bieberbach group of dimension $n$ with holonomy group $(\mathbb{Z}_2)^{n-1}$ we shall call Generalized Hantzsche-Wendt (GHW for short) Bieberbach group of rank $n$. An oriented GHW we shall call Hantzsche-Wendt group (HW for short).

A flat manifold $\mathbb{R}^n/\Gamma$ is GHW (HW) if and only if the Bieberbach group $\Gamma$ is GHW (HW).

Exercise: Let $M^n$ be a flat Hantzsche-Wendt manifold of dimension $n$. Show that $n$ is an odd natural number.
Example

Klein Bottle is GHW, 3-dimensional flat oriented manifold \( M^3 \) with non-cyclic holonomy \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) is HW manifold. J. Conway called it "didicosm".

It can be proved that \( \pi_1(M^3) \) is a Fibonacci group \( F(2, 6) \), where

\[
F(2, 6) = \{x_1, \ldots, x_6 \mid x_1x_2 = x_3, x_2x_3 = x_4, \ldots, x_6x_1 = x_2\}.
\]

It is clear, that any \( n \)-dimensional HW group is related to an element of the second cohomology group \( H^2(\mathbb{Z}_2^{n-1}, \mathbb{Z}^n) \). Hence the number of non-isomorphic GHW groups of given dimension growth exponentially.
Theorem
Let $\Gamma$ be a $n$-dimensional GHW group. Then

$$h_\Gamma((\mathbb{Z}_2)^{n-1}) \subset \text{GL}(n, \mathbb{Z})$$

is a set of the diagonal matrices with $\pm 1$ on diagonal.

The proof is by induction and used the following lemmas.

Lemma
Let $\rho : \mathbb{Z}_2^{n-1} \to \text{GL}(n, \mathbb{Z})$ be a diagonal faithful integral representation with $-\text{Id} \notin \text{Im}(\rho)$. Then there is $g \in \mathbb{Z}_2^{n-1}$ such that $\rho(g) = \text{diag}(-1,-1,...,-1,1,-1,...,-1)$.

Moreover, if furthermore $\text{Im}(\rho) \not\subset \text{SL}(n, \mathbb{Z})$, then there is $g \in \mathbb{Z}_2^{n-1}$ such that $\rho(g) = \text{diag}(1,...,1,-1,1,...,1)$. 
Lemma

Let $\Gamma$ be a $n$-rank Bieberbach group with translation lattice $\Lambda$. Suppose that $(B, b) \in \Gamma$ and $B$ has eigenvalues $1$ and $-1$, with corresponding eigenspaces $V^+$ and $V^-$ of dimension $1$ and $n - 1$ respectively.

Then $\Lambda = (\Lambda \cap V^+) \oplus (\Lambda \cap V^-)$, and the orthogonal projection of $b$ onto $V^+$ lies in $\frac{1}{2}(\Lambda \cap V^+) \setminus (\Lambda \cap V^+)$. 
Rational homology sphere

Theorem

If $M = \mathbb{R}^n / \Gamma$, where $\Gamma$ is HW, then $M$ is a rational homology sphere.

We calculate rational homology of $M$ from the following formula

$$H_i(M, \mathbb{Q}) = (\Lambda^i(\mathbb{Z}^n)) \mathbb{Z}_2^{-1}.$$ 

From definition of the holonomy representation they are zero for all $i \neq 0, n$. Since HW is connected and oriented, then $M$ is a rational homology sphere.
Anosov relation

Let $M$ be a flat manifold and $f : M \to M$ be a continous map. Define the Lefschetz number by

$$L(f) = \sum_{i \geq 0} (-1)^i \text{Tr}(f_* | H_i(M, \mathbb{Q})).$$

Moreover, let $N(f)$ be the Nielsen number of a map $f$.

**Theorem**

(Anosov, 1985) If $f : M \to M$ is a continous map of nilmanifolds, then

$$N(f) = | L(f) |.$$

It is known, that for any non-oriented flat manifold the above theorem is not true.
Theorem

(Bram DE ROCK, 2006) Let \( M \) be any HW flat manifold, then for any continuous map \( f : M \rightarrow M \)

\[
N(f) = |L(f)|.
\]
Abelianization

Theorem
(B.Putrycz, 2006) Let $M$ be any HW flat manifold of dimension $n > 3$, then

$$H_1(M, \mathbb{Z}) = (\mathbb{Z}_2)^{n-1}.$$ 

In the proof are used methods proposed by R. Miatello and J. P. Rossetti.
In 2008 K. Dekimpe and N. Petrosyan calculate, by constructing a free resolution, homology of low dimensional oriented GHW groups/manifolds. In dimension five there are this same (for 2 groups). But in dimension 7 there are four classes on 62 groups. They find an algorithm.
Let us present the table for dimension five (2 groups):

<table>
<thead>
<tr>
<th>$H_k(\Gamma)$</th>
<th>$k = 0$</th>
<th>$k = 1$</th>
<th>$k = 2$</th>
<th>$k = 3$</th>
<th>$k = 4$</th>
<th>$k = 5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathbb{Z}$</td>
<td>$\mathbb{Z}_2^4$</td>
<td>$\mathbb{Z}_2^2$</td>
<td>$\mathbb{Z}_2^4$</td>
<td>0</td>
<td>$\mathbb{Z}$</td>
<td></td>
</tr>
</tbody>
</table>
for dimension seven (62 groups):

<table>
<thead>
<tr>
<th>$H_k(\Gamma)$</th>
<th>$k=0$</th>
<th>$k=1$</th>
<th>$k=2$</th>
<th>$k=3$</th>
<th>$k=4$</th>
<th>$k=5$</th>
<th>$k=6$</th>
<th>$k=7$</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>$\mathbb{Z}$</td>
<td>$\mathbb{Z}_2^6$</td>
<td>$\mathbb{Z}_2^8$</td>
<td>$\mathbb{Z}_4^6$</td>
<td>$\mathbb{Z}_2^8$</td>
<td>$\mathbb{Z}_2^6$</td>
<td>0</td>
<td>$\mathbb{Z}$</td>
</tr>
<tr>
<td>II</td>
<td>$\mathbb{Z}$</td>
<td>$\mathbb{Z}_2^6$</td>
<td>$\mathbb{Z}_2^9$</td>
<td>$\mathbb{Z}_2^{10} \oplus \mathbb{Z}_4^2$</td>
<td>$\mathbb{Z}_2^9$</td>
<td>$\mathbb{Z}_2^6$</td>
<td>0</td>
<td>$\mathbb{Z}$</td>
</tr>
<tr>
<td>III</td>
<td>$\mathbb{Z}$</td>
<td>$\mathbb{Z}_2^6$</td>
<td>$\mathbb{Z}_2^8$</td>
<td>$\mathbb{Z}_2^4 \oplus \mathbb{Z}_4^4$</td>
<td>$\mathbb{Z}_2^8$</td>
<td>$\mathbb{Z}_2^6$</td>
<td>0</td>
<td>$\mathbb{Z}$</td>
</tr>
<tr>
<td>IV</td>
<td>$\mathbb{Z}$</td>
<td>$\mathbb{Z}_2^6$</td>
<td>$\mathbb{Z}_2^8$</td>
<td>$\mathbb{Z}_2^4 \oplus \mathbb{Z}_4^2$</td>
<td>$\mathbb{Z}_2^8$</td>
<td>$\mathbb{Z}_2^6$</td>
<td>0</td>
<td>$\mathbb{Z}$</td>
</tr>
</tbody>
</table>
Theorem
For any generalized Hantzsche-Wendt group $\Gamma$ of dimension $n \geq 3$ and any subgroup $A$ of index two of the maximal abelian subgroup $\mathbb{Z}^n$ we can associate the quadratic function

$$Q^\Gamma_A : (\mathbb{Z}_2)^{n-1} \to \mathbb{Z}_2$$

and its associated alternating, bilinear quadratic form

$$B^\Gamma_A : (\mathbb{Z}_2)^{n-1} \times (\mathbb{Z}_2)^{n-1} \to \mathbb{Z}_2.$$

Moreover, the function $Q^\Gamma_A$ and form $B^\Gamma_A$ corresponds to the following short exact sequence of finite groups

$$0 \to \mathbb{Z}_2 = \mathbb{Z}^n / A \to \Gamma / A \to (\mathbb{Z}_2)^{n-1} \to 0.$$
The proof follows from the lemma.

**Lemma**

Any subgroup \( A \subset \mathbb{Z}^n \) of index two of maximal abelian subgroup of \( \Gamma \) is a normal subgroup of \( \Gamma \).

**Proof:** We have \( \Gamma \subset E(n) = O(n) \ltimes \mathbb{R}^n \). For any \( a \in A \) and \((G, g) \in \Gamma \) we have \((G, g)(I, a)(G, g)^{-1} = (I, G(a))\). Since \((G - I)x \in 2(\mathbb{Z}^n)\), for any \( x \in \mathbb{Z}^n \), then \( G(a) \in A \).
We define the form $B$ in language of crystallographic group $\Gamma$. Let $(X, x), (Y, y) \in \Gamma$ are mapped by $p$ to $X, Y \in V = (\mathbb{Z}_2)^{n-1}$. We have

$$B(X, Y) = (X - I)y - (Y - I)x \in \mathbb{Z}^n$$

and

$$Q(X) = (X, x)^2 = (X + I)x \in \mathbb{Z}^n.$$  

It is easy to see that $B$ and $Q$ are well defined. It means does not depend from the choice of an element $(X, x) \in \Gamma$. By this same argument the bilinear form $B$ is alternating. However, it depends from an index 2 subgroup $A \subset \mathbb{Z}^n$. 
We shall present an example. Let $D \subset \mathbb{Z}^n$ be generated by the following elements:

$$2e_1, e_1 - e_2, e_2 - e_3, \ldots, e_{n-1} - e_n,$$

where $e_i, i = 1, 2, \ldots, n$ is a standard basis of $\mathbb{Z}^n$. $D$ is a subgroup of index two in $\mathbb{Z}^n$. For $n \geq 2$, let $\Gamma_n$ be the subgroup of $E(n)$ generated by the set $\{(B_i, s(B_i)) \mid 1 \leq i \leq n - 1\}$. Recall, $B_i's$ are the $n \times n$ diagonal matrices:

$$B_i = \text{diag}(-1, \ldots, -1, _i^1, -1, \ldots, -1)$$

and

$$s(B_i) = e_i/2 + e_{i+1}/2 \text{ for } 1 \leq i \leq n - 1.$$
From definition we have

\[ B_{D}^{\Gamma_{n}}(B_i, B_j) = \begin{cases} 
0 & \text{if } i = j \\
1 & \text{if } i = j + 1 \\
0 & \text{if } i \geq j + 2
\end{cases} \]

and a matrix

\[ X = \begin{bmatrix}
0 & 1 & 0 & \ldots & 0 \\
1 & 0 & 1 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \ldots & 0 & 1 & 0 & 1 \\
0 & \ldots & 0 & 1 & 0 & 0
\end{bmatrix}. \]

Moreover for any \( B_i \in (Z_2)^{n-1} \), where \( 1 \leq i \leq n - 1 \)

\[ Q_{D}^{\Gamma_{n}}(B_i) = (B_i + I)(e_i/2 + e_{i+1}/2) = e_i \notin D. \]
We want to calculate the Arf invariant $c$ of $Q_D^{F_n}$. For this, we have to transform $X$ to a symplectic matrix. Let us introduce a new basis $f_1, f_2, \ldots, f_k, f_{k+1}, \ldots, f_{2k}$:

$$f_i = B_{2i-1}, 1 \leq i \leq k$$

and

$$f_{k+i} = B_{2i} + B_{2i+2} + B_{2i+4} + \cdots + B_{2k}, 1 \leq i \leq k.$$

It is easy to see that the matrix $X$ with respect to the new basis is a symplectic,

$$\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}.$$
Finally we have

**Proposition**

*For any odd number* $n = 2k + 1$ *the group* $\Gamma_n/D$ *is an extraspecial group with quadratic form of the quaternion type, (for* $k = 2l - 1$, $2l$ *and* $l$-even) *or the real type (for* $k = 2l - 1$, $2l$ *and* $l$-odd).*
Let $M = \mathbb{R}^n / \Gamma$ be a flat manifold.

**Definition**

A holonomy representation $\Psi_\Gamma : H \to GL(n, \mathbb{Z})$ is essentially complex if there exists a matrix $A \in GL(n, \mathbb{R})$, such that,

$$\forall h \in H, A \Psi_\Gamma(h) A^{-1} \in GL\left(\frac{1}{2}n, \mathbb{C}\right).$$
Theorem

(F.E.A. Johnson, E. Rees, 1991) The following conditions are equivalent:

- $M$ is a flat Kähler manifold,
- $\Psi_\Gamma$ is essentially complex,
- $\Gamma$ is a discrete cocompact torsion-free subgroup of $U\left(\frac{1}{2}n\right) \rtimes \mathbb{C}^\frac{1}{2}n$. 
In the same paper is given the following characterization of an essentially complex representation.

\[ \Psi_\Gamma : H \rightarrow GL(n, \mathbb{Z}) \]

is essentially complex if and only if \( n \) is an even number and each \( \mathbb{R} \)-irreducible summand of \( \Psi_\Gamma \) which is also \( \mathbb{C} \)-irreducible occurs with even multiplicity.
**Definition**

A flat manifold has a complex structure if and only if their holonomy representation is essentially complex.

In Algebraic geometry the flat Kähler manifolds are called hyperelliptic varieties.
2-dimensional hyperelliptic varieties:

<table>
<thead>
<tr>
<th>holonomy</th>
<th>CARAT notations</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\mathbb{Z}_2)</td>
<td>18.1.1; 18.1.2</td>
</tr>
<tr>
<td>(\mathbb{Z}_3)</td>
<td>35.1.1; 35.1.2</td>
</tr>
<tr>
<td>(\mathbb{Z}_4)</td>
<td>25.1.2; 27.1.1</td>
</tr>
<tr>
<td>(\mathbb{Z}_6)</td>
<td>70.1.1</td>
</tr>
</tbody>
</table>

There is also a list of all 3-dimensional complex flat manifolds. There are 174 such objects. If a holonomy group of a flat manifold \(M\) is a subgroup of \(SU(n)\), then \(M\) is called Calabi-Yau manifold.
Theorem

(Hodge, 1941) Let $M$ be $n$-dimensional complex Kähler manifold. We have:

1. $H^r(M, \mathbb{C}) = \sum_{p+q=r} H^{p,q}(M)$,
2. if $h^{p,q}(M) = \dim_{\mathbb{C}} H^{p,q}(M)$, then $h^{p,q}(M) = h^{q,p}(M)$,
3. number $b_r(M) = \sum_{p+q=r} h^{p,q}(M)$ is even if $r$ is odd.

The table of numbers $\{h^{p,q}(M), 0 \leq p, q \leq n\}$ is called the Hodge diamond of $M$. 
Definition
A flat Kähler manifold of complex - dimension $n$ with $\mathbb{Z}_2^{n-1}$ holonomy group is called a complex Hantzsche-Wendt manifold.

They are Calabi-Yau manifolds. Here we present their Hodge diamond:
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A. Szczepański
Thank You.