Outer automorphism group of crystallographic groups with trivial center

Andrzej Szczepański
(joint work with R. Lutowski)
University of Gdańsk

University of Chicago, March 10, 2015
Let us denote by $E(n)$ the isometry group $Isom(\mathbb{R}^n) = O(n) \ltimes \mathbb{R}^n$ of the $n$-dimensional Euclidean space.

**Definition**

A crystallographic group of dimension $n$ is a cocompact and discrete subgroup of $E(n)$.

**Example**

1. $\mathbb{Z}^n$

2. If $(B, \begin{pmatrix} 1/2 \\ 0 \end{pmatrix}), (I, \begin{pmatrix} 0 \\ 1 \end{pmatrix}) \in E(2)$, where $B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, then the group $\Gamma \subset E(2)$ generated by the above elements is a crystallographic group of dimension 2.
Let us denote by $E(n)$ the isometry group $\text{Isom}(\mathbb{R}^n) = O(n) \ltimes \mathbb{R}^n$ of the $n$-dimensional Euclidean space.

**Definition**

A crystallographic group of dimension $n$ is a cocompact and discrete subgroup of $E(n)$.

**Example**

1. $\mathbb{Z}^n$

2. If $(B, \begin{pmatrix} 1/2 \\ 0 \end{pmatrix}), (I, \begin{pmatrix} 0 \\ 1 \end{pmatrix}) \in E(2)$, where $B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, then the group $\Gamma \subset E(2)$ generated by the above elements is a crystallographic group of dimension 2.
Crystallographic group

Let us denote by $E(n)$ the isometry group $\text{Isom}(\mathbb{R}^n) = O(n) \ltimes \mathbb{R}^n$ of the $n$-dimensional Euclidean space.

**Definition**

A crystallographic group of dimension $n$ is a cocompact and discrete subgroup of $E(n)$.

**Example**

1. $\mathbb{Z}^n$

2. If $(B, \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}), (I, \begin{pmatrix} 0 \\ 1 \end{pmatrix}) \in E(2)$, where $B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, then the group $\Gamma \subset E(2)$ generated by the above elements is a crystallographic group of dimension 2.
The first part of the eighteenth Hilbert problem was about the description of discrete and cocompact groups of isometries of $\mathbb{R}^n$. The answer for the above Hilbert problem was given by the German mathematician L. Bieberbach in 1913. 

**Theorem**

(Bieberbach) 1. If $\Gamma \subset E(n)$ is a crystallographic group then the set of translations $\Gamma \cap (I \times \mathbb{R}^n)$ is a torsion free and finitely generated abelian group of rank $n$, and is a maximal abelian and normal subgroup of finite index.

2. For any natural number $n$, there are only a finite number of isomorphism classes of crystallographic groups of dimension $n$.

3. Two crystallographic groups of dimension $n$ are isomorphic if and only if they are conjugate in the group $A(n) = GL(n, \mathbb{R}) \rtimes \mathbb{R}^n$. 

Andrzej Szczepański (joint work with R. Lutowski) University of Gdańsk
The first part of the eighteenth Hilbert problem was about the description of discrete and cocompact groups of isometries of $\mathbb{R}^n$. The answer for the above Hilbert problem was given by the German mathematician L. Bieberbach in 1913.

**Theorem**

(Bieberbach) 1. If $\Gamma \subset E(n)$ is a crystallographic group then the set of translations $\Gamma \cap (I \times \mathbb{R}^n)$ is a torsion free and finitely generated abelian group of rank $n$, and is a maximal abelian and normal subgroup of finite index.

2. For any natural number $n$, there are only a finite number of isomorphism classes of crystallographic groups of dimension $n$.

3. Two crystallographic groups of dimension $n$ are isomorphic if and only if they are conjugate in the group $A(n) = GL(n, \mathbb{R}) \rtimes \mathbb{R}^n$. 
The first part of the eighteenth Hilbert problem was about the description of discrete and cocompact groups of isometries of $\mathbb{R}^n$. The answer for the above Hilbert problem was given by the German mathematician L. Bieberbach in 1913.

Theorem

(Bieberbach) 1. If $\Gamma \subset E(n)$ is a crystallographic group then the set of translations $\Gamma \cap (I \times \mathbb{R}^n)$ is a torsion free and finitely generated abelian group of rank $n$, and is a maximal abelian and normal subgroup of finite index.

2. For any natural number $n$, there are only a finite number of isomorphism classes of crystallographic groups of dimension $n$.

3. Two crystallographic groups of dimension $n$ are isomorphic if and only if they are conjugate in the group $A(n) = GL(n, \mathbb{R}) \rtimes \mathbb{R}^n$. 

Andrzej Szczepański (joint work with R. Lutowski) University of Gdańsk

Outer automorphism group of crystallographic groups
The first part of the eighteenth Hilbert problem was about the description of discrete and cocompact groups of isometries of $\mathbb{R}^n$. The answer for the above Hilbert problem was given by the German mathematician L. Bieberbach in 1913.

**Theorem**

1. If $\Gamma \subset E(n)$ is a crystallographic group then the set of translations $\Gamma \cap (I \times \mathbb{R}^n)$ is a torsion free and finitely generated abelian group of rank $n$, and is a maximal abelian and normal subgroup of finite index.

2. For any natural number $n$, there are only a finite number of isomorphism classes of crystallographic groups of dimension $n$.

3. Two crystallographic groups of dimension $n$ are isomorphic if and only if they are conjugate in the group $A(n) = \text{GL}(n, \mathbb{R}) \rtimes \mathbb{R}^n$. 
The first part of the eighteenth Hilbert problem was about the description of discrete and cocompact groups of isometries of $\mathbb{R}^n$. The answer for the above Hilbert problem was given by the German mathematician L. Bieberbach in 1913.

**Theorem**

(Bieberbach) 1. If $\Gamma \subset E(n)$ is a crystallographic group then the set of translations $\Gamma \cap (I \times \mathbb{R}^n)$ is a torsion free and finitely generated abelian group of rank $n$, and is a maximal abelian and normal subgroup of finite index.

2. For any natural number $n$, there are only a finite number of isomorphism classes of crystallographic groups of dimension $n$.

3. Two crystallographic groups of dimension $n$ are isomorphic if and only if they are conjugate in the group $A(n) = GL(n, \mathbb{R}) \rtimes \mathbb{R}^n$. 
**Definition**

A flat manifold $M^n$ of dimension $n$ is a compact connected Riemannian manifold without boundary with sectional curvature equal to zero.

**Example**

1. torus $\mathbb{R}^n/\mathbb{Z}^n \cong S^1 \times S^1 \times \cdots \times S^1$

2. $\mathbb{R}^n/\Gamma$, where $\Gamma \subset E(n)$ is a torsion free crystallographic group

**Remark**

Any flat manifold $M^n \cong \mathbb{R}^n/\Gamma$, where $\Gamma = \pi_1(M^n)$. $\Gamma$ is a torsion free crystallographic group of rank $n$. 

Andrzej Szczepański (joint work with R. Lutowski) University of Gdańsk 

Outer automorphism group of crystallographic groups
A flat manifold $M^n$ of dimension $n$ is a compact connected Riemannian manifold without boundary with sectional curvature equal to zero.

Example

1. torus $\mathbb{R}^n / \mathbb{Z}^n \cong S^1 \times S^1 \times \cdots \times S^1$

2. $\mathbb{R}^n / \Gamma$, where $\Gamma \subset E(n)$ is a torsion free crystallographic group

Remark

Any flat manifold $M^n \cong \mathbb{R}^n / \Gamma$, where $\Gamma = \pi_1(M^n)$. $\Gamma$ is a torsion free crystallographic group of rank $n$. 

Outer automorphism group of crystallographic groups
**Definition**

A flat manifold $M^n$ of dimension $n$ is a compact connected Riemannian manifold without boundary with sectional curvature equal to zero.

**Example**

1. **torus** $\mathbb{R}^n/\mathbb{Z}^n \simeq S^1 \times S^1 \times \cdots \times S^1$

2. $\mathbb{R}^n/\Gamma$, where $\Gamma \subset E(n)$ is a torsion free crystallographic group

**Remark**

Any flat manifold $M^n \simeq \mathbb{R}^n/\Gamma$, where $\Gamma = \pi_1(M^n)$. $\Gamma$ is a torsion free crystallographic group of rank $n$. 
A flat manifold $M^n$ of dimension $n$ is a compact connected Riemannian manifold without boundary with sectional curvature equal to zero.

Example

1. torus $\mathbb{R}^n/\mathbb{Z}^n \simeq S^1 \times S^1 \times \cdots \times S^1$

2. $\mathbb{R}^n/\Gamma$, where $\Gamma \subset E(n)$ is a torsion free crystallographic group

Remark

Any flat manifold $M^n \simeq \mathbb{R}^n/\Gamma$, where $\Gamma = \pi_1(M^n)$. $\Gamma$ is a torsion free crystallographic group of rank $n$. 
From the theorems of Bieberbach the fundamental group \( \pi_1(M^n) = \Gamma \) (Bieberbach group) determines a short exact sequence

\[
0 \rightarrow \mathbb{Z}^n \rightarrow \Gamma \xrightarrow{p} G \rightarrow 0,
\]

where \( \mathbb{Z}^n \) is a torsion free abelian group of rank \( n \) and \( G \) is a finite group with is isomorphic to the holonomy group of \( M^n \).

**Corollary**

Any flat manifold \( M^n \cong \mathbb{R}^n/\Gamma \cong \mathbb{R}^n/\mathbb{Z}^n/\Gamma/\mathbb{Z}^n \cong T^n/G. \)
From the theorems of Bieberbach the fundamental group
\( \pi_1(M^n) = \Gamma \) (Bieberbach group) determines a short exact sequence

\[
0 \rightarrow \mathbb{Z}^n \rightarrow \Gamma \overset{p}{\rightarrow} G \rightarrow 0,
\]

where \( \mathbb{Z}^n \) is a torsion free abelian group of rank \( n \) and \( G \) is a finite
group with is isomorphic to the holonomy group of \( M^n \).

**Collorary**

Any flat manifold \( M^n \cong \mathbb{R}^n/\Gamma \cong \mathbb{R}^n/\mathbb{Z}^n/\Gamma/\mathbb{Z}^n \cong T^n/G \).
From the theorems of Bieberbach the fundamental group 
\( \pi_1(\mathcal{M}^n) = \Gamma \) (Bieberbach group) determines a short exact sequence

\[
0 \rightarrow \mathbb{Z}^n \rightarrow \Gamma \xrightarrow{p} G \rightarrow 0,
\]

where \( \mathbb{Z}^n \) is a torsion free abelian group of rank \( n \) and \( G \) is a finite group with is isomorphic to the holonomy group of \( \mathcal{M}^n \).

**Corollary**

Any flat manifold \( \mathcal{M}^n \cong \mathbb{R}^n/\Gamma \cong \mathbb{R}^n/\mathbb{Z}^n/\Gamma/\mathbb{Z}^n \cong T^n/G \).
From the theorems of Bieberbach the fundamental group \( \pi_1(M^n) = \Gamma \) (Bieberbach group) determines a short exact sequence

\[
0 \rightarrow \mathbb{Z}^n \rightarrow \Gamma \overset{p}{\rightarrow} G \rightarrow 0,
\]

where \( \mathbb{Z}^n \) is a torsion free abelian group of rank \( n \) and \( G \) is a finite group with is isomorphic to the holonomy group of \( M^n \).

**Corollary**

*Any flat manifold* \( M^n \cong \mathbb{R}^n/\Gamma \cong \mathbb{R}^n/\mathbb{Z}^n/\Gamma/\mathbb{Z}^n \cong T^n/G. \)
Let $\Gamma$ be a crystallographic group. We have

\[ 0 \rightarrow \mathbb{Z}^n \rightarrow \Gamma \xrightarrow{p} G \rightarrow 0. \]

Let $h_\Gamma : G \rightarrow GL(n, \mathbb{Z})$ be the integral holonomy representation defined by the formula

\[ \forall g \in G \quad h_\Gamma(g)(e) = \bar{g} e \bar{g}^{-1}, \]

where $\bar{g} \in \Gamma$, $p(\bar{g}) = g$ and $e \in \mathbb{Z}^n$. Since $\mathbb{Z}^n$ is a maximal abelian subgroup, $h_\Gamma$ is a faithfull representation.
Let $\Gamma$ be a crystallographic group. We have

$$0 \to \mathbb{Z}^n \to \Gamma \xrightarrow{p} G \to 0.$$ 

Let $h_\Gamma : G \to GL(n, \mathbb{Z})$ be the integral holonomy representation defined by the formula

$$\forall g \in G \; h_\Gamma(g)(e) = \bar{g} e \bar{g}^{-1},$$

where $\bar{g} \in \Gamma$, $p(\bar{g}) = g$ and $e \in \mathbb{Z}^n$. Since $\mathbb{Z}^n$ is a maximal abelian subgroup, $h_\Gamma$ is a faithfull representation.
Let $\Gamma$ be a crystallographic group. We have

$$0 \to \mathbb{Z}^n \to \Gamma \xrightarrow{p} G \to 0.$$ 

Let $h_\Gamma : G \to GL(n, \mathbb{Z})$ be the integral holonomy representation defined by the formula

$$\forall g \in G \ h_\Gamma(g)(e) = \bar{g} e \bar{g}^{-1},$$

where $\bar{g} \in \Gamma$, $p(\bar{g}) = g$ and $e \in \mathbb{Z}^n$. Since $\mathbb{Z}^n$ is a maximal abelian subgroup, $h_\Gamma$ is a faithfull representation.
Let $\Gamma$ be a crystallographic group. We have

$$0 \rightarrow \mathbb{Z}^n \rightarrow \Gamma \xrightarrow{p} G \rightarrow 0.$$ 

Let $h_\Gamma : G \rightarrow GL(n, \mathbb{Z})$ be the integral holonomy representation defined by the formula

$$\forall g \in G \ h_\Gamma(g)(e) = \bar{g} e \bar{g}^{-1},$$

where $\bar{g} \in \Gamma$, $p(\bar{g}) = g$ and $e \in \mathbb{Z}^n$. Since $\mathbb{Z}^n$ is a maximal abelian subgroup, $h_\Gamma$ is a faithful representation.
Let $\alpha \in H^2(G, \mathbb{Z}^n)$ be an element which corresponds with the above short exact sequence. Let

$$N = \{X \in GL(n, \mathbb{Z}) \mid \forall f \in h_\Gamma(G) \ X f X^{-1} \in h_\Gamma(G)\}.$$ 

$N$ acts on $H^2(G, \mathbb{Z}^n)$ by formula

$$n \ast [c](g_1, g_2) = n^{-1}[c(ng_1 n^{-1}, c(ng_2 n^{-1}),$$

where $[c] \in H^2(G, \mathbb{Z}^n)$, $n \in N$, $g_1, g_2 \in G$. Finally let

$$N_\alpha = \{n \in N \mid n \ast \alpha = \alpha\}.$$
Let $\alpha \in H^2(G, \mathbb{Z}^n)$ be an element which corresponds with the above short exact sequence. Let

\[ N = \{ X \in GL(n, \mathbb{Z}) \mid \forall f \in h_\Gamma(G) \ X f X^{-1} \in h_\Gamma(G) \}. \]

$N$ acts on $H^2(G, \mathbb{Z}^n)$ by formula

\[ n \ast [c](g_1, g_2) = n^{-1}[c(n g_1 n^{-1}, c(n g_2 n^{-1})], \]

where $[c] \in H^2(G, \mathbb{Z}^n)$, $n \in N$, $g_1, g_2 \in G$. Finally let

\[ N_\alpha = \{ n \in N \mid n \ast \alpha = \alpha \}. \]
Let $\alpha \in H^2(G, \mathbb{Z}^n)$ be an element which corresponds with the above short exact sequence. Let

$$N = \{ X \in GL(n, \mathbb{Z}) \mid \forall f \in h_\Gamma(G) \ X f X^{-1} \in h_\Gamma(G) \}.$$ 

$N$ acts on $H^2(G, \mathbb{Z}^n)$ by formula

$$n \ast [c](g_1, g_2) = n^{-1}[c(ng_1n^{-1}, c(ng_2n^{-1}),$$

where $[c] \in H^2(G, \mathbb{Z}^n)$, $n \in N$, $g_1, g_2 \in G$. Finally let

$$N_\alpha = \{ n \in N \mid n \ast \alpha = \alpha \}.$$
Let $\alpha \in H^2(G, \mathbb{Z}^n)$ be an element which corresponds with the above short exact sequence. Let

$$N = \{ X \in GL(n, \mathbb{Z}) \mid \forall f \in h\Gamma(G) \ X f X^{-1} \in h\Gamma(G) \}.$$

$N$ acts on $H^2(G, \mathbb{Z}^n)$ by formula

$$n \ast [c](g_1, g_2) = n^{-1}[c(ng_1n^{-1}, c(ng_2n^{-1})],$$

where $[c] \in H^2(G, \mathbb{Z}^n)$, $n \in N$, $g_1, g_2 \in G$. Finally let

$$N_\alpha = \{ n \in N \mid n \ast \alpha = \alpha \}.$$
Let $\alpha \in H^2(G, \mathbb{Z}^n)$ be an element which corresponds with the above short exact sequence. Let

$$N = \{ X \in GL(n, \mathbb{Z}) \mid \forall f \in h_{\Gamma}(G) \ X f X^{-1} \in h_{\Gamma}(G) \}. $$

$N$ acts on $H^2(G, \mathbb{Z}^n)$ by formula

$$n \ast [c](g_1, g_2) = n^{-1}[c(ng_1n^{-1}, c(ng_2n^{-1}],$$

where $[c] \in H^2(G, \mathbb{Z}^n)$, $n \in N$, $g_1, g_2 \in G$. Finally let

$$N_\alpha = \{ n \in N \mid n \ast \alpha = \alpha \}. $$
From now, all crystallographic groups have a trivial center. It is easy to see, that in this case $(\mathbb{Z}^n)^G = 0$ and $Inn(\Gamma) = \Gamma$. We have a restriction homomorphism $F : Aut(\Gamma) \to Aut(\mathbb{Z}^n)$. In 1973 L. Charlap and A.T. Vasquez proved that the kernel $F_0$ of $F$ is isomorphic to $\mathbb{Z}^1(G, \mathbb{Z}^n)$ and the following diagram has exact rows and columns. This is our main technical tool.
From now, all crystallographic groups have a trivial center. It is easy to see, that in this case \((\mathbb{Z}^n)^G = 0\) and \(\text{Inn}(\Gamma) = \Gamma\). We have a restriction homomorphism \(F : \text{Aut}(\Gamma) \to \text{Aut}(\mathbb{Z}^n)\). In 1973 L. Charlap and A.T. Vasquez proved that the kernel \(F_0\) of \(F\) is isomorphic to \(Z^1(G, \mathbb{Z}^n)\) and the following diagram has exact rows and columns. This is our main technical tool.
From now, all crystallographic groups have a trivial center. It is easy to see, that in this case $(\mathbb{Z}^n)^G = 0$ and $\text{Inn}(\Gamma) = \Gamma$. We have a restriction homomorphism $F : \text{Aut}(\Gamma) \to \text{Aut}(\mathbb{Z}^n)$. In 1973 L. Charlap and A.T. Vasquez proved that the kernel $F_0$ of $F$ is isomorphic to $Z^1(G, \mathbb{Z}^n)$ and the following diagram has exact rows and columns. This is our main technical tool.
From now, all crystallographic groups have a trivial center. It is easy to see, that in this case \((\mathbb{Z}^n)^G = 0\) and \(\text{Inn}(\Gamma) = \Gamma\). We have a restriction homomorphism \(F : \text{Aut}(\Gamma) \to \text{Aut}(\mathbb{Z}^n)\). In 1973 L. Charlap and A.T. Vasquez proved that the kernel \(F_0\) of \(F\) is isomorphic to \(Z^1(G, \mathbb{Z}^n)\) and the following diagram has exact rows and columns. This is our main technical tool.
0 \rightarrow \mathbb{Z}^n \rightarrow \Gamma \rightarrow G \rightarrow 0

\rightarrow \mathbb{Z}^1(G, \mathbb{Z}^n) \rightarrow \text{Aut}(\Gamma) \rightarrow N_\alpha \rightarrow 0

0 \rightarrow H^1(G, \mathbb{Z}^n) \rightarrow \text{Out}(\Gamma) \rightarrow N_\alpha/G \rightarrow 0
Assume $Z(\Gamma) = \{e\}$, then

1. $H^1(G, \mathbb{Z}^n) \cong (\mathbb{Q}^n/\mathbb{Z}^n)^G = H^0(G, \mathbb{Q}^n/\mathbb{Z}^n)$;
2. $Z^1(G, \mathbb{Z}^n) \cong \{m \in \mathbb{Q}^n \mid \forall g \in G \; gm - m \in \mathbb{Z}^n\} = A^0(\Gamma)$ as $N_\alpha$ modules;
3. $A(\Gamma) = N_{\text{Aff}(\mathbb{R}^n)}(\Gamma) = \{a \in \text{Aff}(\mathbb{R}^n) \mid \forall \gamma \in \Gamma \; a\gamma a^{-1} \in \Gamma\} \cong \text{Aut}(\Gamma)$.

Andrzej Szczepański (joint work with R. Lutowski) University of Gdańsk

Outer automorphism group of crystallographic groups
Main Theorem

Theorem

For every $n \geq 2$ there exists a crystallographic group $\Gamma$ of dimension $n$ with $\text{Z}(\Gamma) = \text{Out}(\Gamma) = \{e\}$.

Example

Let $\Gamma_1 = G_1 \ltimes \mathbb{Z}^2$ be the crystallographic group of dimension 2 with holonomy group $G_1 = D_{12}$, where

$$D_{12} = \text{gen} \left\{ \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$$

is the dihedral group of order 12.
A group $G$ is said to be complete if its center $Z(G)$ and outer automorphism group are both trivial.

Complete groups were introduced by O. Hölder in 1895. He has proved already that the symmetric group $S_n$ is complete provided $n \neq 2$ or 6. J. L. Dyer and E. Formanek proved in 1975 that if $F$ is a non-cyclic free group then $Aut(F)$ is complete.
A group $G$ is said to be complete if its center $Z(G)$ and outer automorphism group are both trivial.

Complete groups were introduced by O. Hölder in 1895. He has proved already that the symmetric group $S_n$ is complete provided $n \neq 2$ or 6. J. L. Dyer and E. Formanek proved in 1975 that if $F$ is a non-cyclic free group then $Aut(F)$ is complete.
A group $G$ is said to be complete if its center $Z(G)$ and outer automorphism group are both trivial.

Complete groups were introduced by O. Hölder in 1895. He has proved already that the symmetric group $S_n$ is complete provided $n \neq 2$ or 6. J. L. Dyer and E. Formanek proved in 1975 that if $F$ is a non-cyclic free group then $Aut(F)$ is complete.
In 2005 M. Belolipetsky and A. Lubotzky proved that for every $n \geq 2$ and every finite group $G$ there exist infinitely many discrete, cocompact and torsion free subgroups $\Gamma \subset \text{Isom}(\mathbb{H}^n)$ with $\text{Out}(\Gamma) \simeq G$. In particular for a trivial group $G$ we obtain examples of complete groups.

In 2003 R. Walmüler found an example of complete, torsion free crystallographic group $\Gamma$ of dimension 141 with holonomy group $M_{11}$ (Mathieu group)

$$0 \to \mathbb{Z}^{141} \to \Gamma \to M_{11} \to 0.$$
In 2005 M. Belolipetsky and A. Lubotzky proved that for every $n \geq 2$ and every finite group $G$ there exist infinitely many discrete, cocompact and torsion free subgroups $\Gamma \subset \text{Isom}(\mathbb{H}^n)$ with $\text{Out}(\Gamma) \simeq G$. In particular for a trivial group $G$ we obtain examples of complete groups.

In 2003 R. Walmüler found an example of complete, torsion free crystallographic group $\Gamma$ of dimension 141 with holonomy group $M_{11}$ (Mathieu group)

$$0 \rightarrow \mathbb{Z}^{141} \rightarrow \Gamma \rightarrow M_{11} \rightarrow 0.$$
In 2005 M. Belolipetsky and A. Lubotzky proved that for every \( n \geq 2 \) and every finite group \( G \) there exist infinitely many discrete, cocompact and torsion free subgroups \( \Gamma \subset \text{Isom}(\mathbb{H}^n) \) with \( \text{Out}(\Gamma) \cong G \). In particular for a trivial group \( G \) we obtain examples of complete groups.

In 2003 R. Walmüler found an example of complete, torsion free crystallographic group \( \Gamma \) of dimension 141 with holonomy group \( M_{11} \) (Mathieiu group)

\[
0 \to \mathbb{Z}^{141} \to \Gamma \to M_{11} \to 0.
\]
In 2005 M. Belolipetsky and A. Lubotzky proved that for every $n \geq 2$ and every finite group $G$ there exist infinitely many discrete, cocompact and torsion free subgroups $\Gamma \subset \text{Isom}(\mathbb{H}^n)$ with $\text{Out}(\Gamma) \simeq G$. In particular for a trivial group $G$ we obtain examples of complete groups.

In 2003 R. Walmüler found an example of complete, torsion free crystallographic group $\Gamma$ of dimension 141 with holonomy group $M_{11}$ (Mathieu group)

$$0 \rightarrow \mathbb{Z}^{141} \rightarrow \Gamma \rightarrow M_{11} \rightarrow 0.$$
We shall need a few lemmas.

**Lemma**

*Aut*(\(\Gamma\)) *is a crystallographic group if and only if Out*(\(\Gamma\)) *is a finite group.*

**Proof:** We start with an observation that \(Z^1(G, \mathbb{Z}^n)\) is a free abelian group of rank \(n\) which is a faithful \(N_\alpha\) module. First, assume that *Aut*(\(\Gamma\)) is a crystallographic group with the maximal abelian subgroup \(M\). From Bieberbach’s theorems, \(M\) is the unique normal maximal abelian subgroup of *Aut*(\(\Gamma\)). Hence, \(M = Z^1(G, \mathbb{Z}^n)\), and *Out*(\(\Gamma\)) is a finite group. The reverse implication is obvious.
Proof

We shall need a few lemmas.

**Lemma**

*Aut*(Γ) is a crystallographic group if and only if *Out*(Γ) is a finite group.

**Proof:** We start with an observation that $Z^1(G, \mathbb{Z}^n)$ is a free abelian group of rank $n$ which is a faithful $N_\alpha$ module. First, assume that *Aut*(Γ) is a crystallographic group with the maximal abelian subgroup $M$. From Bieberbach’s theorems, $M$ is the unique normal maximal abelian subgroup of *Aut*(Γ). Hence, $M = Z^1(G, \mathbb{Z}^n)$, and *Out*(Γ) is a finite group. The reverse implication is obvious.
Proof

We shall need a few lemmas.

**Lemma**

$\text{Aut}(\Gamma)$ is a crystallographic group if and only if $\text{Out}(\Gamma)$ is a finite group.

**Proof:** We start with an observation that $Z^1(G, \mathbb{Z}^n)$ is a free abelian group of rank $n$ which is a faithful $N_\alpha$ module. First, assume that $\text{Aut}(\Gamma)$ is a crystallographic group with the maximal abelian subgroup $M$. From Bieberbach’s theorems, $M$ is the unique normal maximal abelian subgroup of $\text{Aut}(\Gamma)$. Hence, $M = Z^1(G, \mathbb{Z}^n)$, and $\text{Out}(\Gamma)$ is a finite group. The reverse implication is obvious.
Proof

**Lemma**

Let \( G, H \) be finite groups and \( H \subset G \subset \text{GL}(n, \mathbb{Z}) \). If the group \( N_{\text{GL}(n, \mathbb{Z})}(H) \) is finite, then \( N_{\text{GL}(n, \mathbb{Z})}(G) \) is finite.

**Proof:** From the assumption, \( \text{Aut}(H) \) and \( \text{Aut}(G) \) are finite and we have monomorphisms:

\[
N_{\text{GL}(n, \mathbb{Z})}(H)/C_{\text{GL}(n, \mathbb{Z})}(H) \xrightarrow{\bar{\phi}} \text{Aut}(H)
\]

and

\[
N_{\text{GL}(n, \mathbb{Z})}(G)/C_{\text{GL}(n, \mathbb{Z})}(G) \xrightarrow{\bar{\phi}} \text{Aut}(G),
\]

where \( \bar{\phi} \) is induced by \( \phi(s)(g) = sgs^{-1}, g \in G, s \in \text{GL}(n, \mathbb{Z}) \). Since \( C_{\text{GL}(n, \mathbb{Z})}(G) \subset C_{\text{GL}(n, \mathbb{Z})}(H) \), our Lemma is proved.
Proof

**Lemma**

Let $G, H$ be finite groups and $H \subset G \subset GL(n, \mathbb{Z})$. If the group $N_{GL(n,\mathbb{Z})}(H)$ is finite, then $N_{GL(n,\mathbb{Z})}(G)$ is finite.

**Proof:** From the assumption, $Aut(H)$ and $Aut(G)$ are finite and we have monomorphisms:

$$N_{GL(n,\mathbb{Z})}(H)/C_{GL(n,\mathbb{Z})}(H) \xrightarrow{\bar{\phi}} Aut(H)$$

and

$$N_{GL(n,\mathbb{Z})}(G)/C_{GL(n,\mathbb{Z})}(G) \xrightarrow{\bar{\phi}} Aut(G),$$

where $\bar{\phi}$ is induced by $\phi(s)(g) = sgs^{-1}, g \in G, s \in GL(n, \mathbb{Z})$. Since $C_{GL(n,\mathbb{Z})}(G) \subset C_{GL(n,\mathbb{Z})}(H)$, our Lemma is proved.
Proof

Lemma

Let \( G, H \) be finite groups and \( H \subset G \subset GL(n, \mathbb{Z}) \). If the group \( N_{GL(n,\mathbb{Z})}(H) \) is finite, then \( N_{GL(n,\mathbb{Z})}(G) \) is finite.

**Proof:** From the assumption, \( Aut(H) \) and \( Aut(G) \) are finite and we have monomorphisms:

\[
N_{GL(n,\mathbb{Z})}(H)/C_{GL(n,\mathbb{Z})}(H) \xrightarrow{\bar{\phi}} Aut(H)
\]

and

\[
N_{GL(n,\mathbb{Z})}(G)/C_{GL(n,\mathbb{Z})}(G) \xrightarrow{\bar{\phi}} Aut(G),
\]

where \( \bar{\phi} \) is induced by \( \phi(s)(g) = sgs^{-1}, g \in G, s \in GL(n, \mathbb{Z}) \). Since \( C_{GL(n,\mathbb{Z})}(G) \subset C_{GL(n,\mathbb{Z})}(H) \), our Lemma is proved.
Lemma

Let $G, H$ be finite groups and $H \subset G \subset \text{GL}(n, \mathbb{Z})$. If the group $N_{\text{GL}(n, \mathbb{Z})}(H)$ is finite, then $N_{\text{GL}(n, \mathbb{Z})}(G)$ is finite.

Proof: From the assumption, $\text{Aut}(H)$ and $\text{Aut}(G)$ are finite and we have monomorphisms:

$$N_{\text{GL}(n, \mathbb{Z})}(H)/C_{\text{GL}(n, \mathbb{Z})}(H) \xrightarrow{\overline{\phi}} \text{Aut}(H)$$

and

$$N_{\text{GL}(n, \mathbb{Z})}(G)/C_{\text{GL}(n, \mathbb{Z})}(G) \xrightarrow{\overline{\phi}} \text{Aut}(G),$$

where $\overline{\phi}$ is induced by $\phi(s)(g) = sgs^{-1}, g \in G, s \in \text{GL}(n, \mathbb{Z})$. Since $C_{\text{GL}(n, \mathbb{Z})}(G) \subset C_{\text{GL}(n, \mathbb{Z})}(H)$, our Lemma is proved.
Collorary

If $|\text{Out}(\Gamma)| < \infty$, then $|\text{Out}(\text{Aut}(\Gamma))| < \infty$. 
Proof

Let $\Gamma$ be a crystallographic group with a holonomy group $G$ of dimension $n$. Assume that the group $H^1(G, \mathbb{Z}^n) = \{e\}$, and the group $Out(\Gamma)$ is finite.

Inductively, put $\Gamma_0 = \Gamma$ and $\Gamma_{i+1} = A(\Gamma_i)$, for $i \geq 0$.

Lemma

$\exists N$ such that $\Gamma_{N+1} = \Gamma_N$.

Proof: We start from observations that for $i > 0$, $\Gamma_i$ is a crystallographic group, $Z(\Gamma_i) = \{e\}$ and $M_0 = M_i$, where $M_i = A^0(\Gamma_{i-1}) \subset \Gamma_i$ is the maximal abelian normal subgroup (a subgroup of translations). Let $G_i = \Gamma_i / M_i$. From definition we can consider $(G_i)$ as a nondecreasing sequence of finite subgroups of $GL(n, \mathbb{Z})$. From Bieberbach theorems and from the diagram, there is only a finite number of possibilties for $G_i$. Hence $\exists N \in \mathbb{N}$ such that $\forall i > N$ $G_i = G_N$. This finishes the proof.
Proof

Let $\Gamma$ be a crystallographic group with a holonomy group $G$ of dimension $n$. Assume that the group $H^1(G, \mathbb{Z}^n) = \{e\}$, and the group $Out(\Gamma)$ is finite.

Inductively, put $\Gamma_0 = \Gamma$ and $\Gamma_{i+1} = A(\Gamma_i)$, for $i \geq 0$.

Lemma

$\exists N$ such that $\Gamma_{N+1} = \Gamma_N$.

Proof: We start from observations that for $i > 0$, $\Gamma_i$ is a crystallographic group, $Z(\Gamma_i) = \{e\}$ and $M_0 = M_i$, where $M_i = A^0(\Gamma_i-1) \subset \Gamma_i$ is the maximal abelian normal subgroup (a subgroup of translations). Let $G_i = \Gamma_i/M_i$. From definition we can consider $(G_i)$ as a nondecreasing sequence of finite subgroups of $GL(n, \mathbb{Z})$. From Bieberbach theorems and from the diagram, there is only a finite number of possibilities for $G_i$. Hence $\exists N \in \mathbb{N}$ such that $\forall i > N \, G_i = G_N$. This finishes the proof.
Proof

Let $\Gamma$ be a crystallographic group with a holonomy group $G$ of dimension $n$. Assume that the group $H^1(G, \mathbb{Z}^n) = \{e\}$, and the group $Out(\Gamma)$ is finite.

Inductively, put $\Gamma_0 = \Gamma$ and $\Gamma_{i+1} = A(\Gamma_i)$, for $i \geq 0$.

Lemma

$\exists N$ such that $\Gamma_{N+1} = \Gamma_N$.

Proof: We start from observations that for $i > 0$, $\Gamma_i$ is a crystallographic group, $Z(\Gamma_i) = \{e\}$ and $M_0 = M_i$, where $M_i = A^0(\Gamma_{i-1}) \subset \Gamma_i$ is the maximal abelian normal subgroup (a subgroup of translations). Let $G_i = \Gamma_i / M_i$. From definition we can consider $(G_i)$ as a nondecreasing sequence of finite subgroups of $GL(n, \mathbb{Z})$. From Bieberbach theorems and from the diagram, there is only a finite number of possibilties for $G_i$. Hence $\exists N \in \mathbb{N}$ such that $\forall i > N \ \ G_i = G_N$. This finishes the proof.
Proof

Let $\Gamma$ be a crystallographic group with a holonomy group $G$ of dimension $n$. Assume that the group $H^1(G, \mathbb{Z}^n) = \{e\}$, and the group $Out(\Gamma)$ is finite.

Inductively, put $\Gamma_0 = \Gamma$ and $\Gamma_{i+1} = A(\Gamma_i)$, for $i \geq 0$.

**Lemma**

$\exists N$ such that $\Gamma_{N+1} = \Gamma_N$.

**Proof:** We start from observations that for $i > 0$, $\Gamma_i$ is a crystallographic group, $Z(\Gamma_i) = \{e\}$ and $M_0 = M_i$, where $M_i = A^0(\Gamma_{i-1}) \subset \Gamma_i$ is the maximal abelian normal subgroup (a subgroup of translations). Let $G_i = \Gamma_i / M_i$. From definition we can consider $(G_i)$ as a nondecreasing sequence of finite subgroups of $GL(n, \mathbb{Z})$. From Bieberbach theorems and from the diagram, there is only a finite number of possibilities for $G_i$. Hence $\exists N \in \mathbb{N}$ such that $\forall i > N \quad G_i = G_N$. This finishes the proof.
Proof

Example

Let $\Gamma_2 = G_2 \ltimes \mathbb{Z}^3$ be the crystallographic group of dimension 3, with holonomy group $G_2 = S_4 \times \mathbb{Z}_2$ generated by matrices

$$
\begin{bmatrix}
0 & 1 & 0 \\
0 & -1 & -1 \\
1 & 1 & 0
\end{bmatrix}, \quad
\begin{bmatrix}
0 & 0 & 1 \\
0 & -1 & -1 \\
-1 & 0 & 1
\end{bmatrix}.
$$

Here $S_4$ denotes the symmetric group on four letters. For $i = 1, 2$ we have

$$N_{GL(n_i, \mathbb{Z})}(G_i) = G_i \text{ and } H^1(G_i, \mathbb{Z}^{n_i}) = 0,$$

where $n_i$ is the rank of $\Gamma_i$. Hence $A(\Gamma_i) = \Gamma_i$, and $Out(\Gamma_i) = \{e\}$, for $i = 1, 2$. 

Andrzej Szczepański (joint work with R. Łutowski) University of Gdańsk

Outer automorphism group of crystallographic groups
Proof

We shall use the following observation.

**Theorem**

Let $\Gamma_i$, $i = 1,2,\ldots,k$ be mutually nonisomorphic directly indecomposable torsion free crystallographic groups with trivial center. Let $n_i \in \mathbb{N}$, $i = 1,2,\ldots,k$. Then

$$\text{Out}(\Gamma_1^{n_1} \times \Gamma_2^{n_2} \times \cdots \times \Gamma_k^{n_k}) \cong \text{Out}(\Gamma_1) \wr S_{n_1} \times \cdots \times \text{Out}(\Gamma_k) \wr S_{n_k}.$$
Now we are ready to finish the proof of Theorem. The cases $n = 2, 3$ are done in the above examples. Assume $n \geq 4$. Let $n = 2k + 3i$, where $i \in \{0, 1\}$. Put $\Gamma' = \Gamma_1^k \times \Gamma_2^i$. Then $\Gamma'$ is centerless and by the last theorem the bottom exact sequence of the diagram looks as follows

$$0 \to 0 \to \text{Out}(\Gamma') \to S_k \to 0.$$ 

Hence, $\Gamma'$ satisfies the assumption of Lemma 0.4 and the sequence $\Gamma_0 = \Gamma'$, $\Gamma_{i+1} = A(\Gamma_i)$ stabilizes, i.e., $\exists N$ such that $\forall i \geq N \Gamma_i = \Gamma_N$. Finally, $\text{Out}(\Gamma_N) = \{e\}$ and $\text{Z}(\Gamma_N) = \{e\}$. 

Andrzej Szczepański (joint work with R. Lutowski) University of Gdańsk 
Outer automorphism group of crystallographic groups
Now we are ready to finish the proof of Theorem. The cases \( n = 2, 3 \) are done in the above examples. Assume \( n \geq 4 \). Let \( n = 2k + 3i \), where \( i \in \{0, 1\} \). Put \( \Gamma' = \Gamma_1^k \times \Gamma_2^i \). Then \( \Gamma' \) is centerless and by the last theorem the bottom exact sequence of the diagram looks as follows

\[
0 \rightarrow 0 \rightarrow \text{Out}(\Gamma') \rightarrow S_k \rightarrow 0.
\]

Hence, \( \Gamma' \) satisfies the assumption of Lemma 0.4 and the sequence \( \Gamma_0 = \Gamma', \ \Gamma_{i+1} = A(\Gamma_i) \) stabilizes, i.e., \( \exists N \) such that \( \forall i \geq N \ \Gamma_i = \Gamma_N \). Finally, \( \text{Out}(\Gamma_N) = \{e\} \) and \( Z(\Gamma_N) = \{e\} \).
Now we are ready to finish the proof of Theorem. The cases $n = 2, 3$ are done in the above examples. Assume $n \geq 4$. Let $n = 2k + 3i$, where $i \in \{0, 1\}$. Put $\Gamma' = \Gamma_1^k \times \Gamma_2^i$. Then $\Gamma'$ is centerless and by the last theorem the bottom exact sequence of the diagram looks as follows

$$0 \to 0 \to \text{Out}(\Gamma') \to S_k \to 0.$$ 

Hence, $\Gamma'$ satisfies the assumption of Lemma 0.4 and the sequence $\Gamma_0 = \Gamma'$, $\Gamma_{i+1} = \text{A}(\Gamma_i)$ stabilizes, i.e., $\exists N$ such that $\forall i \geq N \quad \Gamma_i = \Gamma_N$. Finally, $\text{Out}(\Gamma_N) = \{e\}$ and $\text{Z}(\Gamma_N) = \{e\}$. 

Andrzej Szczepański (joint work with R. Lutowski) University of Gdańsk

Outer automorphism group of crystallographic groups