Flat manifolds with only finitely many affinities

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1 Introduction

Let $X$ denote a compact, connected, flat Riemannian manifold (flat manifold for short) of dimension $n$ with fundamental group $\Gamma$. Then $\Gamma$ is a Bieberbach group of rank $n$, i.e., $\Gamma$ is torsion free and there is a short exact sequence of groups

$$0 \to L \to \Gamma \to G \to 1,$$

where $G$ is finite, the so-called holonomy group of $\Gamma$, and $L$ is a free abelian group of rank $n$. Moreover, $L$ is a maximal abelian subgroup of $\Gamma$.

It is known that $X$ is determined by $\Gamma$ up to affine equivalence [2]. The set $\text{Aff}(X)$ of affine self equivalences of $X$ is a Lie group. Let $\text{Aff}_0(X)$ denote its identity component. Then $\text{Aff}_0(X)$ is a torus whose dimension equals the first Betti number of $X$, and $\text{Aff}(X)/\text{Aff}_0(X)$ is isomorphic to $\text{Out}(\Gamma)$, the group of outer automorphisms of $\Gamma$ [2, Chapter V]. In this note we investigate some flat manifolds $X$ for which $\text{Aff}(X)$ is finite. It is natural to ask which finite groups can occur as $\text{Aff}(X)$ for some flat manifold $X$. In particular, is there a flat manifold whose group of affinities is trivial? Related questions for a larger class of manifolds have been investigated by Malfait in [9].

By the remarks above, $\text{Aff}(X)$ is finite, if and only if $\text{Out}(\Gamma)$ is finite and the first Betti number of $X$ is zero. If some (non-trivial) cyclic Sylow subgroup of $G$ has a normal complement, then by [4, Theorem 0.1], the first Betti number of $X$ is non-zero, in which case $\text{Aff}(X)$ is infinite. In Section 2

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we show that \( \text{Aff}(X) \) is non-trivial if \( G \) is \( p \)-nilpotent, i.e., has a normal \( p \)-complement, for some prime \( p \) dividing the order of \( G \).

In [6] we gave the following criterion, due independently also to Brown, Neubüser, and Zassenhaus, for a Bieberbach group to have a finite outer automorphism group. The conjugation action of \( \Gamma \) gives \( L \) the structure of a \( \mathbb{Z}G \)-lattice (a \( \mathbb{Z}G \)-module which is free and finitely generated as abelian group), the so-called translation lattice of \( \Gamma \). Since \( \Gamma \) is torsion free and \( L \) is maximal abelian, \( G \) is faithfully represented on \( L \) (see [2, III.1]). In other words, there is an injective group homomorphism of \( G \) into \( \text{GL}(L) \). Put \( L^Q := \mathbb{Q} \otimes_{\mathbb{Z}} L \). Then \( \text{Out}(\Gamma) \) is finite if and only if \( L^Q \) is multiplicity free as a \( \mathbb{Q}G \)-module, and \( R \otimes_{\mathbb{Q}} V \) is irreducible for every irreducible constituent \( V \) of \( L^Q \) [6]. The first Betti number of \( X \) equals the number of trivial constituents of \( L^Q \) (see, e.g., [4, Proposition 1.4]).

It follows from [6, Lemma 2.3] that if \( X_1 \) and \( X_2 \) are flat manifolds with \( \text{Aff}(X_i) \) finite, \( i = 1, 2 \), then the group of affinities of the product manifold \( X_1 \times X_2 \) is also finite. In particular, taking the \( m \)-fold product of such a flat manifold with itself, we obtain a flat manifold with a finite group of affinities containing the symmetric group on \( m \) letters as a subgroup.

To give a more explicit description of \( \text{Out}(\Gamma) \) let \( N \) denote the normalizer in \( \text{GL}(L) \) of \( G \) (viewed as a subgroup of \( \text{GL}(L) \) via the monomorphism discussed above). Then \( N \) acts in a natural way on \( H^2(G, L) \). Let \( \alpha \in H^2(G, L) \) denote the cohomology class giving rise to the extension (1), and let \( N_\alpha \) denote its stabilizer in \( N \). Then \( G \) is a normal subgroup of \( N_\alpha \) and we have a short exact sequence (see [2, Theorem V.1.1])

\[
0 \to H^1(G, L) \to \text{Out}(\Gamma) \to N_\alpha/G \to 1. \tag{2}
\]

In order to construct a flat manifold \( X \) with trivial group of affinities, one must find a Bieberbach group \( \Gamma \) with \( H^1(G, L) = 0 \) and \( N_\alpha = G \). In Section 2 we shall construct one example where \( H^1(G, L) = 0 \) and \( N_\alpha/G \) is a group with two elements, and another example where \( H^1(G, L) \) has two elements and \( N_\alpha/G \) is trivial. So far we have not been able to construct a flat manifold \( X \) with a trivial group of affinities.

In Section 3 of our paper we compute the group of affinities of the generalized Hantzsche-Wendt manifolds introduced in [10, 13], thus giving, for the first time, examples of \( \text{Aff}(X) \) for flat manifolds \( X \) with non-cyclic holonomy in any dimension.
Finally, in Section 4, we use the generalized Hantzsche-Wendt manifolds to construct a family of flat manifolds \( X_n \) of dimension \( 2n + 1 \), whose holonomy groups are certain extraspecial 2-groups of order \( 2^{2n+1} \) and with \( \text{Aff}(X_n) \) finite for all \( n \).

We close the introduction with a few words on notation. If \( M \) is a finite set, \( |M| \) denotes the number of its elements. If \( G \) is a group acting on a set \( M \), we write \( M^G \) for the set of \( G \)-fixed points of \( M \). Finally, \( C_n \) denotes the cyclic group of order \( n \), and \( (C_n)^m \) is the direct product of \( m \) copies of \( C_n \).

## 2 Flat manifolds with few symmetries

Let \( G \) be a finite group and \( M \) a \( \mathbb{Z}G \)-lattice of finite rank. We start with the following observation.

**Lemma 2.1** Suppose that \( H^0(G, M) = 0 \). Let \( p \) be a prime. Then \( p \) divides \( |H^1(G, M)| \) if and only if the \( \mathbb{F}_pG \)-module \( M/pM \) has a trivial submodule.

In particular \( H^1(G, M) = 0 \), if and only if \( M/qM \) has no trivial submodule for all primes \( q \) dividing \( |G| \).

**Proof:** The hypothesis \( H^0(G, M) = 0 \) yields

\[
H^1(G, M) \cong H^0(G, \mathbb{Q} \otimes \mathbb{Z} M/M) \cong (\mathbb{Q} \otimes \mathbb{Z} M/M)^G.
\]

Suppose that \( p \) divides \( |H^1(G, M)| \). Then there exists \( 0 \neq x \in (\mathbb{Q} \otimes \mathbb{Z} M/M)^G \) such that \( px = 0 \). We thus have \( x = x_0 + M, x_0 \in \mathbb{Q} \otimes \mathbb{Z} M \) and \( px_0 \in M \). Then \( x_0 \in \frac{1}{p} M \subset \mathbb{Q} \otimes \mathbb{Z} M \) and \( x_0 \not\in M \). Hence \( M/pM \) has a trivial submodule. The other direction is proved similarly.

This lemma has some interesting consequences.

**Proposition 2.2** Suppose that \( G \) is \( p \)-nilpotent, i.e., has a normal \( p \)-complement, for some prime \( p \) dividing \( |G| \). If \( H^0(G, M) = 0 \) and \( p \) divides \( |H^2(G, M)| \), then \( p \) also divides \( |H^1(G, M)| \).

**Proof:** We have \( H^2(G, \mathbb{Z}_p \otimes \mathbb{Z} M) \neq 0 \), since \( p \) divides \( |H^2(G, M)| \). Let \( U \) be an indecomposable direct summand of \( \mathbb{Z}_p \otimes \mathbb{Z} M \) which lies in the principal \( p \)-block of \( G \). Then every indecomposable direct summand of \( U/pU \) lies in the principal \( p \)-block of \( G \). Since \( G \) is \( p \)-nilpotent, \( U/pU \) contains a non-trivial vector fixed by \( G \) (see, e.g., [3, § 63A]). The result follows from the lemma.
Corollary 2.3 Let $X$ be a flat manifold whose holonomy group $G$ is $p$-nilpotent for some prime $p$ dividing $|G|$. Then $\text{Aff}(X)$ is non-trivial.

Proof: Let $M$ denote the translation lattice of the fundamental group of $X$. If $H^0(G, M) \neq 0$, then $\text{Aff}(X)$ contains a torus, hence is infinite. If $H^0(G, M) = 0$, then $H^1(G, M) \neq 0$ by the above proposition ($p$ divides the order of $H^2(G, M)$, since this cohomology group contains a special element). As $H^1(G, M)$ is isomorphic to a subgroup of $\text{Aff}(X)$, the result follows.

It follows in particular that flat manifolds with nilpotent holonomy have a non-trivial group of affinities. A somewhat weaker version of Corollary 2.3 has also been obtained by Malfait [9, Proposition 5.9]. Moreover, this author conjectures that the assumption on the holonomy group is in fact unnecessary, in other words that $\text{Aff}(X)$ is non-trivial for every flat manifold $X$ ([9, Conjecture 5.12]).

Example 2.4 Let $G = \text{SL}_3(2)$. Then $G$ has a presentation $G = \langle a, b | a^2, b^3, (ab)^7, [b, a]^4 \rangle$ (see [7, p. 290]). We use the notation of [7, Chapter 6, Section 11]. In particular, we consider right $\mathbb{Z}G$-modules to make it easier to match our results with the tables in [7].

Let

$$L := L_3^6 \oplus L_8^7 \oplus L_4^8,$$

where $L_m$ is the lattice of dimension $m$ (and, in case $m = 6$, with character $\chi_6$), which is denoted by $L_i$ in [7]. It can easily be checked with Lemma 2.1 that $H^1(G, L) = 0$.

We identify $G$ with its image in $\text{GL}(L) \cong \text{GL}_{21}(\mathbb{Z})$. Since $L_8^7$ is not invariant under the outer automorphism of $G$, it follows that $N := N_{\text{GL}(L)}(G) = G C_{\text{GL}(L)}(G)$. We have $C_{\text{GL}(L)}(G) \cong (C_2)^3$.

Let $\alpha \in H^2(G, L_3^6)$ be defined by the 1-cocycle $\delta \in Z^1(G, \mathbb{Q} \otimes \mathbb{Z} L_3^6/L_3^5)$ determined by

$$\delta(a) = \frac{1}{2}[0, 1, 0, 0, 0, 0], \quad \delta(b) = 0.$$

Then $\alpha$ has order 2.

Let $\beta \in H^2(G, L_8^7)$ be defined by the 1-cocycle $\delta \in Z^1(G, \mathbb{Q} \otimes \mathbb{Z} L_8^7/L_8^6)$ determined by

$$\delta(a) = \frac{1}{3}[2, 0, 0, 0, 0, 0, 0], \quad \delta(b) = \frac{1}{3}[0, 0, 0, 0, 0, 0, 1].$$
Then $\beta$ has order 3.

Finally, let $\gamma \in H^2(G, L^8_4)$ be defined by the 1-cocycle $\delta \in Z^1(G, \mathbb{Q} \otimes \mathbb{Z} L^8_4/L^8_4)$ determined by
\[
\delta(a) = \frac{1}{7}[0, 2, 0, 5, 0, 0, 0, 5], \quad \delta(b) = \frac{1}{7}[6, 0, 0, 0, 0, 0, 0, 0].
\]
Then $\gamma$ has order 7.

Put $\sigma := \alpha + \beta + \gamma \in H^2(G, L)$. Then $\sigma$ is a special cocycle since $\text{res}_{G \langle a \rangle}^G(\alpha) \neq 0$, $\text{res}_{G \langle b \rangle}^G(\beta) \neq 0$ and $\text{res}_{G \langle ab \rangle}^G(\gamma) \neq 0$. It is clear that $N_\sigma = GC$, where $C = \langle -\text{id}_{L^8_4} + \text{id}_{L^7_4} + \text{id}_{L^5_4} \rangle \leq C_{GL(L)}(G)$.

Let $\Gamma$ denote the Bieberbach group obtained by extending $L$ by $G$ with the cocycle $\sigma$, and let $X$ be the flat manifold with fundamental group $\Gamma$. Then $\text{Aff}(X) \cong C$ is a group of order 2.

**Example 2.5** Let $\tilde{G} = \text{SL}_3(2),2$, the automorphism group of $G$. Then $\tilde{G}$ has a presentation $\tilde{G} = \langle a, b, c | a^2, b^3, c^2, (ab)^7, [b, a]^4, [a, c], (acb^2)^2 \rangle$.

Let $\tilde{L} := \tilde{L}^7_4 \oplus \tilde{L}^7_{10} \oplus \tilde{L}^8_4$, where $\tilde{L}^m_i$ is an extension to $\tilde{G}$ of the $\mathbb{Z}G$-lattice $L^m_i$ (with the notation of Example 2.4). More precisely, the trace of $c$ on these lattices equals $-1, 1, -2$, in the respective cases.

We find $H^1(\tilde{G}, \tilde{L}^7_4) = C_2$ and $H^1(\tilde{G}, \tilde{L}^7_{10}) = 0 = H^1(\tilde{G}, \tilde{L}^8_4)$. Thus $H^1(\tilde{G}, \tilde{L}) = C_2$.

Let $\alpha \in H^2(\tilde{G}, L^7_4)$ be defined by the 1-cocycle $\delta \in Z^1(\tilde{G}, \mathbb{Q} \otimes \mathbb{Z} L^7_4/L^7_4)$ determined by
\[
\delta(a) = \frac{1}{6}[3, 5, 3, 3, 3, 0, 1],
\delta(b) = \frac{1}{6}[0, 4, 2, 2, 0, 0, 0],
\delta(c) = \frac{1}{6}[3, 3, 0, 3, 0, 0, 0].
\]
Then $\alpha$ has order 6.

Let $\beta \in H^2(\tilde{G}, L^7_{10})$ be defined by the 1-cocycle $\delta \in Z^1(\tilde{G}, \mathbb{Q} \otimes \mathbb{Z} L^7_{10}/L^7_{10})$ determined by
\[
\delta(a) = \frac{1}{12}[6, 0, 6, 6, 0, 6, 0, 6],
\]

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\[ \delta(b) = \frac{1}{12}[0,8,0,0,0,1,1], \]
\[ \delta(c) = \frac{1}{12}[2,8,8,2,6,9,9]. \]

Then \( \beta \) has order 12.

Let \( \gamma \in H^2(\tilde{G}, L_8^\Lambda) \) be defined by the 1-cocycle \( \delta \in Z^1(\tilde{G}, Q \otimes_\mathbb{Z} L_8^\Lambda/L_8^\Lambda) \) determined by
\[ \begin{align*}
\delta(a) &= \frac{1}{7}[0,3,0,4,0,0,5,2], \\
\delta(b) &= \frac{1}{7}[0,5,0,0,5,0,0,0], \\
\delta(c) &= \frac{1}{7}[0,0,0,0,3,3,0,0].
\end{align*} \]

Then \( \gamma \) has order 7.

Let \( \sigma := \alpha + \beta + \gamma \). Then \( \sigma \) is special in \( H^2(\tilde{G}, \tilde{L}) \), since \( \text{res}_{\Gamma_2}^\tilde{G}(\alpha) \neq 0 \), \( \text{res}_{\Gamma_2}^\tilde{G}(\beta) \neq 0 \) and \( \text{res}_{\Gamma_2}^\tilde{G}(\gamma) \neq 0 \).

Since \( \text{Aut}(\tilde{G}) = \tilde{G} \), we have \( N_{\text{GL}(\tilde{L})}(\tilde{G}) = \tilde{G} \text{C}_{\text{GL}(\tilde{L})}(\tilde{G}) \). No non-trivial element of \( \text{C}_{\text{GL}(\tilde{L})}(\tilde{G}) \cong (C_2)^3 \) fixes \( \sigma \). Hence \( N_{\sigma} = \tilde{G} \).

Let \( \tilde{\Gamma} \) denote the extension of \( \tilde{L} \) by \( \tilde{G} \) determined by \( \sigma \), and let \( \tilde{X} \) be the flat manifold with fundamental group \( \tilde{\Gamma} \). Then \( \text{Aff}(\tilde{X}) \) is a group of order 2.

The computations in these examples have been performed with Maple [1] and GAP [12].

### 3 Generalized Hantzsche-Wendt manifolds

In this section we shall calculate the group \( \text{Out}(\Gamma_{2n}) \) where \( \Gamma_{2n} \) is the fundamental group of the generalized Hantzsche-Wendt manifold of dimension \( 2n + 1 \) introduced in [10, 13].

Let us recall the definition of \( \Gamma_{2n} \), which we shall call Hantzsche-Wendt group for short. We denote by \( a_i, 1 \leq i \leq 2n + 1 \), the \((2n+1) \times (2n+1)-\)diagonal matrices over \( \mathbb{Z} \) with diagonal entries 1 on position \( i \) and \(-1\) on the other positions (see [13, p. 292]). Let \( A \) be the subgroup of \( \text{GL}_{2n+1}(\mathbb{Z}) \) generated by \( a_i, i = 1, 2, \ldots, 2n \). Let \( L = \mathbb{Z}^{2n+1} \) with standard (column) basis
vectors \( u_1, u_2, \ldots, u_{2n+1} \). We view \( L \) as a left \( \mathbb{Z}A \)-lattice. The Hantzsche-Wendt group \( \Gamma_{2n} \) is an extension of \( L \) by \( A \). This extension is given by the element

\[
[s] \in H^1(A, Q \otimes \mathbb{Z} L/L) \cong H^2(A, L),
\]

represented by the 1-cocycle \( s \in Z^1(A, Q \otimes \mathbb{Z} L/L) \) with

\[
s(a_i) = \frac{1}{2}(u_i + u_{i+1}) + L, \quad 1 \leq i \leq 2n
\]

(see [13, p. 294]). We remark that Maxwell has also defined a Bieberbach group with holonomy group \( A \) and \( \mathbb{Z}A \)-lattice \( L \) in Part (e) of the proof of [10, Proposition 6]. It is not difficult to see that Maxwell’s cocycle is cohomologous to the one in (3).

We shall use (2) to calculate \( \text{Out}(\Gamma_{2n}) \). First we must find the normalizer \( N = N_{GL_{2n+1}(\mathbb{Z})}(A) \) of the holonomy group \( A \cong (C_2)^{2n} \) in \( GL_{2n+1}(\mathbb{Z}) \). By immediate calculations one can prove the following lemma.

**Lemma 3.1** Let \( S_{2n+1} \leq GL_{2n+1}(\mathbb{Z}) \) denote the group of permutation matrices, and let \( \hat{A} = \langle -I_{2n+1}, A \rangle \), where \( I_{2n+1} \) denotes the identity matrix. Then \( N = \hat{A}S_{2n+1} \). In particular, \( N \cong C_2 \wr S_{2n+1} \), the wreath product of \( C_2 \) and \( S_{2n+1} \).

**Example 3.2** If \( n = 1 \), then the cohomology class defined by (3) is the only (up to conjugation by \( N \)) special element of \( H^2(A, L) \).

However, for \( n = 2 \), there are exactly two \( N \)-orbits of special elements in \( H^2(A, L) \). Let \( t : A \to Q \otimes \mathbb{Z} L/L \) be the 1-cocycle defined by

\[
\begin{align*}
t(a_1) &= \frac{1}{2}(u_1 + u_3) + L, \\
t(a_2) &= \frac{1}{2}u_2 + L, \\
t(a_3) &= \frac{1}{2}(u_3 + u_4) + L, \\
t(a_4) &= \frac{1}{2}(u_4 + u_5) + L,
\end{align*}
\]

and let \([t]\) denote the corresponding cohomology class in \( H^2(A, L) \). Then it is easily checked that \([t]\) is special. Moreover, the stabilizer of \([t]\) in \( S_5 \) is trivial.
We shall show below that this is not the case for \([s]\), so that \([s]\) and \([t]\) are not in the same \(N\)-orbit. Finally, it is not hard to prove that there are no other \(N\)-orbits containing special cocycles.

We expect that the number of \(N\)-orbits of special cocycles in \(H^2(A, L)\) grows as \(n\) grows. We have not attempted to classify these special orbits.

Let us recall the definition of the action \(*\) of \(N\) on \(H^1(A, Q \otimes_{\mathbb{Z}} L/L)\). Let \(n \in N\) and let \(c : A \to Q \otimes_{\mathbb{Z}} L/L\) represent a cohomology class from \(H^1(A, Q \otimes_{\mathbb{Z}} L/L)\). Then

\[
(n \ast c)(x) = nc(n^{-1}xn)
\]

for \(x \in A\). We shall prove:

**Lemma 3.3** For \(n \geq 2\), \(N[s] := \{n \in N \mid n \ast [s] = [s]\} = \tilde{A}F\), where \(F\) is the cyclic subgroup of \(S_{2n+1}\) generated by the \((2n+1)\)-cycle \((1, 2, \ldots, 2n, 2n+1)\). For \(n = 1\), \(N[s] = N\).

**Proof:** It is obvious that \(n \ast s = s\) for \(n \in \tilde{A}\). Let \(\sigma \in S_{2n+1}\). From \(\sigma^{-1}a_i\sigma = a_{\sigma^{-1}(i)}\) we obtain

\[
\sigma \ast s(a_i) = \sigma(s(a_{\sigma^{-1}(i)})) = \frac{1}{2}(u_i + u_{\sigma(\sigma^{-1}(i) + 1)}).
\]  

(4)

(Here, and in the remainder of the proof, the subscripts have to be read modulo \(2n+1\).) Note that \(\sigma \in F\) if and only if \(\sigma(i) + 1 = \sigma(i + 1)\) for all \(1 \leq i \leq 2n + 1\). Thus (4) shows that \(F \leq N[s]\).

We shall prove that \(\sigma \notin N[s]\) for \(\sigma \notin F\) and \(n \geq 2\). Let \(\sigma \notin F\) and suppose that \(\sigma \in N[s]\). Then there is an element \(v = \sum_{i=1}^{2n+1} v_iu_i \in Q \otimes_{\mathbb{Z}} L/L \cong (Q/\mathbb{Z})^{2n+1}\) such that

\[
(\sigma \ast s-s)(a_i) = \frac{1}{2}(u_{\sigma(\sigma^{-1}(i) + 1)} - u_{i+1}) = (a_i - I_{2n+1})v, \quad 1 \leq i \leq 2n+1. \tag{5}
\]

Since \(\sigma^{-1} \notin F\), there is an \(i\) such that \(\sigma(\sigma^{-1}(i) + 1) \neq i + 1\). Therefore, \((\sigma \ast s-s)(a_i) \neq 0\) and \(v_{i+1} = 1/4 \in Q/\mathbb{Z}\). Let \(1 \leq j \leq 2n + 1\), \(j \neq i + 1\). Then the \((i+1)\)st component of \((a_j - I_{2n+1})v\) equals \(1/2 \in Q/\mathbb{Z}\). Equation (5) implies that \(\sigma(\sigma^{-1}(j) + 1) = i + 1\) or \(j + 1 = i + 1\). Thus \(2n + 1 \leq 3\), i.e., \(n \leq 1\). It is easy to see that \(N[s] = N\) if \(n = 1\).
From the definition and the properties of the holonomy representation we conclude that

\[ H^1(A, L) \simeq (C_2)^{2n+1}. \]

We describe the action of \( N_{s_l}/A = \langle -I_{2n+1} \rangle \times F \) on \( H^1(A, L) \) following [2, Example V.6.1]. \( F \) acts by permuting the direct factors, and \( \langle -I_{2n+1} \rangle \) acts trivially. We can now formulate the main result of this section. It follows from Lemmas 3.1 and 3.3. (For \( n = 1 \), see [5, pp. 128,129] and [15, pp. 321–323].)

**Theorem 3.4** Let \( M^{2n+1}, n \geq 2 \), be the generalized Hantzsche-Wendt flat manifold of dimension \( 2n + 1 \). Then \( \text{Aff}(M^{2n+1}) \) is a split extension of \( H^1(A, L) \) and \( \langle -I_{2n+1} \rangle \times F \), i.e., it is isomorphic to \( C_2 \times (C_2 \wr F) \). The subgroup of \( \text{Aff}(M^{2n+1}) \) which preserves orientation is isomorphic to \( C_2 \wr F \).

**Proof:** Lemma 3.3 and the sequence (2) show that \( \text{Aff}(M^{2n+1}) \) is an extension of \( H^1(A, L) \) by \( \langle -I_{2n+1} \rangle \times F \). Since \( |F| \) is odd, the extension \( H^1(A, L) \) by \( F \) splits, so \( \text{Aff}(M^{2n+1}) \) has a subgroup \( B \) of index 2 isomorphic to \( C_2 \wr F \).

We embed \( \Gamma_{2n} \) in the usual way into the group \( \mathcal{A}_{2n+1} \) of affine motions of \( \mathbb{R}^{2n+1} \). Then \( M^{2n+1} = \mathbb{R}^{2n+1}/\Gamma_{2n} \) and \( \text{Aff}(M^{2n+1}) \cong N_{\mathcal{A}_{2n+1}}(\Gamma_{2n})/\Gamma_{2n} \) (see [2, Lemma V.6.1]). It is clear that \( B \) can be represented by elements of \( \mathcal{A}_{2n+1} \) whose linear part has determinant 1. Thus \( -I_{2n+1} \), having determinant \( -1 \), gives rise to a non-trivial element of \( \text{Aff}(M^{2n+1}) \) not lying in \( B \). It also follows that \( B \) is equal to the subgroup of orientation preserving elements of \( \text{Aff}(M^{2n+1}) \).

### 4 Extra-special 2-groups

We continue with the notation of the previous section. Let \( T_{2n} \) be the sublattice of \( L \) generated by

\[ 2u_1, u_1 - u_2, u_2 - u_3, \ldots, u_{2n} - u_{2n+1}. \]

Then \( T_{2n} \) has index 2 in \( L \) and the group \( \mathcal{G}_{2n} := \Gamma_{2n}/T_{2n} \) is a finite group of order \( 2^{2n+1} \). It is proved in [11] that \( \mathcal{G}_{2n} \) is extraspecial (of type varying with \( n \)). We can use the construction of the generalized Hantzsche-Wendt manifolds to prove that the extraspecial groups \( \mathcal{G}_{2n} \) occur as holonomy groups of Bieberbach groups with finite outer automorphism groups. Such groups were called \( \mathcal{R}_1 \)-groups in [6].

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Proposition 4.1 For each \( n \geq 1 \), \( G_{2n} \) is an \( R_1 \)-group.

Proof: It is easy to give a list of all irreducible complex representations of the group \( G_{2n} \). There are \( 2^{2n} \) 1-dimensional representations arising from the commutator factor group and one faithful representation of dimension \( 2^n \) (see \[8, Problem 2.13\]). If \( G_{2n} \) is a central product of dihedral groups of order 8, the complex irreducible representation of degree \( 2^n \) can be realized over \( Q \). Otherwise, it has Frobenius-Schur indicator \(-1\), and thus can not be realized over \( R \). It follows that any \( Q \)-irreducible representation of \( G_{2n} \) is \( R \)-irreducible.

Let \( K \) be a faithful integral representation of \( G_{2n} \) which, as a rational representation, is multiplicity free and does not contain a 1-dimensional direct summand. Then the \( ZG_{2n} \)-module \( T_{2n} \oplus K \) is a translation lattice for a Bieberbach group with finite outer automorphism group and holonomy \( G_{2n} \).

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