Flat manifolds with holonomy group $\mathbb{Z}_2^k$ of
diagonal type

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1 Introduction

Let $M^n$ be a flat manifold of dimension $n$. By definition, this is a compact
connected, Riemannian manifold without boundary with sectional curvature
equal to zero. From the theorems of Bieberbach ([2]) the fundamental group
$\pi_1(M^n) = \Gamma$ determines a short exact sequence:

$$0 \to \mathbb{Z}^n \to \Gamma \xrightarrow{p} G \to 0,$$

(1)

where $\mathbb{Z}^n$ is a torsion free abelian group of rank $n$ and $G$ is a finite group
which is isomorphic to the holonomy group of $M^n$. The universal covering
of $M^n$ is the Euclidean space $\mathbb{R}^n$ and hence $\Gamma$ is isomorphic to a discrete
cocompact subgroup of the isometry group $Isom(\mathbb{R}^n) = O(n) \times \mathbb{R}^n = E(n)$. Conversely, given a short exact sequence of the form (1), it is known that the
group $\Gamma$ is (isomorphic to) the fundamental group of a flat manifold if and
only if $\Gamma$ is torsion free. In this case $\Gamma$ is called a Bieberbach group. We can
define a holonomy representation $\phi : G \to GL(n, \mathbb{Z})$ by the formula:

$$\forall g \in G, \phi(g)(e_i) = \tilde{g}e_i(\tilde{g})^{-1},$$

(2)

where $e_i \in \Gamma$ are generators of $\mathbb{Z}^n$ for $i = 1, 2, \ldots, n$, and $\tilde{g} \in \Gamma$ such that
$p(\tilde{g}) = g$. In this article we shall consider only the case

$$G = \mathbb{Z}_2^k, 1 \leq k \leq n - 1, \text{ with } \phi(\mathbb{Z}_2^k) \subset D \subset GL(n, \mathbb{Z}),$$

(3)

where $D$ is the group of all diagonal matrices. We want to consider relations
between two families of flat manifolds with the above property (3): the family $\mathcal{RB}M$ of real Bott manifolds and the family $\mathcal{GHW}$ of generalized Hantzsche-
Wendt manifolds. In particular, we shall prove (Proposition 1) that the
intersection $\mathcal{GHW} \cap \mathcal{RB}M$ is not empty.
In the next section we consider some class of real Bott manifolds without Spin and Spin$^C$ structure. There are given conditions (Theorem 1) for the existence of such structures. As an application a list of all 5-dimensional oriented real Bott manifolds without Spin structure is given, see Example 2. In this case we generalize the results of L. Auslander and R. H. Szczarba, [1] from 1962, cf. Remark 1. At the end we formulate a question about cohomological rigidity of $\mathcal{GHW}$ manifolds.

2 Families

2.1 Generalized Hantzsche-Wendt manifolds

We start with the definition of generalized Hantzsche-Wendt manifold.

Definition 1 ([16, Definition]) A generalized Hantzsche-Wendt manifold (for short $\mathcal{GHW}$-manifold) is a flat manifold of dimension $n$ with holonomy group $(\mathbb{Z}_2)^{n-1}$.

Let $M^n \in \mathcal{GHW}$. In [16, Theorem 3.1] it is proved that the holonomy representation (2) of $\pi_1(M^n)$ satisfies (3).

The simple and unique example of an oriented 3-dimensional generalized Hantzsche-Wendt manifold is a flat manifold which was considered for the first time by W. Hantzsche and H. Wendt in 1934, [8].

Let $M^n \in \mathcal{GHW}$ be an oriented, $n$-dimensional manifold (a HW-manifold for short). In 1982, see [16], the second author proved that for odd $n \geq 3$ and for all $i, H^i(M^n, \mathbb{Q}) \cong H^i(S^n, \mathbb{Q})$, where $\mathbb{Q}$ are the rational numbers and $S^n$ denotes the $n$-dimensional sphere. Moreover, for $n \geq 5$ the commutator subgroup of the fundamental group $\pi_1(M^n) = \Gamma$ is equal to the translation subgroup $[\Gamma, \Gamma] = \Gamma \cap \mathbb{R}^n$, [15]. The number $\Phi(n)$ of affine non equivalent HW-manifolds of dimension $n$ grows exponentially, see [13, Theorem 2.8], and for $m \geq 7$ there exist many isospectral manifolds non pairwise homeomorphic, [13, Corollary 3.6]. The manifolds have an interesting connection with Fibonacci groups [17] and the theory of quadratic forms over the field $\mathbb{F}_2$, [18]. HW-manifolds have no Spin-structure, [12, Example 4.6 on page 4593].

The (co)homology groups and cohomology rings with coefficients in $\mathbb{Z}$ or $\mathbb{Z}_2$, of generalized Hantzsche-Wendt manifolds are still not known, see [4] and [5]. We finish this overview with an example of generalized Hantzsche-Wendt manifolds which have been known already in 1974.
Example 1  Let $M^n = \mathbb{R}^n/\Gamma_n$, $n \geq 2$ be manifolds defined in [11] (see also [16, page 1059]), where $\Gamma_n \subset E(n)$ is generated by $\gamma_0 = (I = \text{id}, (1, 0, ..., 0))$ and

$$
\gamma_i = \begin{pmatrix}
1 & 0 & 0 & \ldots & 0 \\
0 & 1 & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & \ldots & 1 & 0 & 0 \\
0 & \ldots & 0 & -1 & 0 \\
0 & \ldots & 0 & 0 & 1 \\
\vdots & \vdots & \ddots & \ddots & \ddots \\
0 & \ldots & 0 & 0 & 0 & 1
\end{pmatrix},
\begin{pmatrix}
0 \\
0 \\
\vdots \\
0 \\
0 \\
\frac{1}{2} \\
\vdots \\
0
\end{pmatrix} \in E(n),
$$

(4)

where the $-1$ is placed in the $(i, i)$ entry and the $\frac{1}{2}$ as an $(i + 1)$ entry, $i = 1, 2, ..., n - 1$. $\Gamma_2$ is the fundamental group of the Klein bottle.

2.2 Real Bott manifolds

We follow [3], [10] and [14]. To define the second family let us introduce a sequence of $\mathbb{R}P^1$-bundles

$$M_n \to \mathbb{R}P^1 M_{n-1} \to \ldots \to \mathbb{R}P^1 M_1 \to \mathbb{R}P^1 M_0 = \{\text{a point}\}
$$

(5)

such that $M_i \to M_{i-1}$ for $i = 1, 2, ..., n$ is the projective bundle of a Whitney sum of a real line bundle $L_{i-1}$ and the trivial line bundle over $M_{i-1}$. We call the sequence (5) a real Bott tower of height $n$, [3].

Definition 2 ([10]) The top manifold $M_n$ of a real Bott tower (5) is called a real Bott manifold.

Let $\gamma_i$ be the canonical line bundle over $M_i$ and set $x_i = w_1(\gamma_i)$. Since $H^1(M_{i-1}, \mathbb{Z}_2)$ is additively generated by $x_1, x_2, ..., x_{i-1}$ and $L_{i-1}$ is a line bundle over $M_{i-1}$, one can uniquely write

$$w_1(L_{i-1}) = \Sigma_{k=1}^{i-1} a_{k,i} x_k
$$

(6)

with $a_{k,i} \in \mathbb{Z}_2 = \{0, 1\}$ and $i = 2, 3, ..., n$.

From above $A = [a_{k,i}]$ is an upper triangular matrix \(^1\) of size $n$ whose diagonal entries are 0 and other entries are either 0 or 1. Summing up, we can say that the tower (5) is completely determined by the matrix $A$.

\(^1a_{k,i} = 0\) unless $k < i$.  

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From [10, Lemma 3.1] we can consider any real Bott manifold $M(A)$ in the following way. Let $M(A) = \mathbb{R}^n / \Gamma(A)$, where $\Gamma(A) \subset E(n)$ is generated by elements

$$s_i = \begin{pmatrix}
1 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & 1 & 0 \\
0 & \cdots & 0 & 0 & (-1)^{a_{i,i+1}} \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & 0 & 0 \\
0 & \cdots & 0 & \cdots & \cdots \\
\end{pmatrix}
\begin{pmatrix}
0 \\
0 \\
\frac{1}{2} \\
\vdots \\
0 \\
0 \\
\end{pmatrix}
\in E(n), \quad (7)
$$

where $(-1)^{a_{i,i+1}}$ is placed in $(i + 1, i + 1)$ entry and $\frac{1}{2}$ as an $(i)$ entry, $i = 1, 2, \ldots, n - 1$. $s_n = (I, (0, 0, \ldots, 0, 1)) \in E(n)$. From [10, Lemma 3.2,3.3] $s_1^2, s_2^2, \ldots, s_n^2$ commute with each other and generate a free abelian subgroup $\mathbb{Z}^n$. It is easy to see that it is not always a maximal abelian subgroup of $\Gamma(A)$. Moreover, we have the following commutative diagram

$$
\begin{array}{cccccccc}
0 & \rightarrow & N & \rightarrow & \Gamma(A) & \rightarrow & \mathbb{Z}_2^k & \rightarrow & 0 \\
& & \uparrow i & & \downarrow p & & \uparrow & & \\
0 & \rightarrow & \mathbb{Z}^n & \rightarrow & \Gamma(A) & \rightarrow & \mathbb{Z}_2^n & \rightarrow & 0 \\
\end{array}
$$

where $k = rk_{\mathbb{Z}_2}(A), N$ is the maximal abelian subgroup of $\Gamma(A)$, and $p : \Gamma(A)/\mathbb{Z}^n \rightarrow \Gamma(A)/N$ is a surjection induced by the inclusion $i : \mathbb{Z}^n \rightarrow N$. From the first Bieberbach theorem, see [2], $N$ is a subgroup of all translations of $\Gamma(A)$ i.e. $N = \Gamma(A) \cap \mathbb{R}^n = \Gamma(A) \cap \{(I, a) \in E(n) \mid a \in \mathbb{R}^n\}$.

**Definition 3** ([3]) A binary square matrix $A$ is a Bott matrix if $A = PBP^{-1}$ for a permutation matrix $P$ and a strictly upper triangular binary matrix $B$.

Let $\mathcal{B}(n)$ be the set of Bott matrices of size $n$. Since two different upper triangular matrices $A$ and $B$ may produce (affinely) diffeomorphic ($\sim$) real Bott manifolds $M(A), M(B)$, see [3] and [10], there are three operations on $\mathcal{B}(n)$, denoted by (Op1), (Op2) and (Op3), such that $M(A) \sim M(B)$ if and only if the matrix $A$ can be transformed into $B$ through a sequence of the above operations, see [3, part 3]. The operation (Op1) is a conjugation by a permutation matrix,
(Op2) is a bijection $\Phi_k : B(n) \to B(n)$

$$\Phi_k (A)_{*,j} := A_{*,j} + a_{kj} A_{*,k},$$

(8)

for $k, j \in \{1, 2, \ldots, n\}$ such that $\Phi_k \circ \Phi_k = 1_{B(n)}$.

Finally (Op3) is, for distinct $l, m \in \{1, 2, \ldots, n\}$ and the matrix $A$ with $A_{*,l} = A_{*,m}$

$$\Phi^{l,m}(A)_{i,*} := \begin{cases} A_l_{*,*} + A_{m,*} & \text{if } i = m \\ A_{i,*} & \text{otherwise} \end{cases}$$

(9)

Here $A_{*,j}$ denotes j-th column and $A_{i,*}$ denotes i-th row of the matrix $A$.

Let us start to consider the relations between these two classes of flat manifolds. We start with an easy observation

$$\mathcal{RBM}(n) \cap \mathcal{GHW}(n) = \{ M(A) \mid rank_{\mathbb{Z}_2} A = n - 1 \} = \{ M(A) \mid a_{1,2} a_{2,3} \ldots a_{n-1,n} = 1 \}. $$

These manifolds are classified in [3, Example 3.2] and for $n \geq 2$

$$\# (\mathcal{RBM}(n) \cap \mathcal{GHW}(n)) = 2^{(n-2)(n-3)/2}. \quad (10)$$

There exists the classification, see [16] and [3], of diffeomorphism classes of $\mathcal{GHW}$ and $\mathcal{RBM}$ manifolds in low dimensions. For dim $\leq 6$ we have the following table.

<table>
<thead>
<tr>
<th>dim</th>
<th>number of $GHW$ manifolds</th>
<th>number of $RBM$ manifolds</th>
<th>number of $GHW \cap RBM$ manifolds</th>
</tr>
</thead>
<tbody>
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<td></td>
<td>total</td>
<td>oriented</td>
<td>total</td>
</tr>
<tr>
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<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>4</td>
<td>12</td>
<td>0</td>
<td>12</td>
</tr>
<tr>
<td>5</td>
<td>123</td>
<td>2</td>
<td>54</td>
</tr>
<tr>
<td>6</td>
<td>2536</td>
<td>0</td>
<td>472</td>
</tr>
</tbody>
</table>

**Proposition 1** $\Gamma_n \in \mathcal{GHW} \cap \mathcal{RBM}$.

**Proof:** It is enough to see that the group $(G,0) \Gamma_n (G,0)^{-1} = \Gamma (A)$, where $G = [g_{ij}], 1 \leq i, j \leq n$,

$$g_{ij} := \begin{cases} 1 & \text{if } j = n - i + 1 \\ 0 & \text{otherwise} \end{cases}$$

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and $A = [a_{ij}], 1 \leq i, j \leq n$, with
\[
a_{ij} := \begin{cases} 
1 & \text{if } j = i + 1 \\
0 & \text{otherwise}
\end{cases}
\]

3 Existence of Spin and Spin$^C$ structures on real Bott manifolds

In this section we shall give some condition for the existence of Spin and Spin$^C$ structures on real Bott manifolds. We use notations from the previous sections. There are a few ways to decide whether there exists a Spin structure on an oriented flat manifold $M^n$, see [6]. We start with the following. A closed oriented differential manifold $N$ has such a structure if and only if the second Stiefel-Whitney class $w_2(N) = 0$.

In the case of an oriented real Bott manifold $M(A)$ we have the formula for $w_2$.

Recall, see [10], that for the Bott matrix $A$

\[
H^*(M(A); \mathbb{Z}_2) = \mathbb{Z}_2[x_1, x_2, ..., x_n]/(x_j^2 = x_j \sum_{i=1}^n a_{i,j}x_i \mid j = 1, 2, ..., n) \quad (11)
\]
as graded rings. Moreover, from [11, (3.1) on page 3] the $k$-th Stiefel-Whitney class

\[
w_k(M(A)) = (B(p))^* \sigma_k(y_1, y_2, ..., y_n) \in H^k(M(A); \mathbb{Z}_2), \quad (12)
\]
where $\sigma_k$ is the $k$-th elementary symmetric function,

\[p : \pi_1(M(A)) \to G \subset O(n)\]
a holonomy representation, $B(p)$ is a map induced by $p$ on the classification spaces and $y_i \overset{(6)}{=} w_1(L_{i-1})$. Hence,

\[
w_2(M(A)) = \sum_{1 \leq i < j \leq n} y_i y_j \in H^2(M(A); \mathbb{Z}_2). \quad (13)
\]

There exists a general condition, see [4, Theorem 3.3], for the calculation of the second Stiefel-Whitney for flat manifolds with $(\mathbb{Z}_2)^k$ holonomy of diagonal type but we prefer the above explicit formula (13). Its advantage follows from the knowledge of the cohomology ring (11) of real Bott manifolds.

\[\text{We use it in Example 2.}\]
An equivalent condition for the existence of a Spin structure is as follows. An oriented flat manifold \( M^n \) (a Bieberbach group \( \pi_1(M^n) = \Gamma \)) has a Spin structure if and only if there exists a homomorphism \( \epsilon : \Gamma \to \text{Spin}(n) \) such that \( \lambda_n \epsilon = p \). Here \( \lambda_n : \text{Spin}(n) \to SO(n) \) is the covering map, see [6]. We have a similar condition, under assumption \( H^2(M^n, \mathbb{R}) = 0 \), for the existence of Spin\(^C\) structure, [6, Theorem 1]. In this case \( M^n \) (a Bieberbach group \( \Gamma \)) has a Spin\(^C\) structure if and only if there exists a homomorphism \( \bar{\epsilon} : \Gamma \to \text{Spin}^C(n) \) (14) such that \( \bar{\lambda}_n \bar{\epsilon} = p \). \( \bar{\lambda}_n : \text{Spin}^C(n) \to SO(n) \) is the homomorphism induced by \( \lambda_n \), see [6]. We have the following easy observation. If there exists \( H \subset \Gamma \), a subgroup of finite index, such that the finite covering \( \tilde{M}^n \) with \( \pi_1(\tilde{M}^n) = H \) has no Spin (Spin\(^C\)) structure, then \( M^n \) has also no such structure.

We shall prove.

**Theorem 1** Let \( A \) be a matrix of an orientable real Bott manifold \( M(A) \) of dimension \( n \).

**I.** Let \( l \in \mathbb{N} \) be an odd number. If there exist \( 1 \leq i < j \leq n \) and rows \( A_{i,*}, A_{j,*} \) such that
\[
\# \{ m \mid a_{i,m} = a_{j,m} = 1 \} = l
\]
and
\[
a_{i,j} = 0,
\]
then \( M(A) \) has no Spin structure.

Moreover, if
\[
\# \{ J \subset \{1, 2, ..., n\} \mid \# J = 2, \Sigma_{j \in J} A_{s,j} = 0 \} = 0,
\]
then \( M(A) \) has no Spin\(^C\) structure.

**II.** If there exist \( 1 \leq i < j \leq n \) and rows
\[
A_{i,*} = (0, ..., 0, a_{i,i_1}, ..., a_{i,i_{2k}}, 0, ..., 0),
\]
\[
A_{j,*} = (0, ..., 0, a_{j,i_{2k}+1}, ..., a_{j,i_{2k}+2l}, 0, ..., 0)
\]
such that \( a_{i,i_1} = a_{i,i_2} = ... = a_{i,i_{2k}} = 1, a_{i,m} = 0 \) for \( m \notin \{ i_1, i_2, ..., i_{2k} \} \)
\( a_{j,i_{2k}+1} = a_{j,i_{2k}+2} = ... = a_{j,i_{2k}+2l} = 1, a_{j,r} = 0 \) for \( r \notin \{ i_{2k}+1, i_{2k}+2, ..., i_{2k}+2l \} \) and \( l, k \) odd then \( M(A) \) has no Spin structure.
**Proof:** From [10, Lemma 2.1] the manifold \( M(A) \) is orientable if and only if for any \( i = 1, 2, \ldots, n \),
\[
\sum_{k=i+1}^{n} a_{i,k} = 0 \mod 2.
\]
Assume that \( \epsilon : \pi_1(M(A)) \rightarrow \text{Spin}(n) \) defines a Spin structure on \( M(A) \).
Let \( a_{i,i_1}, a_{i,i_2}, \ldots, a_{i_i,i_{2m}}, a_{j,j_1}, a_{j,j_2}, \ldots, a_{j_j,j_{2p}} = 1 \) and let \( s_i, s_j \) be elements of \( \pi_1(M(A)) \) which define rows \( i, j \) of \( A \), see (7). Then
\[
\epsilon(s_i) = \pm e_{i_1}e_{i_2}\ldots e_{i_{2m}},
\]
\[
\epsilon(s_j) = \pm e_{j_1}e_{j_2}\ldots e_{j_{2p}}
\]
and
\[
\epsilon(s_is_j) = \pm e_{k_1}e_{k_2}\ldots e_{k_{2r}}.
\]
From (15) \( 2r = 2m + 2p - 2l \). Moreover \( \epsilon(s_i^2) = (-1)^m, \epsilon(s_j^2) = (-1)^p \) and \( \epsilon((s_is_j)^2) = (-1)^{m+p-l} = (-1)^{m+p+l} \). Since from (16) (see also [10, Lemma 3.2]) \( s_is_j = s_js_i \) we have \( \epsilon((s_i)^2)\epsilon((s_j)^2) = \epsilon((s_is_j)^2) \). Hence
\[
(-1)^{m+p} = (-1)^{m+p+l}.
\]
This is impossible since \( l \) is an odd number and we have a contradiction.

For the existence of the Spin\(^C\) structure it is enough to observe that the condition (17) is equivalent to equation \( H^2(M(A), \mathbb{R}) = 0 \), see [3, formula (8.1)]. Hence, we can apply the formula (14). Let us assume that \( \bar{\epsilon} : \pi_1(M(A)) \rightarrow \text{Spin}^C(n) \) defines a Spin\(^C\) structure. Using the same arguments as above, see [6, Proposition 1], we obtain a contradiction. This finished the proof of \( \mathbf{I} \).

For the proof \( \mathbf{II} \) let us observe that \( s_i^2 = (s_is_j)^2 \). Hence \( (-1)^k = \epsilon((s_i)^2 = \epsilon((s_is_j)^2)) = (-1)^{k+l} = 1 \). This is impossible.

\( \square \)

In the above theorem rows of number \( i \) and \( j \) correspond to generators \( s_i, s_j \) which define a finite index subgroup \( H \subset \pi_1(M(A)) \). It is a Bieberbach group with holonomy group \( \mathbb{Z}_2 \oplus \mathbb{Z}_2 \). We proved that \( H \) (if it exists) has no Spin (Spin\(^C\)) structure, (see the discussion before Theorem 1). In the next example we give the list of all 5-dimensional real Bott manifolds (with) without Spin(Spin\(^C\)) structure.

**Example 2** From [14] we have the list of all 5-dimensional oriented real Bott manifolds. There are 7 such manifolds without the torus. Here are
their matrices:

\[ A_4 = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad A_{23} = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \]

\[ A_{29} = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad A_{37} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \]

\[ A_{40} = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad A_{48} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \]

\[ A_{49} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \]

From the first part of Theorem 1 above, for \( i = 1, j = 2 \) the manifold \( M(A_4) \) has no Spin\(^\mathbb{C}\) structure. For the same reasons (for \( i = 1, j = 2 \)) manifolds \( M(A_{40}) \) and \( M(A_{48}) \) have no Spin structures. The manifold \( M(A_{23}) \) has no a Spin structure, because it satisfies for \( i = 1, j = 3 \) the second part of the Theorem 1. Since any flat oriented manifold with \( \mathbb{Z}_2 \) holonomy has Spin structure, [9, Theorem 3.1] manifolds \( M(A_{29}), M(A_{49}) \) have it. Our last example, the manifold \( M(A_{37}) \) has Spin structure and we leave it as an exercise.

In all these cases it is possible to calculate the \( w_2 \) with the help of (6), (13) and (11). In fact, \( w_2(M(A_4)) = (x_2)^2 + x_1 x_3, w_2(M(A_{23})) = x_1 x_3, w_2(M(A_{40})) = w_2(M(A_{48})) = x_1 x_2. \) In all other cases \( w_2 = 0. \)
Example 3 Let

\[
A = \begin{bmatrix}
0 & 0 & 1 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & * & * & * & * \\
0 & 0 & 0 & 0 & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix},
\]

be a family of Bott matrices, with \( * \in \mathbb{Z}_2 \). It is easy to check that the first two rows satisfy the condition of Theorem 1. Hence the oriented real Bott manifolds \( M(A) \) have no the Spin structure.

Remark 1 In [1] on page 6 an example of the flat (real Bott) manifold \( M \) without Spin structure is considered. By an immediate calculation the Bott matrix of \( M \) is equal to

\[
\begin{bmatrix}
0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}.
\]

4 Concluding Remarks

The tower (5) is an analogy of a Bott tower

\[
W_n \to W_{n-1} \to ... \to W_1 = \mathbb{C}P^1 \to W_0 = \{ \text{a point} \}
\]

where \( W_i \) is a \( \mathbb{C}P^1 \) bundle on \( W_{i-1} \) i.e.; \( W_2 = P(1 \oplus L_{i-1}) \) and \( L_{i-1} \) is a holomorphic line bundle over \( W_{i-1} \). As in (5) \( P(1 \oplus L_{i-1}) \) is projectivisation of the trivial linear bundle and \( L_{i-1} \). It was introduced by Grossberg and Karshon [7]. As is well known, see [3] for the complete bibliography, \( W_n \) is a toric manifold.

There is an open problem: Is it true that two toric manifolds are diffeomorphic (or homeomorphic) if their cohomology rings with integer coefficients are isomorphic as graded rings? In some cases it has partial affirmative solutions (see [10]). For real Bott manifolds the following is true.

Theorem ([10, Theorem 1.1]) Two real Bott manifolds are diffeomorphic if and only if their cohomology rings with \( \mathbb{Z}_2 \) coefficients are isomorphic as graded rings. Equivalently, they are cohomological rigid.
All of this suggests the following:

**Question** Are $\mathcal{GHW}$-manifolds cohomological rigid?

The answer to the above question is positive for manifolds from $\mathcal{GHW} \cap \mathcal{RBM}$. It looks the most interesting for oriented $\mathcal{GHW}$-manifolds. However, for $n = 5$ there are two oriented Hantzsche-Wendt manifolds. From direct calculations with the help of a computer we know that they have different cohomology rings with $\mathbb{Z}_2$ coefficients.

**References**


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