Abstract: We formulate a condition for an existence of a Spin\(^{\mathbb{C}}\)-structure on an oriented flat manifold \(M^n\) with \(H_2(M^n, \mathbb{R}) = 0\). We prove that \(M^n\) has a Spin\(^{\mathbb{C}}\)-structure if and only if there exist a homomorphism \(\epsilon: \pi_1(M^n) \rightarrow \text{Spin}^{\mathbb{C}}(n)\) such \(\bar{\lambda}_n \circ \epsilon = h\), where \(h: \pi_1(M^n) \rightarrow SO(n)\) is a holonomy homomorphism and \(\bar{\lambda}_n: \text{Spin}^{\mathbb{C}}(n) \rightarrow SO(n)\) is a standard homomorphism defined on page 2. As an application we shall prove that all cyclic Hantzsche - Wendt manifolds have not the Spin\(^{\mathbb{C}}\)-structure.

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1. INTRODUCTION

Let \(M^n\) be a flat manifold of dimension \(n\). By definition, this is a compact connected, Riemannian manifold without boundary with sectional curvature equal to zero. From the theorems of Bieberbach ([2]) the fundamental group \(\pi_1(M^n) = \Gamma\) determines a short exact sequence:

\[
0 \rightarrow \mathbb{Z}^n \rightarrow \Gamma \xrightarrow{h} F \rightarrow 0,
\]

where \(\mathbb{Z}^n\) is a torsion free abelian group of rank \(n\) and \(F\) is a finite group which is isomorphic to the holonomy group of \(M^n\). The universal covering of \(M^n\) is the Euclidean space \(\mathbb{R}^n\) and hence \(\Gamma\) is isomorphic to a discrete cocompact subgroup of the isometry group \(\text{Isom}(\mathbb{R}^n) = O(n) \ltimes \mathbb{R}^n = E(n)\). In the above short exact sequence \(\mathbb{Z}^n \cong (\Gamma \cap \mathbb{R}^n)\) and \(h\) can be considered as the projection \(h : \Gamma \rightarrow F \subset O(n) \subset E(n)\) on the first component. Conversely, given a short sequence of the form (1), it is known that the group \(\Gamma\) is (isomorphic to) a Bieberbach group if and only if \(\Gamma\) is torsion free.

By Hantzsche-Wendt manifold (for short HW-manifold) \(M^n\) we shall understand any oriented flat manifold of dimension \(n\) with a holonomy group \((\mathbb{Z}_2)^{n-1}\). It is easy to see that \(n\) is always an odd number. Moreover, (see [12] and [17]) HW-manifolds are rational homology spheres and its holonomy representation \(^1\) is diagonal, [16]. Hence \(\pi_1(M^n)\) is

\(^1\) That is a representation \(\phi_T : F \rightarrow \text{GL}(n, \mathbb{Z})\), given by a formula \(\phi_T(f)(z) = fzf^{-1}\), where \(f \in \Gamma, f \in F, z \in \mathbb{Z}^n\) and \(p(f) = f\).
generated by $\beta_i = (B_i, b_i) \in \text{SO}(n) \ltimes \mathbb{R}^n$, $1 \leq i \leq n$, where
\begin{equation}
B_i = \text{diag}(-1, -1, \ldots, -1, \underbrace{1, \ldots, 1}_{i-1}, -1, -1, \ldots, -1) \quad \text{and} \quad b_i \in \{0, 1/2\}^n.
\end{equation}

Let us recall some other properties of $M^n$. For $n \geq 5$ the commutator subgroup of the fundamental group is equal to the translation subgroup $[\Gamma, \Gamma] = \Gamma \cap \mathbb{R}^n$, ([14]). The number $\Phi(n)$ of affinian not equivalent HW-manifolds of dimension $n$ grows exponentially, see [12, Theorem 2.8] and for $m \geq 7$ there exist many pairs of isospectral manifolds all not homeomorphic to each other, [12, Corollary 3.6]. These manifolds have interesting connection with Fibonacci groups [18] and the theory of quadratic forms over a field $\mathbb{F}_2$, [19]. HW-manifolds have not a Spin-structure, [11, Example 4.6 on page 4593]. Hence tangent bundles of HW-manifolds are not trivial. There are still not known their (co)homology groups with coefficients in $\mathbb{Z}$. We send reader to [4] where are presented results for low dimensions and an algorithm. Finally, let us mention about properties related to the theory of fixed points. HW-manifolds satisfy so called Anosov relation. This means for any continuous map $f : M^n \to M^n, | L(f) | = N(f)$, where $L(f)$ is the Lefschetz number of $f$ and $N(f)$ is the Nielsen number of $f$, see [3].

In this note we are interested in properties of the tangent bundle of HW-manifolds. We shall prove that they are line element parallelizable (Proposition 1) and we shall define an infinite family of HW-manifolds without Spin$^C$-structure (Theorem 2). However, the main result of this article is related to an existence Spin$^C$-structures on oriented flat manifolds. The group Spin$^C(n)$ is given by $\text{Spin}^C(n) = (\text{Spin}(n) \times S^1)/\{1, -1\}$ where $\text{Spin}(n) \cap S^1 = \{1, -1\}$. Moreover, there is a homomorphism of groups $\tilde{\lambda}_n : \text{Spin}^C(n) \to \text{SO}(n)$ given by $\tilde{\lambda}_n[g, z] = \lambda_n(g)$, where $g \in \text{Spin}(n), z \in S^1$ and $\lambda_n : \text{Spin}(n) \to \text{SO}(n)$ is the universal covering. We shall prove:

**Theorem 1** Let $M$ be an oriented flat manifold with $H^2(M, \mathbb{R}) = 0$. $M$ has a Spin$^C$-structure if and only if there exists a homomorphism $\epsilon : \Gamma \to \text{Spin}^C(n)$ such that
\begin{equation}
\tilde{\lambda}_n \circ \epsilon = h.
\end{equation}

As an application we prove Theorem 2.

**Theorem 2** All cyclic HW-manifolds have not the Spin$^C$-structure.

For a description of cyclic HW-manifolds see Definition 2. We conjecture that all HW-manifolds have not the Spin$^C$-structure.

2. **Hantzsche-Wendt manifolds are line element parallelizable**

We keep notations from the introduction. For any discrete group $G$, we have a universal principal $G$-bundle with the total space $EG$ and the base space $BG$. $BG$ is called the classifying space of a group $G$ and is unique up to homotopy. In our case $\mathbb{R}^n$ is the total space of a principal $\Gamma$-bundle with a base space $M^n$. Here $\mathbb{E} \Gamma = \mathbb{R}^n$ and $\mathbb{B} \Gamma = M^n$, see [20, page 369]. Now $G \to BG$ behaves more or less like a functor, and in particular, from the surjection $h : \Gamma \to h(\Gamma) = F$ we can construct a corresponding map $B(h) : \mathbb{B} \Gamma \to \mathbb{B} F$. Finally, the inclusion $i_n : F \to \text{O}(n)$ yields a map $B(i_{n}) : \mathbb{B} F \to \mathbb{B} \text{O}(n))$. The universal
$n$-dimensional vector bundle over $B(O(n))$ yields, via this map a vector bundle $\eta_n$ over $BF$.

**Lemma 1.** ([20, Proposition 1.1]) $B(h)^*(\eta_n)$ is equivalent to the tangent bundle of $M^n$.

**Proof:** (See [20, page 369]) We have a commutative diagram as follows

$$
\begin{array}{ccc}
\mathbb{R}^n = E \Gamma & \xrightarrow{E(h)} & EF \\
\downarrow & & \downarrow \\
M^n = B \Gamma & \xrightarrow{B(h)} & BF
\end{array}
$$

where $E(h)(g \cdot e) = h(g) \cdot E(h)(e)$ for all $g \in \Gamma$ and $e \in E \Gamma = \mathbb{R}^n$. Let the total space of $\eta_n$ be $EF \times \mathbb{R}^n / F$ where $f \in F$ acts via $f(e, v) = (f \cdot e, f \cdot v)$. Now clearly the total space $\tau$ of the tangent bundle of $M^n = B \Gamma$ can be taken to be $\mathbb{R}^n \times \mathbb{R}^n / \Gamma$ where $\Gamma$ acts via $g(v_1, v_2) = (gv_1, h(g)v_2)$. Thus we have a commutative diagram as follows:

$$
\begin{array}{ccc}
\tau = \mathbb{R}^n \times \mathbb{R}^n / \Gamma & \xrightarrow{E \times \mathbb{R}^n / F} & EF \times \mathbb{R}^n / F \\
\downarrow & & \downarrow \\
M^n = \mathbb{R}^n / \Gamma & \xrightarrow{B(h)} & BF
\end{array}
$$

where $F$ acts on $EF \times \mathbb{R}^n$ as followings $\{v_1, v_2\} \rightarrow \{E(h)(v_1), v_2\}$. This finishes the proof. \(\square\)

**Remark 1.** From the above Lemma we can observe that the tangent bundle is flat in sense of [1, page 272].

Let us present the main result of this section.

**Proposition 1.** Let $M^n$ be a HW-manifold of dimension $n$. Then its tangent bundle is line element parallelizable, (is a sum of line bundles).

**Proof:** By definition the fundamental group $\Gamma = \pi_1(M^n)$ is a subgroup of $SO(n) \times \mathbb{R}^n$ and $h(\Gamma) = (\mathbb{Z}_2)^{n-1} \subset SO(n)$ is a group of all diagonal orthogonal matrices. It is also an image of the holonomy representation $\phi_\Gamma : (\mathbb{Z}_2)^{n-1} \rightarrow SO(n)$. Let us recall a basic facts about line bundles. It is well known that the classification space for line bundles is $\mathbb{R}P^\infty$, the infinite projective space. Hence any line bundle $\xi : L \rightarrow M^n$ is isomorphic to $f^*(\eta_1)$, where

$$f \in [M^n, \mathbb{R}P^\infty] \simeq H^1(M^n, \mathbb{Z}_2) \simeq Hom(\Gamma, \mathbb{Z}_2) \xrightarrow{(*)} (\mathbb{Z}_2)^{n-1}$$

is a classification map and $\eta_1 \in H^1(\mathbb{R}P^\infty, \mathbb{Z}_2) = \mathbb{Z}_2$ is not a trivial element. Here $\eta_1$ represents the universal line vector bundle and the isomorphism $(*)$ follows from [14, Cor. 3.2., ]. Since $(\mathbb{Z}_2)^{n-1}$ is an abelian group,

$$\phi_\Gamma = \bigoplus_{i=1}^{n}(\phi_i),$$
where \((\phi_T)_i : (\mathbb{Z}_2)^{n-1} \to \{\pm 1\}\) are irreducible representations of \((\mathbb{Z}_2)^{n-1}\), for \(i = 1, 2, ..., n\). From Lemma 1 and [7, Theorem 8.2.2] the tangent bundle

\[
\tau(M^n) = B(h)^*(\eta_n) = \bigoplus_{i=1}^{n} B(h_i)^*(\eta_i),
\]

where \(h_i = (\phi_T)_i \circ h\). This finishes the proof. □

3. Spin\(^C\)-structure

It is well known (see [11, Example 4.6 on page 4593]) that HW-manifolds have not the Spin-structure. In this section we shall consider the question: Do HW-manifolds have the Spin\(^C\)-structure?

On the beginning let us recall some facts about the group Spin\(^C\), which was defined in the Introduction. We start with homomorphisms ([5, page 25]):

- \(i : \text{Spin}(n) \to \text{Spin}^C(n)\) is the natural inclusion \(i(g) = [g, 1]\).
- \(j : S^1 \to \text{Spin}^C(n)\) is the natural inclusion, \(j(z) = [1, z]\).
- \(l : \text{Spin}^C(n) \to S^1\) is given by \(l[g, z] = z^2\).
- \(p : \text{Spin}^C(n) \to \text{SO}(n) \times S^1\) is given by \(p([g, z]) = (\lambda_n(g), z^2)\). Hence \(p = \lambda_n \times l\).

Since \(S^1 = \text{SO}(2)\), there is a natural map \(k : \text{SO}(n) \times \text{SO}(2) \to \text{SO}(n+2)\). Then we can describe \(\text{Spin}^C(n)\) as the pullback by this map of the covering map

\[
\begin{array}{ccc}
\text{Spin}^C(n) & \to & \text{Spin}(n+2) \\
\downarrow & & \downarrow \lambda_n \\
\text{SO}(n) \times \text{SO}(2) & \xrightarrow{k} & \text{SO}(n+2)
\end{array}
\]

Let \(W^n\) be an \(n\)-dimensional, compact oriented manifold and let \(\delta : W^n \to \text{BSO}(n)\) be the classification map of its tangent bundle \(TW^n\). We now recall the definition of a Spin\(^C\)-structure ([9, page 34], [5, page 47]).

**Definition 1.** A Spin\(^C\)-structure on the manifold \(W^n\) is a lift of \(\delta\) to \(B\text{Spin}^C(n)\), giving a commutative diagram:

\[
\begin{array}{ccc}
B\text{Spin}^C(n) & \to & \text{BSO}(n) \\
\downarrow b(\lambda_n) & & \downarrow \\
W^n & \xrightarrow{\delta} & \text{BSO}(n).
\end{array}
\]
Remark 2.

(1) (See [5, Remark, d on page 49].) $W^n$ has the Spin$^C$-structure if and only if the Stiefel-Whitney class $w_2 \in H^2(W^n, \mathbb{Z}_2)$ is $\mathbb{Z}_2$-reduction of an integral Stiefel-Whitney class $\tilde{w}_2 \in H^2(W^n, \mathbb{Z})$.

(2) Let $K(Z, 2)$ and $K(Z_2, 2)$ be the Eilenberg-Maclane spaces. From the homotopy theory

$$H^2(W^n, \mathbb{Z}) = [W^n, BS^1] = [W^n, K(Z, 2)]$$

and $H^2(W^n, \mathbb{Z}_2) = [W^n, K(Z_2, 2)]$. Hence the above condition defines a commutative diagram

$$\begin{array}{ccc}
K(Z, 2) & \rightarrow & K(Z_2, 2) \\
\downarrow & & \downarrow \\
\tilde{w}_2 & \Rightarrow & w_2 \end{array}$$

where the vertical arrow is induced by an epimorphism $\mathbb{Z} \rightarrow \mathbb{Z}_2$.

From previous sections (Lemma 1) an oriented flat manifold $M = B\Gamma$, and $\delta = B(h)$ where $\Gamma = \pi_1(M)$ and $h : \Gamma \rightarrow SO(n)$ is a holonomy homomorphism. Let us recall (see [15, page 323] and Remark 3) that an oriented manifold $M$ has a Spin-structure if and only if there exists a homomorphism $e : \Gamma \rightarrow \text{Spin}(n)$ such that

(4) $\lambda_n \circ e = h$.

Hence, a condition of existence of the Spin$^C$-structure on $M$ is very similar to the above condition (4).

Theorem 1. Let $M$ be an oriented flat manifold with $H^2(M, \mathbb{R}) = 0$. $M$ has a Spin$^C$-structure if and only if there exists a homomorphism $\epsilon : \Gamma \rightarrow \text{Spin}^C(n)$ such that

(5) $\bar{\lambda}_n \circ \epsilon = h$.

Proof: Let us assume that there exists a homomorphism $\epsilon : \Gamma \rightarrow \text{Spin}^C(n)$ such that $\bar{\lambda}_n \epsilon = h$. We claim that conditions of Definition 1 are satisfied. In fact, $B(\bar{\lambda}_n)B(\epsilon) = B(h)$ up to homotopy. To go the other way, let us assume that $M = B\Gamma$ admits a Spin$^C$-structure. We have a commutative diagram.
Diagram 1

where $\Gamma_0$ is defined by the second Stiefel-Whitney class $w_2 \in H^2(\Gamma, \mathbb{Z}_2)$ and $\Gamma_1$ is defined by the element $r_*(w_2) \in H^2(\Gamma, S^1)$. Here $r : \mathbb{Z}_2 \to S^1$ is a group monomorphism. Let $h^2 : H^2(\text{SO}(n), K) \to H^2(\Gamma, K)$ be a homomorphism induced by the holonomy homomorphism $h$, for $K = \mathbb{Z}_2, S^1$. From definition, (see [10, Chapter 23.6]) there exists an element \[ x_2 \in H^2(\text{SO}(n), \mathbb{Z}_2) = \mathbb{Z}_2 \] such that $h^2(x_2) = w_2$ and $h^2(r_*(x_2)) = r_*(h^2(x_2)) = r_*(w_2)$. Moreover we have two infinite sequences of cohomology which are induced by the following commutative diagram of groups

\[
\begin{array}{ccccccc}
1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z}_2 & \longrightarrow & 1 \\
\uparrow & & \uparrow & & \uparrow r & & \\
1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{R} & \longrightarrow & S^1 & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow r_* & & \downarrow & & \\
\cdots & \longrightarrow & H^2(\Gamma, \mathbb{Z}) & \longrightarrow & H^2(\Gamma, \mathbb{Z}_2) & \longrightarrow & H^2(\Gamma, \mathbb{Z}) & \longrightarrow & \cdots \\
\downarrow & & \downarrow \text{red} & & \downarrow r_* & & \downarrow & & \\
\cdots & \longrightarrow & H^2(\Gamma, \mathbb{Z}) & \longrightarrow & H^2(\Gamma, \mathbb{R}) & \longrightarrow & H^2(\Gamma, S^1) & \longrightarrow & H^2(\Gamma, \mathbb{Z}) & \longrightarrow & \cdots
\end{array}
\]

From Remark 2 $\text{red}(\bar{w}_2) = w_2$ and since $H^2(\Gamma, \mathbb{R}) = 0, r_*(w_2) = 0$. It follows that the row $0 \to S^1 \to \Gamma_1 \to \Gamma \to 0$ of the Diagram 1 splits. Hence there exists a homomorphism $\epsilon : \Gamma \to \text{Spin}^{C}(n)$ which satisfies (5). This proves the theorem.

As an immediate corollary we have.

**Corollary 1.** Let $M$ be an oriented flat manifold with the fundamental group $\Gamma$. If there exists a homomorphism $\epsilon : \Gamma \to \text{Spin}^{C}(n)$ such that

\[ \bar{\lambda}_n \circ \epsilon = h. \]

then $M$ has a Spin$^C$-structure.

---

\[ ^2H^{*}(\text{SO}(n), \mathbb{Z}_2) = \mathbb{Z}_2[x_2, x_3, ..., x_n]. \]
Remark 3. A condition (4) of an existence of the Spin-structure for oriented flat manifolds also follows from the proof of the above Theorem 1.

Question: Is the assumption $H^2(M, \mathbb{R}) = 0$ about the second cohomology group necessary?

Example 1.

(1) Because of the inclusion $i : \text{Spin}(n) \to \text{Spin}^C(n)$ each Spin-structure on $M$ induces a Spin$^C$-structure.

(2) If $M$ is any smooth compact manifold with an almost complex structure, then $M$ has a canonical Spin$^C$-structure, see [5, page 27].

Example 2. Any oriented compact manifold of dimension up to four has a Spin$^C$-structure, see [6, page 49].

From Example 2 and [15, Theorem on page 324] we have immediately.

Corollary 2. There exist three four dimensional flat manifolds without Spin-structure but with Spin$^C$-structure.

In [5, Example on page 50] is given a compact 5-dimensional manifold $Q$, without Spin$^C$-structure. However the fundamental group $\pi_1(Q) = 1$. There are also two other non-simply connected 5-dimensional examples, see [8, Examples page 438]. The first one is hypersurface in $\mathbb{R}P^2 \times \mathbb{R}P^4$ defined by the equation $x_0y_0 + x_1y_1 + x_2y_2 = 0$ where $[x_0 : x_1 : x_2]$ and $[y_0 : y_1y_2 : y_3 : y_4]$ are homogeneous coordinates in $\mathbb{R}P^2$ and $\mathbb{R}P^4$ respectively. The second one is the Dold’s manifold $P(1, 2) = \mathbb{C}P^2 \times S^1 / \sim$, where $\sim$ is an involution, which acts on $\mathbb{C}P^2$ by complex conjugation and antipodally on $S^1$. Our next result gives examples of 5-dimensional flat manifolds without Spin$^C$-structure.

Proposition 2. Two HW-manifolds $M_1$ and $M_2$ of dimension five have not the Spin$^C$-structure.

Proof: Since $H^2(M_i, \mathbb{R}) = 0, i = 1, 2, ([4], [16])$ we can apply a condition from Theorem 1. Let $\Gamma_1 = \pi_1(M_1)$. It has the CARAT number 1-th 219.1.1, see [13].

It is generated by

\[
\alpha_1 = ([1, 1, 1, -1, -1], (0, 0, 1/2, 1/2, 0)), \quad \alpha_2 = ([1, 1, -1, -1, 1], (0, 1/2, 0, 0, 0)),
\]

\[
\alpha_3 = ([1, -1, 1, 1, 1], (0, 0, 0, 0, 1/2)), \quad \alpha_4 = ([1, 1, -1, -1, 1], (1/2, 0, 0, 0, 0))
\]

and translations. We assume that there exists a homomorphism $\epsilon : \Gamma_1 \to \text{Spin}^C(5)$ such that $\lambda_n \circ \epsilon = h$. From definition

\[
\alpha_2 \alpha_3 = \alpha_3 \alpha_2
\]

and $(\alpha_2 \alpha_3)^2 = (\alpha_2)^2 (\alpha_3)^2$. Put $\epsilon(\alpha_i) = [a_i, z_i] \in \text{Spin}^C(5), a_i \in \text{Spin}(5), z_i \in S^1, i = 1, 2, 3$. Then

\[
\epsilon((\alpha_2 \alpha_3)^2) = [-1, z_2^2 z_3^2] = \epsilon((\alpha_2)^2) \epsilon((\alpha_3)^2) = [-1, z_2^2] [-1, z_3^2] = [1, z_2^2 z_3^2]
\]

\footnote{Here we use the name CARAT for tables of Bieberbach groups of dimension $\leq 6$, see [13].}
and \(-z_1^2 z_2^2 = z_2^2 z_1^2\). We obtain contradiction.

Now, let us consider the second five dimensional HW-group \(\Gamma_2 = \pi_1(M_2)\) which has a number 2-th. 219.1.1., (see [13]). It is generated by
\[
\beta_1 = (B_1, (1/2, 1/2, 0, 0, 0)), \quad \beta_2 = (B_2, (0, 1/2, 1/2, 0, 0)), \\
\beta_3 = (B_3, (0, 0, 1/2, 1/2, 0)) \quad \text{and} \quad \beta_4 = (B_4, (0, 0, 0, 1/2, 1/2)).
\]
Put \(\beta_5 = (\beta_1 \beta_2 \beta_3 \beta_4)^{-1} = (B_5, (1/2, 0, 0, 0, -1/2))\). Assume that there exists a homomorphism \(\epsilon : \Gamma_2 \rightarrow \text{Spin}^C(5)\) which defines the \(\text{Spin}^C\)-structure on \(M_2\). Let \(\epsilon(\beta_i) = [a_i, z_i] \in \text{Spin}^C(5) = (\text{Spin}(5) \times S^1)/\{1, -1\}\). Let \(t_i = (I, (0, ..., 0, 1, 0, ..., 0), i = 1, 2, 3, 4, 5, 8)\).

Since \(\epsilon\) is a homomorphism
\[
\forall 1 \leq i \leq 5, \quad \epsilon((\beta_i \beta_{i+2})^2) = [a_i a_{i+2} a_i a_{i+2}, z_i^2 z_{i+2}^2] = [-1, z_i^2 z_{i+2}^2] .
\]
Moreover, by easy computation
\[
\forall 1 \leq i \leq 5, \quad (\beta_i)^2 = t_i, (\beta_i \beta_{i+2})^2 = t_{i+1} t_{i+3}^{-1} \quad \text{and} \quad \epsilon(t_i) = [\pm 1, z_i^2] .
\]
From (7), (8) and (11)
\[
[-1, z_1^2 z_2^2] = [1, z_3^2 z_4^2] = [-1, z_3^2 z_4^2] = [1, z_2^2 z_3^2] = [-1, z_2^2 z_3^2] = [1, z_1^2 z_2^2] ,
\]
which is impossible. Here indexes we read modulo 5. This finishes the proof.

\[
\square
\]

**Definition 2.** The HW-manifold \(M^n\) of dimension \(n\), is cyclic if and only if \(\pi_1(M^n)\) is generated by the following elements (see [18, Lemma 1]):
\[
\beta_i = (B_i, (0, 0, 0, ..., 0, 1/2, 1/2, 0, ..., 0)), 1 \leq i \leq n - 1, \\
\beta_n = (\beta_1 \beta_2 \ldots \beta_{n-1})^{-1} = (B_n, (1/2, 0, ..., 0, -1/2)).
\]
We have.

**Theorem 2.** Cyclic HW-manifolds have not the Spin\(^C\)-structure.

**Proof:** Since the above group \(\Gamma_2\) satisfies our assumption the proof is generalization of arguments from the Proposition 2. Let \(\Gamma\) be a fundamental group of the cyclic HW-manifold of dimension \(\geq 5\), with set of generators \(\beta_i = (B_i, b_i), i = 1, 2, ..., n\). Since \(H^2(\Gamma, \mathbb{R}) = 0, ([4])\) we can apply a condition from Theorem 1. Let us assume that there exist a homomorphism \(\epsilon : \Gamma \rightarrow \text{Spin}^C(n)\), which defines the Spin\(^C\)-structure and
\[
\epsilon(\beta_i) = [a_i, z_i], a_i \in \text{Spin}(n), z_i \in S^1.
\]
From [14] the maximal abelian subgroup \(\mathbb{Z}^n\) of \(\Gamma\) is exactly the commutator subgroup \(\Gamma\). Hence \(\epsilon([\Gamma, \Gamma]) \subset \{i \text{ Spin}(n)\} \subset \text{Spin}^C(n)\). Since \(\forall i, \epsilon((\beta_i)^2) = [a_i^2, z_i^2] \quad \text{and} \quad (\beta_i)^2 \in [\Gamma, \Gamma], \quad z_i^2 = \pm 1, \quad \text{for} \quad i = 1, 2, ..., n\). It follows that
\[
\forall i, z_i \in \{\pm 1, \pm i\} .
\]
Let \(t_i = (I, (0, ..., 0, 1, 0, ..., 0), i = 1, 2, ..., n\). From (10)
\[
\epsilon(t_i) = \epsilon((\beta_i)^2) = [\pm 1, z_i^2], i = 1, 2, ..., n
\]
and also
\[(13)\] 
\[\forall 1 \leq i \leq n \quad \epsilon \left( (\beta_i \beta_{i+2})^2 \right) = [-1, z_i^2 z_{i+2}^2].\]

Moreover
\[(14)\] 
\[\forall 1 \leq i \leq n \quad (\beta_i \beta_{i+2})^2 = t_{i+1} t_{i+3}^{-1}.\]

From equations (12), (13) and (14) we have
\[\left[ -1, z_i^2 z_{i+2}^2 \right] = \left[ 1, z_{i+1}^2 z_{i+3}^2 \right]\]

and
\[\forall 1 \leq i \leq n \quad z_i^2 z_{i+2}^2 = -z_{i+1}^2 z_{i+3}^2 = z_{i+2}^2 z_{i+4}^2.\]

Since \( n \) is odd \( z_i^2 z_{i+2}^2 = -z_{i+1}^2 z_{i+3}^2 = -z_{i+2}^2 z_{i+4}^2 \), contradiction, (cf. (9)) \(^4\). This finishes the proof.

\[\square\]

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**References**


\(^4\)The indexes should be read modulo \( n \).