Introduction to flat manifolds

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Let $\mathbb{R}^n$ be $n$-dimensional Euclidean space, with isometry group $E(n) = O(n) \rtimes \mathbb{R}^n$.

**Definition**

$\Gamma$ is a crystallographic group of rank $n$ iff it is a discrete and cocompact subgroup of $E(n)$.

A Bieberbach group is a torsion free crystallographic group.
Basic properties

Theorem
(Bieberbach, 1910)

- If $\Gamma$ is a crystallographic group of dimension $n$, then the set of all translations of $\Gamma$ is a maximal abelian subgroup of a finite index.
- There is only a finite number of isomorphic classes of crystallographic groups of dimension $n$.
- Two crystallographic groups of dimension $n$ are isomorphic if and only if there are conjugate in the group affine transformations $A(n) = GL(n, \mathbb{R}) \ltimes \mathbb{R}^n$. 

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Theorem
(Zassenhaus, 1947) A group $\Gamma$ is a crystallographic group of dimension $n$ if and only if, it has a normal maximal abelian subgroup $\mathbb{Z}^n$ of a finite index.
Definition
Let $\Gamma$ be a crystallographic group of dimension $n$ with translations subgroup $A \cong \mathbb{Z}^n$. A finite group $\Gamma/A = G$ we shall call a holonomy group of $\Gamma$.

Let $(A, a) \in E(n)$ and $x \in \mathbb{R}^n$. $\Gamma$ acts on $\mathbb{R}^n$ in the following way:

$$(A, a)(x) = Ax + a.$$
Definition
Let $\Gamma$ be $n$-dimensional Bieberbach group. We have the following short exact sequence of groups.

$$0 \to \mathbb{Z}^n \to \Gamma \xrightarrow{p} \Gamma/\mathbb{Z}^n = H \to 0.$$ 

Let us define a homomorphism $h_\Gamma : H \to GL(n, \mathbb{Z})$. Put

$$\forall h \in H, h_\Gamma(h)(e_i) = \bar{h}^{-1} e_i \bar{h},$$

where $p(\bar{h}) = h$ and $e_i \in \mathbb{Z}^n$ is a standard basis. $h_\Gamma$ is called a holonomy representation of a group $\Gamma$. 

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Flat manifold

Let $\Gamma \subset E(n)$ be a torsion free crystallographic group. Since $\Gamma$ is cocompact and discrete subgroup, then the orbit space $\mathbb{R}^n / \Gamma$ is a manifold. If $\Gamma$ is not torsion free then the orbit space $\mathbb{R}^n / \Gamma$ is an orbifold.

Definition
The above manifolds (orbifolds) we shall call "flat".
From elementary covering theory any compact Riemannian manifold (orbifold) with sectional curvature equal to zero is flat.
Example

Flat surfaces:

- torus $S^1 \times S^1$,
- Klein bottle $S^1 \times S^1 / \mathbb{Z}_2$

We shall see that many properties of the Bieberbach Groups correspond to properties of flat manifolds.
Classification

From the second Bieberbach theorem there is only a finite number of flat manifolds of given dimension. For example in dimension 3 there are 10 flat manifolds. Here classification was made in 1936. Then we have a computer program CARAT see https://www.mathb.rwth-aachen.de/carat/index.html.

We have:
in dimension 4 - 74,
in dimension 5 - 1060,
in dimension 6 - 38746.

**Remark:** There exists $G \subset GL(6, \mathbb{Z})$ s.t. $|H^2(G, \mathbb{Z}^6)| = 2^{30} = 1073741824$. 
Theorem (Calabi - 1957) Let $\Gamma$ be a torsion free crystallographic group of dimension $n$ with an epimorphism $f : \Gamma \to \mathbb{Z}$. Then $\ker f$ is a torsion free crystallographic group of dimension $n - 1$.

"the induction method of Calabi"

1. classify all torsion free crystallographic groups of rank $< n$;
2. classify all torsion free crystallographic groups of dimension $n$ with finite abelianization;
3. classify all torsion free c.g. $\Gamma$ of dimension $n$ defined by the short exact sequence

$$0 \to \Gamma_{n-1} \to \Gamma \to \mathbb{Z} \to 0,$$

where $\Gamma_{n-1}$ is from point 1.
Definition
Let $M$ be a flat manifold with the fundamental group $\Gamma$, which acts by isometries on a flat torus $T^k$. Then $\Gamma$ also acts by isometries on the space $\tilde{M} \times T^k$. We shall call the space $(\tilde{M} \times T^k)/\Gamma$ a flat toral extension of the manifold $M$.

Theorem
(A.T.Vasquez - 1970) For any finite group $G$ there exists a natural number $n(G)$ with the following property: if $M$ is any flat manifold with holonomy group $G$, then $M$ is a flat toral extension of some flat manifold of dimension $\leq n(G)$. 
The third way of the classification is called the Auslander-Vasquez method. It is related only to flat manifolds with given holonomy group $G$ and has the following steps:
1. calculate the Vasquez invariant $n(G)$;
2. describe all flat toral extensions of the manifolds of dimension $\leq n(G)$;
3. classify all flat manifolds of dimension $\leq n(G)$.
For example for $p$-group, $n(G) = \Sigma_{C \in \mathcal{X}} | G : C |$, where $\mathcal{X}$ is a set of representatives of conjugacy classes of subgroups of $G$ of prime order; G. Cliff, A. Weiss 1989. $n(A_5) = 16$; G. Cliff, Hongliu Zheng 1996.
Boundary problem

**Theorem**

We can ask: Does every compact Riemannian flat manifold bound a compact hyperbolic manifold?

**Example**
Let $V$ be a hyperbolic (complete Riemannian with constant sectional curvature $-1$) manifold with one cusp. After cut a cusp we have a compact hyperbolic manifold $V'$ with boundary $\partial V'$, where $\partial V'$ is flat.

We can ask: Let $M^n$ be a flat manifold of dimension $n$. Is there some $V^{(n+1)}$ such that $\partial((V')(^{n+1})) \cong M^n$?
In 2000 D. D. Long and A. Reid proved:

**Theorem**

*Let $V$ be a hyperbolic manifold with one cusp of dimension $4n$. If a flat manifold $M^{(4n-1)}$ of dimension $(4n - 1)$ has geometric realization as $\partial V'$, then $\eta(M^{(4n-1)}) \in \mathbb{Z}$.***

The proof is consequence of the Atiyah, Patodi and Singer theorem. Here we consider $\eta$-invariant of signature operator. Already in dimension 3 there exists a flat manifold $M^3$ such that $\eta(M^3) = 3/4 \notin \mathbb{Z}$. 
A Bieberbach group $\Gamma \subset SO(n) \ltimes \mathbb{R}^n$ has a spin structure if and only if there exists a homomorphism $\epsilon : \Gamma \to Spin(n)$ such that $\text{pr}_1 = \lambda_n \circ \epsilon$. Here $\text{pr}_1$ is a projection on the first component, and $\lambda_n : Spin(n) \to SO(n)$ is a universal covering.

It is well known that $H^2(SO(n), \mathbb{Z}_2) = \mathbb{Z}_2$. We define a group $Spin(n)$ as a middle group in a non-trivial short exact sequence

$$0 \to \mathbb{Z}_2 \to Spin(n) \xrightarrow{\lambda_n} SO(n) \to 0.$$
Let $C_n$ be Clifford’s algebra over the real numbers. By definition it is an associative algebra with unity, generated by elements

$$\{e_1, e_2, \ldots, e_n\}$$

and with relations

$$\forall i, \ e_i^2 = -1,$$

$$\forall i, j, \ e_i e_j = -e_j e_i,$$

where $1 \leq i, j \leq n$. We define $C_0 = \mathbb{R}$. It is easy to see that $C_1 = \mathbb{C}$ and $C_2 = \mathbb{H}$, where $\mathbb{H}$ is the four-dimensional quaternion algebra. Moreover, $\mathbb{R}^n \subset C_n$ and $\dim \mathbb{R} C_n = 2^n$, where $\mathbb{R}^n$ is $n$-dimensional $\mathbb{R}$-vector space with the basis $e_1, e_2, \ldots, e_n$. 

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We have the following homomorphisms (involutions) on $C_n$:

(i) $*: e_{i_1} e_{i_2} \ldots e_{i_k} \mapsto e_{i_k} e_{i_{k-1}} \ldots e_{i_2} e_{i_1}$,
(ii) $': e_i \mapsto -e_i$,
(iii) $-: a \mapsto (a')^*$, $a \in C_n$.

Suppose $C_n^0 = \{x \in C_n \mid x' = x\}$. It is easy to observe that

$$\forall a, b \in C_n, (ab)^* = b^* a^*.$$ 

We define subgroups of $C_n$,

$$Pin(n) = \{x_1 x_2 \ldots x_k \mid x_i \in S^{n-1} \subset \mathbb{R}^n \subset C_n, i = 1, 2, \ldots k\},$$

$$Spin(n) = Pin(n) \cap C_n^0.$$
Theorem
(B. Putrycz, J. P. Rossetti, 2009) Let $\Gamma$ be an oriented Hanztsche-Wendt group of dimension $2n + 1 \geq 5$, then $\Gamma$ has not a spin structure.

A few words about the proof.
Let $\beta_i = (B_i, b_i) \in \Gamma$ be generators of $\Gamma$, for $i = 1, 2, \ldots, 2n$, where $B_i = \text{diag}[−1, −1, \ldots, 1, \underbrace{−1, \ldots, −1}_{i}]$. It is easy to see that $\lambda_n(\pm e_1 e_2, \ldots, e_{i−1} e_{i+1} \ldots e_{2n+1}) = B_i$ and $\lambda_n(\pm e_i e_j) = \text{diag}[1, \ldots, 1, \underbrace{−1}_{i}, 1, \ldots, 1, \underbrace{−1}_{j}, 1, \ldots 1]$. Moreover $\forall i, j \ (e_i e_j)^2 = −1$ and $(e_{i_1} e_{i_2} \ldots e_{i_{2m}})^2 = (−1)^m \mod 2$. 
Let $n = 2m + 1$ and let $\epsilon : \Gamma \to Spin(n)$ be a homomorphism s.t. $\lambda_n \circ \epsilon = \text{pr}_1$. From above $\forall i \; \epsilon(\beta_i) = \pm e_1 e_2 \ldots e_{i-1} e_{i+1} \ldots e_{2m+1}$. 

Hence $\epsilon(t_i) = \epsilon((\beta_i)^2) = (-1)^m \mod 2$.

We consider two cases. For $m$ even $\epsilon(Z^n) = 1$. We have $\epsilon(\beta_1 \beta_2) = \pm e_1 e_2$. Finally $\epsilon((\beta_1 \beta_2)^2) = 1 = (\pm e_1 e_2)^2 = -1$, and we have a contradiction.

For $m$ odd a proof is rather more difficult.
Spin structures in dimensions 4, 5 and 6

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<th># orientable flat manifolds</th>
<th># spin flat manifolds</th>
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Thank You.