Cohomological rigidity of oriented Hantzsche–Wendt manifolds

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By Hantzsche–Wendt manifold (for short HW-manifold) we understand any oriented closed Riemannian manifold of dimension \( n \) with a holonomy group \((\mathbb{Z}_2)^{n-1}\). Two HW-manifolds \( M_1 \) and \( M_2 \) are cohomological rigid if and only if a homeomorphism between \( M_1 \) and \( M_2 \) is equivalent to an isomorphism of graded rings \( H^*(M_1, \mathbb{F}_2) \) and \( H^*(M_2, \mathbb{F}_2) \). We prove that HW-manifolds are cohomological rigid.

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1. Introduction

Let \( M^n \) be a flat manifold of dimension \( n \). By definition, this is a compact connected, Riemannian manifold without boundary with sectional curvature equal to zero. From

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the theorems of Bieberbach ([1,8]) the fundamental group \( \pi_1(M^n) = \Gamma \) determines a short exact sequence:

\[
0 \to \mathbb{Z}^n \to \Gamma \overset{p}{\to} G \to 0, \tag{1}
\]

where \( \mathbb{Z}^n \) is a torsion free abelian group of rank \( n \) and \( G \) is a finite group which is isomorphic to the holonomy group of \( M^n \). The universal covering of \( M^n \) is the Euclidean space \( \mathbb{R}^n \) and hence \( \Gamma \) is isomorphic to a discrete cocompact subgroup of the isometry group \( \text{Isom}(\mathbb{R}^n) = \text{O}(n) \ltimes \mathbb{R}^n = \text{E}(n) \). In the above short exact sequence \( \mathbb{Z}^n \cong (\Gamma \cap \mathbb{R}^n) \) and \( p \) can be considered as the projection \( p : \Gamma \to G \subset \text{O}(n) \subset \text{E}(n) \) on the first component. An orthogonal representation \( p \) is equivalent (see [8]) to a holonomy representation. That is a homomorphism \( \phi_\Gamma : G \to \text{GL}(n, \mathbb{Z}) \), given by a formula \( \phi_\Gamma(g)(z) = \bar{g}z\bar{g}^{-1} \), where \( \bar{g} \in \Gamma, g \in G, z \in \mathbb{Z}^n \) and \( p(g) = g \). Conversely, given a short sequence of the form (1), it is known that the group \( \Gamma \) is (isomorphic to) a Bieberbach group if and only if \( \Gamma \) is torsion free.

By Hantzsche–Wendt manifold (for short \( \text{HW-manifold} \) \( M^n \) we understand any oriented flat manifold of dimension \( n \) with a holonomy group \( (\mathbb{Z}_2)^{n-1} \). It is easy to see that \( n \) is always an odd number. Moreover, any HW-manifold has a diagonal holonomy representation, see [7]. It means \( \pi_1(M^n) \) is generated by \( \beta_i = (B_i, b_i) \in \text{SO}(n) \ltimes \mathbb{R}^n, 1 \leq i \leq n \), where

\[
B_i = \text{diag}(-1, -1, ..., -1, 1, -1, ..., -1)
\]

and \( b_i \in \{0, 1/2\}^n \). For other properties of \( M^n \) we send a reader to [8] and to next sections. We shall need

**Definition 1.** (See [4].) Two flat manifolds \( M_1 \) and \( M_2 \) are cohomological rigid if and only if a homeomorphism between \( M_1 \) and \( M_2 \) is equivalent to an isomorphism of graded rings \( H^*(M_1, \mathbb{F}_2) \) and \( H^*(M_2, \mathbb{F}_2) \).

Our main result is the following theorem.

**Theorem.** Hantzsche–Wendt manifolds are cohomological rigid.

The Theorem answers the question from [2, problem 4.3].

For the proof we introduce a new presentation of \( \text{HW-manifolds} \). We consider these manifolds rather as a finite quotient of the torus than a quotient of the \( \mathbb{R}^n \). Here, we use an obvious equivalence \( \mathbb{R}^n / \Gamma = (\mathbb{R}^n / \mathbb{Z}^n) / G = T^n / G \), where \( \Gamma \) is a Bieberbach group from (1). According to the definition of \( n \)-dimensional \( \text{HW-manifold} \) we shall define a \( (n \times n) \)-\( \text{HW-matrix} \) \( A \). The analysis of properties of the matrix \( A \) is used in the proof. Moreover, we apply the Lyndon–Hochschild–Serre spectral sequence \( \{E^r_{p,q}, d_r\} \) of the covering \( T^n \to T^n / G \) with \( \mathbb{F}_2 \) coefficients. Since a holonomy representation \( \Phi_\Gamma \) is diagonal
$E^{p,q}_2 = H^p((\mathbb{Z}_2)^{n-1}) \otimes H^q(\mathbb{Z}^n)$. We shall only use the multiplicative structure of the first and second cohomology group. In particular, we shall consider the properties of the transgression homomorphism $d_2 : H^1(\mathbb{Z}^n) \to H^2((\mathbb{Z}_2)^{n-1})$. Finally, another important point of the proof is an isomorphism of cohomology groups $H^1((\mathbb{Z}_2)^{n-1})$ and $H^1(\Gamma)$, which was proved in [6, Theorem 3.1]. Hence, we can consider elements of the image of the transgression homomorphism $d_2$ as homogeneous polynomials of degree two which are equivalent to polynomial functions.

Let us present a structure of the paper. In the next section, we give a “new-old” definition of HW-manifold and we outline the proof of the theorem. In section 3 we define HW-matrix and prove some of its properties.

At the last section, we present the proof of the Main Lemma.

2. Proof of the Main Theorem

Let $\mathcal{D} = \{g_i \mid i = 0, 1, 2, 3\}$, where $g_i : S^1 \to S^1$, and $\forall z \in S^1 \subset \mathbb{C}$,

$$g_0(z) = z, g_1(z) = -z, g_2(z) = \bar{z}, g_3(z) = -\bar{z}.$$ (3)

Equivalently, if $S^1 = \mathbb{R}/\mathbb{Z}, \forall [t] \in \mathbb{R}/\mathbb{Z},$

$$g_0([t]) = [t], g_1([t]) = [t + \frac{1}{2}], g_2([t]) = [-t], g_3([t]) = [-t + \frac{1}{2}].$$ (4)

Let $(t_1, t_2, ..., t_n) \in \mathcal{D}^n$ and $(z_1, z_2, ..., z_n) \in T^n = S^1 \times S^1 \times ... \times S^1$. It is easy to see that $\mathcal{D} = \mathbb{Z}_2 \times \mathbb{Z}_2$, and $g_3 = g_1 g_2$. For $k = 1, 2, 3$ we have different projections

$$p^{(k)} : \mathcal{D} \to \mathbb{F}_2 = \{0, 1\}$$ (5)

such that $p^{(k)}(g_k) = 1$ and for $i = 1, 2, ..., n$ we have homomorphisms

$$p^{(k)} \circ pr_i : \mathcal{D}^n \to \mathcal{D}^{p^{(k)}} \overset{p^{(k)}}{\to} \mathbb{F}_2$$ (6)

given by the formula $p^{(k)} \circ pr_i(t_1, t_2, ..., t_i, ..., t_n) = p^{(k)}(t_i)$.

We summing up values of the projections $p^{(2)}$ and $p^{(3)}$ in Table 1.

<table>
<thead>
<tr>
<th>Table 1</th>
<th>Values of the projections from $\mathcal{D} \to \mathbb{F}_2$.</th>
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<tbody>
<tr>
<td></td>
<td>$g_0$</td>
</tr>
<tr>
<td>$p^{(2)}$</td>
<td>0</td>
</tr>
<tr>
<td>$p^{(3)}$</td>
<td>0</td>
</tr>
</tbody>
</table>
The next, obvious formula
\[ \forall x \in D \ x = p^{(2)}(x)2 + p^{(3)}(x)3 \] (7)
will be useful later. We can define an action \( D^n \) on \( T^n \) as follows:
\[ (t_1, t_2, \ldots, t_n)(z_1, z_2, \ldots, z_n) = (t_1 z_1, t_2 z_2, \ldots, t_n z_n). \] (8)
We have

**Proposition 1.** Let \( M^n \) be a HW-manifold of dimension \( n \). Then there exists a subgroup \( (\mathbb{Z}_2)^{n-1} \subset D^n \) such that \( M^n = T^n/(\mathbb{Z}_2)^{n-1} \), where the action \( (\mathbb{Z}_2)^{n-1} \) on \( T^n \) is defined by (2) and (8).

**Proof.** Let \( \pi_1(M^n) = \Gamma \) and \( (B_l, b_l) \in \Gamma \) be the generators (2), \( l = 1, 2, \ldots, n \). On each coordinate, (4) defines \( g_j \in D, j = 0, 1, 2, 3 \) which are determined by projections \( p^{(1)} \circ pr_i, p^{(2)} \circ pr_i, p^{(3)} \circ pr_i \). □

Let us start to prove that the graded ring \( H^*(M^n, \mathbb{F}_2) \) defines a manifold \( M^n \). We have an exact sequence
\[ 0 \rightarrow \mathbb{Z}^n \rightarrow \Gamma \overset{p}{\rightarrow} (\mathbb{Z}_2)^{n-1} \rightarrow 0, \] (9)
where \( \Gamma = \pi_1(M^n) \). As we mentioned already in the introduction the image of a holonomy representation \( \Phi_\Gamma((\mathbb{Z}_2)^{n-1}) \), is a subgroup of the group of all diagonal matrices of \( GL(n, \mathbb{Z}) \). Moreover (see [6]) \( H^1(\Gamma, \mathbb{F}_2) = (\mathbb{F}_2)^{n-1} \) for any Hantsche–Wendt group \( \Gamma \) of dimension \( n \). That is an observation which we shall use during the proof.

Since \( (\mathbb{Z}_2)^{n-1} \subset D^n \) the above maps \( p^{(k)} \circ pr_i, k = 1, 2, 3 \) define homomorphisms from \( (\mathbb{Z}_2)^{n-1} \rightarrow \mathbb{F}_2 \in Hom((\mathbb{Z}_2)^{n-1}, \mathbb{F}_2) = H^1((\mathbb{Z}_2)^{n-1}, \mathbb{F}_2) \overset{[6]}{=} H^1(M^n, \mathbb{F}_2) \). Hence we can define elements
\[ T_i = (p^{(2)} \circ pr_i) \cup (p^{(3)} \circ pr_i) \in H^2((\mathbb{Z}_2)^{n-1}, \mathbb{F}_2), \]
where \( \cup \) is a cup product. It is well known that \( H^*((\mathbb{Z}_2)^{n-1}, \mathbb{F}_2) \) is isomorphic to \( \mathbb{F}_2[x_1, x_2, \ldots, x_{n-1}] \). Hence the elements \( p^{(k)} \circ pr_i = p^{(k)}_{i} \) correspond to
\[ \sum_{j=1}^{n-1} p^{(k)}(pr_i(b_j))x_j = \sum_{j=1}^{n-1} p^{(k)}(A_{ji})x_j \in \mathbb{F}_2[x_1, x_2, \ldots, x_{n-1}], \] (10)
where \( b_1, b_2, \ldots, b_{n-1} \) is the basis of \( (\mathbb{Z}_2)^{n-1} \) and \( k = 2, 3; i = 1, 2, \ldots, n \). Here the matrix \( A_{ij}, i = 1, 2, \ldots, n-1; j = 1, 2, \ldots, n \) is related to \( HW\)-matrix (**Definition 2**) from the next section.
We shall apply the Lyndon–Hochschild–Serre spectral sequence \( \{ E_{p,q}^{r}, d_{r} \} \) of (9). Since a holonomy representation \( \Phi_{\Gamma} \) is diagonal \( E_{2}^{p,q} = H^{p}(\mathbb{Z}_{2}^{n-1}) \otimes H^{q}(\mathbb{Z}^{n}) \). Hence (see [3, Corollary 7.2.3 on p. 77]) we have an exact sequence (see [2, p. 770])

\[
H^{1}(\mathbb{Z}^{n}, \mathbb{F}_{2}) \xrightarrow{d_{2}} H^{2}(\mathbb{Z}_{2}^{n-1}, \mathbb{F}_{2}) \xrightarrow{p^{*}} H^{2}(\Gamma, \mathbb{F}_{2}),
\]

where \( d_{2} \) is a transgression and \( p^{*} \) is induced by the above homomorphism \( p : \Gamma \rightarrow (\mathbb{Z}_{2})^{n-1} \). In what follows we shall prove (see also [2, Theorem 2.7]) that a rank of

\[
\text{Im}(d_{2}) \subset H^{2}(\mathbb{Z}_{2}^{n-1}, \mathbb{F}_{2}) \subset H^{*}(\mathbb{Z}_{2}^{n-1}, \mathbb{F}_{2}) \simeq \mathbb{F}_{2}[x_{1}, x_{2}, \ldots, x_{n-1}]
\]

is equal to \( n \).

Let us define a basis \( \hat{t}_{i}, i = 1, 2, \ldots, n \) of \( H^{1}(\mathbb{Z}^{n}, \mathbb{F}_{2}) = \text{Hom}(\mathbb{Z}^{n}, \mathbb{F}_{2}) \). For \( k \in \mathbb{Z} \), we shall write \( \bar{k} = 0 \) if \( k \) is even and \( \bar{k} = 1 \) if \( k \) is odd. Let \( (k_{1}, k_{2}, \ldots, k_{n}) \in \mathbb{Z}^{n} \) and let

\[
\hat{t}_{i}(k_{1}, k_{2}, \ldots, k_{n}) = \bar{k}, i = 1, 2, \ldots, n.
\]

We have

**Proposition 2.** \( d_{2}(\hat{t}_{i}) = T_{i} = (p^{(2)} \circ pr_{i}) \cup (p^{(3)} \circ pr_{i}) \). Moreover elements \( T_{i}, i = 1, 2, \ldots, n \) are a basis of \( \text{Im}(d_{2}) \).

**Proof.** By Theorem 2.5 (ii) and Proposition 1.3 of [2] and using (10) it follows that

\[
d_{2}(\hat{t}_{i}) = \sum_{A_{il}=1} x_{i}^{2} + \sum_{i \neq j} x_{i} x_{j},
\]

where the second sum is taken for such \( i, j \) that

\[
(A_{il}, A_{jl}) \in \{(1, 2), (2, 1), (1, 3), (3, 1), (3, 2), (2, 3)\}.
\]

On the other hand

\[
T_{i} = p_{l}^{(2)} p_{l}^{(3)} = \sum_{i=1}^{n-1} p^{(2)}(A_{il})p^{(3)}(A_{il})x_{i}^{2} + \sum_{1 \leq i < j \leq n-1} (p^{(2)}(A_{il})p^{(3)}(A_{jl}) + p^{(2)}(A_{jl})p^{(3)}(A_{il})x_{i}x_{j}.
\]

Comparing coefficients of the above two polynomials finishes the proof. \( \square \)

The main idea of the proof of rigidity is an application of the above Proposition 2. It means, we show that any \( \text{HW-manifold} \ M \), of dimension greater than three, define
elements in the cohomology ring $H^*(M, \mathbb{F}_2)$ which determines $M$ up to affine equivalence. In the **Main Lemma**, we shall prove an existence of $n$ linear independence elements $T_1, T_2, \ldots, T_n \in \text{Im}(d_2)$ such that for any $i = 1, 2, \ldots, n$ $T_i = p_i q_i$. At the end of this section we give a method of a reconstruction of **HW-group** from the set $\{T_i\}_{i=1,2,\ldots,n}$.

Let us define

$$D = \{y \in \text{Im}(d_2) \mid y \text{ is a product of two polynomials of degree 1}\}. \tag{13}$$

We shall prove that $D$ has less than $n + 2$ elements from which we can reconstruct the basis $T_1, T_2, \ldots, T_n$ of $\text{Im}(d_2)$.

**Main Lemma.** Let $n > 3$, then there are the following possibilities for the structure of the set $D$:

1. $D = \{T_1, T_2, \ldots, T_n\}$;
2. $D = \{T_1, T_2, \ldots, T_n, T_i + T_j\}$, and we can find a polynomial $p$ of degree one such that $p \mid T_i$ and $p \mid T_j$ for some $1 \leq i, j \leq n$. In the second case we can rediscover the set of generators $T_1, T_2, \ldots, T_n$. \hfill $\square$

Let $M$ be **HW-manifold** of dimension $n$. From the **Main Lemma**, we know that there is a set $D = \{T_1, T_2, \ldots, T_n\} \subset \text{Im}(d_2)$ such that any $T_i$ is a product of two polynomials $p_i$ and $q_i$, $i = 1, 2, \ldots, n$ of a degree one. Let $V$ be $(n - 1)$-dimensional $\mathbb{F}_2$ vector space. We define a linear map $h : V^* \to D^n$, which simple version is (7) such that

$$h_i(x) = p_i(x)2 + q_i(x)3, \text{ for } i = 1, 2, \ldots, n, \tag{14}$$

where $p_i, q_i \in V \simeq V^{**}$. Hence, through formulas (10), (12) and the Table 1, $\text{Im}(h)$, defines a Hantzsche–Wendt group.

**Example 1.** 1. Let $V = \text{gen}\{x_1, x_2, x_3\}$ and $D = \{x_1^2 + x_1 x_2, x_1 x_2 + x_1 x_3 + x_2^2 + x_2 x_3\}$. Put $p_1 = x_1, q_1 = x_1 + x_2, p_2 = x_1 + x_2, q_2 = x_2 + x_3$. Hence a homomorphism $h(x_1^*) = (1, 2), h(x_2^*) = (3, 1)$ and $h(x_3^*) = (0, 3)$. Here $x_1^*, x_2^*, x_3^*$ is a dual basis of $V^*$. Finally we define a subgroup of $D^2$ which generators are rows of the matrix

$$\begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 0 & 3 \end{bmatrix}.$$

2. Let $\mathbb{Z}_2^{n-1} \subset D^n$ be a **HW-group**, and $D$ a set from the **Proposition 2**. Assume that $D = \{p_1 q_1, p_2 q_2, \ldots, p_n q_n\}$. Then

$$h_i(x) = p_i(x)2 + q_i(x)3 = p(x_i)2 + q(x_i)3 = x_i.$$  

Hence for $x \in \mathbb{Z}_2^{n-1}, h(x) = x$ and $\text{Im}(h) = \mathbb{Z}_2^{n-1}$. 


Let $\phi : H^*(M_1, \mathbb{F}_2) \to H^*(M_2, \mathbb{F}_2)$ be an isomorphism of cohomology rings of $HW$-manifolds $M_1$ and $M_2$. From the Main Lemma for the both manifolds we have the sets of elements $D_1$ and $D_2$ such that $\phi(D_1) = D_2$. Hence we obtain the affine equivalence manifolds $M_1$ and $M_2$. $\square$

3. Properties of Hantzsche–Wendt matrices

Let us illustrate the Proposition 1 on two $HW$-manifolds of dimension 5, (see [8]). We shall denote by $\Gamma_1$ and $\Gamma_2$ its fundamental groups.

Example 2. A group $\Gamma_1 \subset E(5)$ is generated by

$$(B_1, (1/2, 1/2, 0, 0, 0)), (B_2, (0, 1/2, 1/2, 0, 0)),$$

$$(B_3, (0, 0, 1/2, 1/2, 0)), (B_4, (0, 0, 0, 1/2, 1/2)).$$

From above $\mathbb{R}^5/\Gamma_1 \simeq T^5/(\mathbb{Z}_2)^4$, where $(\mathbb{Z}_2)^4 \subset D^5$ is defined by

$$(g_1, g_3, g_2, g_2, g_2), (g_2, g_1, g_3, g_2, g_2),$$

$$(g_2, g_2, g_1, g_3, g_2), (g_2, g_2, g_1, g_3).$$

Moreover a group $\Gamma_2 \subset E(5)$ is generated by

$$(B_1, (1/2, 0, 1/2, 1/2, 0)), (B_2, (0, 1/2, 1/2, 1/2, 2/2)),$$

$$(B_3, (1/2, 1/2, 1/2, 1/2, 1/2)), (B_4, (1/2, 0, 1/2, 1/2, 1/2)).$$

Hence, $\mathbb{R}^5/\Gamma_2 \simeq T^5/(\mathbb{Z}_2)^4$ where generators of a group $(\mathbb{Z}_2)^4 \subset D^5$ are following

$$(g_1, g_2, g_3, g_3, g_2), (g_2, g_1, g_3, g_3, g_3),$$

$$(g_3, g_3, g_1, g_3, g_3), (g_3, g_2, g_3, g_1, g_3).$$

In what follows we shall write $i$ for $g_i$, $i = 0, 1, 2, 3$. Let $A$ be a $(n \times m)$ matrix with coefficients $A_{ij} \in D$. For short $A \in D^{n \times m}$. Let $A_i$ ($A^j$) denote $i$-row ($j$-column) of a matrix $A$.

Definition 2. By $HW$-matrix we shall understand a matrix $A \in D^{n \times n}$ such that $A_{ii} = 1$, $A_{ij} \in \{2, 3\}$ for $i \neq j$, $1 \leq i, j \leq n$ and if $X \subset \{1, 2, \ldots, n\}$ and $1 \leq \#X \leq n - 1$ then the row $\sum_{i \in X} A_i$ has 1 on a some position.

Lemma 1. Any $HW$-manifold of dimension $n$ defines a $(n \times n)$ $HW$-matrix.

Proof. Let $(\beta_i, b_i)$, $1 \leq i \leq n - 1$ be generators of the fundamental group of some $n$-dimensional $HW$-manifold $M$. Then $i$-generator defines $i$-row of some $(n \times n)$
**HW-matrix**, cf. (2), (4). See also Example 2 and Proposition 1. The last row is defined by the product $\beta_1\beta_2\ldots\beta_{n-1}$ or equivalently is a sum of the first $(n - 1)$ rows. It is easy to see that the first property of the above matrix follows from a definition, see [5, p. 4]. Since a holonomy group $(\mathbb{Z}_2)^{n-1}$ acts free on $T^n$ (or equivalently $\pi_1(M)$ is a torsion free group) the last part of lemma follows. □

We shall present some properties of **HW-matrices**.

**Remark 1.** Let $\sigma \in S_n$ and let $P_\sigma$ be the corresponding permutation matrix. It is not difficult to see that if $A$ is **HW-matrix** then $P_\sigma A P_\sigma^{-1}$ also satisfies conditions of the Definition 2. Moreover, if $A'$ is a conjugation matrix of $A$, where conjugation means exchange at some column numbers 2 for 3, then $A'$ is again a **HW-matrix**. The **HW-matrix** is related to the matrix defined on page 6 of [5].

**Remark 2.** Let $A$ be a $(n \times n)$ **HW-matrix**. Then

$$(p^{(2)} + p^{(3)})(A) = \begin{bmatrix} 0 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 0 & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \cdots & 1 & 0 & 1 \\ 1 & 1 & \cdots & 1 & 1 & 0 \end{bmatrix}. \quad (15)$$

Let $A \in \mathcal{D}^{m \times n}$ be a $(m \times n)$ matrix with coefficients in $\mathcal{D}$ and $(\alpha_1, \alpha_2, \ldots, \alpha_n) \in \{2, 3\}^n$. By $p^{(\alpha)}(A)$ we shall understand a $(m \times n)$-matrix with coefficients in $\mathbb{F}_2$ which a $i$-column is equal to $p^{(\alpha_i)}(A^i)$.

Let $M$ be a matrix. By defect of $M$ we shall understand a number

$$d(M) = \{\text{number of columns of } M\} - \text{rk}(M).$$

**Lemma 2.** 1. Let $M_1$ be a matrix $M$ from which we remove some columns. Then

$$d(M_1) \leq d(M).$$

2. If $A$ is a **HW-matrix** of dimension $n$ and $\alpha \in \{2, 3\}$, then

$$d(p^{(\alpha)}(A)) \leq 1.$$ 

**Proof.** The first statement is clear. For the proof of a second one, let us assume that $d(p^{(\alpha)}(A)) > 1$. Hence $\text{rk}(p^{(\alpha)}(A)) < n - 1$. By definition there exists a non-trivial $X \subset \{1, 2, \ldots, n - 1\}$, such that $\sum_{i \in X} p^{(\alpha)}(A_i) = 0$. Finally $p^{(\alpha)}(\sum_{i \in X} A_i) = \sum_{i \in X} p^{(\alpha)}(A_i) = 0$. This contradicts the Definition 2. □

**Lemma 3.** Let $m < n$ and $W \in \mathcal{D}^{m \times n}$ is a sub-matrix of some $(n \times n)$ **HW-matrix**. Then $\text{rk}(p^{(\alpha)}(W)) = m$. 


Proof. Similar to the proof of the last Lemma. □

A symmetric \((m \times m)\) matrix \(A \in (\mathbb{F}_2)^{m \times m}\) defines a nonoriented graph, \(\text{graph}(A)\) with set of vertices \(\{1, 2, \ldots, m\}\) and two different vertices \(i\) and \(j\) are connected if and only if \(A_{ij} = 1\). We say that a matrix \(A\) is connected if a graph(\(A\)) is connected. Let \(A \in \mathcal{D}^{m \times m}\) be a symmetric matrix, then \(p^{(i)}(A)\) are symmetric with coefficients in \(\mathbb{F}_2\), \(i = 2, 3\). We shall write \(i \sim_2 j\) if \(i, j\) are at the same connected component of a matrix \(p^{(2)}(A)\). Similar definition is for a relation \(i \sim_3 j\).

Lemma 4. Let a HW-matrix \(M\) have the following decomposition on the blocks:

\[
M = \begin{bmatrix}
* & 2 & * \\
C & A & D \\
* & 3 & *
\end{bmatrix}, \quad (16)
\]

where \(A\) is a symmetric matrix and 2, 3 are block matrices with all rows and columns equal 2 and 3 correspondingly. Then

(I) if \(i \sim_2 j\) \(\implies\) \(D_i = D_j\);
(II) if \(i \sim_3 j\) \(\implies\) \(C_i = C_j\).

Proof. For the proof of (I) let us assume that \(i, j\) (where \(i < j\)) are connected by a 2-edge; i.e. \(A_{i,j} = 2\). Let \(r\) be some column of a matrix \(D\). Let us consider a diagonal submatrix of the matrix \(M\) related to \((i, j, r)\). It looks like

\[
\begin{bmatrix}
1 & 2 & a \\
2 & 1 & b \\
3 & 3 & 1
\end{bmatrix}, \quad (17)
\]

The sums of the first two columns are zero. Since Lemma 3 a sum of elements of the last one is not zero. Hence \(a = b\). We have just proved that if \(A_{i,j} = 2\) then \(D_i = D_j\). It also means that if \(i \sim_2 j\) then \(D_i = D_j\). The proof of the second point of the lemma is similar. □

The next lemmas are about possibilities of complement of some matrices to a HW-matrix. We shall first consider an odd case.

Lemma 5. Let \(A \in \mathcal{D}^{m \times m}\) be a symmetric matrix with 1 on the diagonal and \(\{2, 3\}\) off the diagonal with a column sums equal to 1. Assume that \(m > 1\). Then a matrix

\[
K_A = \begin{bmatrix}
2 \\
A \\
3
\end{bmatrix}, \quad (18)
\]

cannot be complement to HW-matrix.

Proof. Let us assume that there exists a HW-matrix
From assumption $m$ is an odd number and heights of the blocks 2 and 3 are also odd. We shall use induction. For $m = 3$

$$A = \begin{bmatrix} 1 & a & a \\ a & 1 & a \\ a & a & 1 \end{bmatrix}.$$  \hfill (20)

Here $a = 2$ or 3. If $a = 3$ then $\text{rk}(p^{(2)}(A)) = 1$ and $d(p^{(2)}(A)) = 3 - 1 = 2 > 1$. From Lemma 2 it is impossible. For $a = 3$ the proof is the same. Let us assume that $m > 3$.

1. We shall consider a matrix $p^{(2)}(A)$. We claim that there is no such decomposition as

$$p^{(2)}(A) = B \oplus E,$$

such that a dimension of a matrix $B$ is odd and $> 1$. In fact, in that case

$$A = \begin{bmatrix} \tilde{B} & 3 \\ 3 & E \end{bmatrix}.$$  \hfill (21)

Since a column sums of $A$ are equal to 1 and height of a block 3 under $\tilde{B}$ is even, a column sums of $\tilde{B}$ are 1. If $K_A$ has complement then $K_{\tilde{B}}$ has a complement (where a dimension of a block 3 is greater on a dimension of $E$). But by induction it is impossible, since $1 < \text{dimension}(\tilde{B}) < m$.

2. We claim that there is no such a nontrivial decomposition as

$$p^{(2)}(A) = B \oplus E \oplus F.$$ 

In fact since $m$ is odd we have two possibilities:

(a) dimension of one component is odd and other components have dimension even
(b) dimension of all components are odd.

In the case (a) $\text{dim}(B \oplus E) > 1$ and odd. Hence we consider decomposition $p^{(2)}(A) = (B \oplus E) \oplus F$. But it is a previous case 1.

In case (b), since $m > 3$ there exists a component (for example $B$) which dimension is $> 1$. In that case we have a decomposition $p^{(2)}(A) = B \oplus (E \oplus F)$ which was already considered in the point 1.

3. By definition we have a decomposition

$$p^{(2)}(A) = B_1 \oplus \ldots \oplus B_s,$$

where all components are connected matrices. From the above we can assume that $s = 2$ and odd component has a graph equal to one point or $s = 1$. Equivalently,
(a) \(A = \begin{bmatrix} 1 & 3 \\ 3 & B \end{bmatrix}\) and \(p^{(2)}(B)\) is connected or
(b) \(p^{(2)}(A)\) is connected.

In the first case

\[ p^{(3)}(A) = \begin{bmatrix} 1 & p^{(3)}(B) \\ 1 & 1 \end{bmatrix}. \]  

Hence \(p^{(3)}(A)\) is connected. Summing up, we have

(a) \(A = \begin{bmatrix} 1 & 3 \\ 3 & B \end{bmatrix}\) and both \(p^{(2)}(B)\) and \(p^{(3)}(A)\) are connected or
(b) \(p^{(2)}(A)\) is connected.

If we exchange \(p^{(2)}\) for \(p^{(3)}\) in the above points 1., 2. and 3. with the similar arguments, we obtain finally two cases:

(a) \(A = \begin{bmatrix} 1 & 3 \\ 3 & B \end{bmatrix}\) and both \(p^{(2)}(B)\) and \(p^{(3)}(A)\) are connected or
(b) both \(p^{(2)}(A)\) and \(p^{(3)}(A)\) are connected.

We come back to the beginning of the proof. We shall try to figure out matrices \(C\) and \(D\). From definition of \(\sim_3\) and because \(p^{(3)}(A)\) is connected we conclude that all rows of the matrix \(C\) are identical. By conjugation we can assume that \(C = 2\). Using the same arguments and definition of \(\sim_2\) together with a connectedness of \(p^{(2)}(B)\) we conclude that with exception of the first row, all rows of the matrix \(D\) are the same. By conjugation and permutation we can assume that the first row of the matrix \(D\) is equal to \([2, ..., 2, 3, ..., 3]\). All other rows of a matrix \(D\) consist only 3. Summing up a matrix

\[ W = \begin{bmatrix} C & A & D \end{bmatrix} \]

is following

\[ \begin{bmatrix} 2 & \begin{bmatrix} 1 & 3 \\ 3 & B \end{bmatrix} & 2 & 3 \\ 2 & 3 & 3 \end{bmatrix}. \]  

Apply homomorphisms: \(p^{(3)}, [p^{(2)}, p^{(3)}], p^{(2)}, p^{(2)}\) to the corresponding columns we get a matrix

\[ W' = \begin{bmatrix} 0 & \begin{bmatrix} 1 & 1 \\ 0 & p^{(3)}(B) \end{bmatrix} & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \]  

We have \(\text{rk}W' = 1 + \text{rk}(p^{(3)}(B))\). From assumption sums of columns of a matrix \(A\) are equal to 1. Hence sums of columns of a matrix

\[ (p^{(2)}, p^{(3)})A = \begin{bmatrix} 1 & p^{(3)}(B) \\ 0 & 1 \end{bmatrix} \]  

(25)
are also equal to 1 and sums of columns of a matrix $p^{(3)}(B)$ are equal to 0. It means $\text{rk}(p^{(3)}(B)) < m - 1$ and also $\text{rk}(W') < m$. From Lemma 3

$$\text{rk}(W') = \text{rk}(W) = \text{number of rows} \ (W) = m.$$ 

Hence a matrix $W$ cannot be a matrix of some rows of $\text{HW-matrix}$.

We have to still consider a case when matrices $p^{(2)}(A)$ and $p^{(3)}(A)$ are connected. Similar to the above consideration, using relation $\sim_2$ and $\sim_3$ plus conjugation we can assume that

$$[C \ A \ D] = [2 \ A \ 3].$$ 

Hence all nonempty sums of rows of a matrix $A$ include 1. For $m > 1$ it is impossible.  

The next lemma is an even version of the Lemma 5.

**Lemma 6.** Let $A \in D^{m \times m}$ be a symmetric matrix with 1 on the diagonal and \{2, 3\} off the diagonal with a column sums equal to 3. Assume that $m > 1$. Then a matrix

$$K_A = \begin{bmatrix} 2 \\ A \\ 3 \end{bmatrix},$$

cannot be a complement to some $\text{HW-matrix}$.

**Proof.** As in the proof of the previous lemma let us assume that there exists a $\text{HW-matrix}$

$$\begin{bmatrix} * & 2 & * \\ C & A & D \\ * & 3 & * \end{bmatrix}.$$ (27)

From assumption and Definition 2 $m$ is an even number and a height of the block 2 is even and 3 is odd. We shall use induction. For $m = 4$.

1. On the beginning let us consider the case, where $p^{(2)}(A)$ is not connected. We have two cases of matrices of dimension 4:

(a) $A = \begin{bmatrix} 1 & 3 \\ 3 & B \end{bmatrix}$, where $B$ has a dimension 3 and

(b) $A = \begin{bmatrix} B & 3 \\ 3 & E \end{bmatrix}$, where matrices $A, B$ have rank two.

The case (a) is impossible since $1 + 3 \neq 3$. In the case (b) matrices $A$ and $B$ are symmetric with columns sums equal to 3. Hence $B = E = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$, and

$$p^{(2)}(A) = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$ (28)
From the other side a matrix \( p^{(2)}(K_A) \) has rows of 1 \( (p^{(2)}(2) = 1) \) and rows of 0 \( (p^{(2)}(3) = 0) \). These rows are linear combination of rows of \( p^{(2)}(A) \) and

\[
\text{rk} p^{(2)}(K_A) = \text{rk} p^{(2)}(A) = 2.
\]

Finally \( d(K_A) = 4 - 2 = 2 > 1 \) and from Lemma 2 we are done.

2. As the second step let us consider the case where \( p^{(3)}(A) \) is not connected. We have to consider two cases of matrices of dimension 4:

(a) \( A = \begin{bmatrix} 1 & 2 \\ 2 & B \end{bmatrix} \), and

(b) \( A = \begin{bmatrix} B & 2 \\ 2 & E \end{bmatrix} \), and \( B \) and \( E \) have dimension 2.

In the case (a) a matrix \( B \) is symmetric of dimension 3 with sums of columns 1. If \( K_A \) has complement to \( HW\)-matrix then also a matrix \( K_B \) has this possibility. But it is impossible by Lemma 5. In case (b) matrices \( B, E \) are symmetric with sums of columns 3. Hence \( B = E = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \) and

\[
p^{(2)}(A) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}.
\] (29)

In the matrix \( p^{(2)}(K_A) \) we have rows of 1 and 0. They are linearly dependent from the rows of \( p^{(2)}(A) \). Hence

\[
\text{rk} p^{(2)}(K_A) = \text{rk} p^{(2)}(A) = 1
\]

and

\[
d(K_A) = 4 - 1 = 3 > 1.
\]

From Lemma 2 the matrix \( K_A \) has not complement to the \( HW\)-matrix.

3. By the above points 1. and 2. we have that \( p^{(2)}(A) \) and \( p^{(3)}(A) \) are connected matrices. As in the proof of Lemma 5 using relations \( \sim_2, \sim_3 \) and conjugations of matrices we can assume that

\[
[C \ A \ D] = [2 \ A \ 3].
\]

By assumption a sum of all rows of the above matrix has 1 on a some position. We can see easily that it is impossible at the first and the third block. For a matrix \( A \) it is also impossible since \( m \) is even. This contradicts our assumption that \( m < n \).

Let us assume that \( m > 4 \). We shall consider three steps.

1. Assume that \( p^{(2)}(A) \) is not connected. We have to consider two cases:
(a) \( p^{(2)}(A) \) is a direct sum of two odd blocks,
(b) \( p^{(2)}(A) \) is a direct sum of two even blocks.

Hence \( A = \begin{bmatrix} B & 3 \\ 3 & E \end{bmatrix} \). In the case (a) since dimensions of \( B, E \) are odd and sums of column of \( A \) are 3 we obtain that sums of column of \( B \) and \( E \) are 0. Moreover, if \( B \) is an odd diagonal submatrix of \( HW\)-matrix then by Definition 2 a sum of rows of \( B \) should enclose 1. But this is impossible and also case (a) is impossible.

In case (b) since dimensions of \( B, E \) are even and sums of column of \( A \) are 3 we obtain that sums of column of \( B \) and \( E \) are 3. Moreover either the matrix \( B \) or the matrix \( E \) has rank > 2. Assume the matrix \( B \) has such a property. If a matrix \( K_A \) has complement, then a matrix \( K_B \) has complement to \( HW\)-matrix. But by induction it is impossible.

2. Assume that \( p^{(3)}(A) \) is not connected. We have to consider two cases. The same as in the step 1.

(a) \( p^{(3)}(A) \) is a direct sum of two odd blocks,
(b) \( p^{(3)}(A) \) is a direct sum of two even blocks.

Hence \( A = \begin{bmatrix} B & 2 \\ 2 & E \end{bmatrix} \). In the first case since dimensions of \( B, E \) are odd and sums of column of \( A \) are 3 we obtain that sums of column of \( B \) and \( E \) are 1. Moreover, either the matrix \( B \) or the matrix \( E \) has rank > 2. Assume the matrix \( B \) has such a property. If a matrix \( K_A \) has complement then (after permutation of indexes) a matrix \( K_B \) has complement to \( HW\)-matrix. But by Lemma 5 it is impossible. In the second case, since dimensions of \( B, E \) are even and sums of column of \( A \) are 3 we obtain that sums of column of \( B \) and \( E \) are 3. Moreover, either the matrix \( B \) or \( E \) has rank > 2. Assume the matrix \( B \) has such a property: If a matrix \( K_A \) has complement then a matrix \( K_B \) has complement to \( HW\)-matrix. But by induction it is impossible.

We can assume that matrices \( p^{(2)}(A) \) and \( p^{(3)}(A) \) are connected. As in the previous cases we can assume that

\[
[C \ A \ D] = [2 \ A \ 3].
\]

By Definition 2 a sum of all rows should enclose 1. Since \( m \) is even and \( m < n \) we have a contradiction. □

4. Proof of the Main Lemma

We keep the notation from previous sections, but we also need a new definitions. Denote by \( \mathcal{P}_n \) an algebra of all subsets of the set \( \{1, 2, \ldots, n\} \). Let \( |U| \) denote the number of elements of a set \( U \in \mathcal{P}_n \) modulo two. We have an isomorphism of algebras \( I : \mathbb{F}_2^n \to \mathcal{P}_n \), where

\[
I(x) = \{i \mid x_i = 1\}, x \in \mathbb{F}_2^n
\]  \hspace{1cm} (30)

is an indicator.
Definition 3. Let $A$ be a $HW$-matrix. The function $J : \mathcal{P}_n \rightarrow \mathcal{P}_n$ is defined by

$$J(U) = \{ s \mid \sum_{i \in U} A_{is} = 1 \},$$

where $U \in \mathcal{P}_n$.

Remark 3. In what follows we shall use a formula (10) with a basis $b_i, 1 \leq i \leq n - 1$. Let us consider a map $l : \mathcal{P}_n \rightarrow \mathbb{F}_2[x_1, \ldots, x_{n-1}]$ given by a formula

$$l_Z := \sum_{i \in Z} x_i.$$  

In this language the formula (10) for $k = 2, 3$ we can write as

$$\sum_{j=1}^{n-1} p^{(k)} A_{j} x_j = l_S$$

where $S = \{ p^{(k)}(A_{1,i}), p^{(k)}(A_{2,i}), \ldots, p^{(k)}(A_{n-1,i}) \}$.

Proposition 3. The map $J$ has the following properties:

1. $U \neq 0, 1$ then $J(U) \neq 0$, here 0, 1 denote the trivial additive and multiplicative element of the algebra $\mathcal{P}_n$ respectively;
2. $J(U + 1) = J(U)$ where $U + 1 = U'$ denotes a complement of the subset $U$ in the set $
\{1, 2, \ldots, n\}$;
3. $J(\{i\}) = \{i\}, i = 1, 2, \ldots, n$;
4. if $|U| = 1$ then $J(U) \subset U$;
5. if $|U| = 0$ then $J(U) \subset U'$.

Proof. Elementary calculations with support of the matrix (15). □

Any polynomial of $\mathbb{F}_2[x_1, x_2, \ldots, x_n]$ we shall identify with a polynomial map $\mathbb{F}_2^n \rightarrow \mathbb{F}_2$. Hence by indicator function (30) the formula (32) has the following presentation $l_Z(e_j) = \{ j \in Z \}$, where $Z \in \mathcal{P}_n$. Since the transgressive elements $T_i \in \mathbb{F}_2[x_1, \ldots, x_{n-1}]$ we define a split monomorphism of rings $\mathbb{F}_2[x_1, \ldots, x_{n-1}] \stackrel{\phi}{\rightarrow} \mathbb{F}_2[x_1, \ldots, x_n]$ such that $T_i = \phi(T_i) \in \mathbb{F}_2[x_1, \ldots, x_n], i = 1, \ldots, n$. Here, $\phi(x_i) = x_i + x_n, i = 1, 2, \ldots, n - 1$. Obviously $\# D = \# \phi(D)$.

From definition, for polynomial functions $\bar{T}_i$ we have $\bar{T}_i(e_j) = \delta_{ij}$, where $1 \leq i, j \leq n$ and $e_i \in (\mathbb{F}_2)^n$ is the standard basis. Hence, by the isomorphism (30) a map $J$ (see Definition 3) is equivalent to a function $T : \mathbb{F}_2^n \rightarrow \mathbb{F}_2^n, T(x) = (\bar{T}_1(x), \bar{T}_2(x), \ldots, \bar{T}_n(x))$, where $x \in \mathbb{F}_2^n$. Hence and from an equation (12) we have a commutative diagram
We shall use these observations in the proof of the Main Lemma. Moreover, we shall apply a remark that homogeneous polynomials of degree 2 are recognized by their polynomial functions. Let $S, Z_1, Z_2 \in \mathcal{P}_n$. From definition if
\[
\sum_{i \in S} \bar{T}_i = l_{Z_1} \cdot l_{Z_2}
\]
then $S = Z_1 \cap Z_2$.

**Proposition 4.** The following conditions are equivalent.

1. $\sum_{i \in S} \bar{T}_i = l_{Z_1} \cdot l_{Z_2}$
2. $\forall U \in \mathcal{P}_n |J(U)S| = |UZ_1| \cdot |UZ_2|$.

**Proof.** We shall use (33) and an isomorphism $I$. Let $x \in \mathbb{F}_2^n, U = I(x)$. We have
\[
\sum_{i \in S} \bar{T}_i(x) = \sum_{i \in S \cap I(T(x))} 1 = |I(T(x)) \cap S| = |J(I(x)) \cap S| = |J(U) \cap S|.
\]
From the other side
\[
l_{Z_1}(x) \cdot l_{Z_2}(x) = \sum_{i \in Z_1 \cap I(x)} 1 \cdot \sum_{i \in Z_2 \cap I(x)} 1 = |UZ_1| \cdot |UZ_2|.
\]
This finishes a proof. \(\square\)

**Corollary 1.** Let us assume the condition (ii) of Proposition 4, then

1. $|Z_1|$ or $|Z_2|$ is even,
2. if $S \neq 0$ then $|Z_1|$ and $|Z_2|$ are even
3. if $S \neq Z_1$ and $S \neq Z_2$ then $Z_1 \cup Z_2 = 1$.

**Proof.** 1. Since $J(1) = J(\{1, 2, \ldots, n\}) = 0$ the condition is true.
2. Since $J(U) = J(U') = J(U + 1)$ we have
\[
|UZ_1||UZ_2| = |(1 + U)Z_1||(1 + U)Z_2|.
\]
Hence
|Z_1||Z_2| + |Z_1||UZ_2| + |Z_2||UZ_1| = 0.

From a point 1. we can assume that |Z_1| = 0 (or |Z_2| = 0) and |Z_2||UZ_1| = 0. If |Z_2| = 1 then \(\forall U \in \mathcal{P}_n, |UZ_1| = 0\) and \(Z_1 = 0\). Since \(S = Z_1 \cdot Z_2 \neq 0\) we have a contradiction.

3. Let \(a \in Z_1 \setminus S, b \in Z_2 \setminus S\) and \(c \notin Z_1 \cup Z_2\). Put \(U = \{a, b, c\}\). We have \(J(U)S \subset US = 0\) and \(UZ_1 = \{a\}, UZ_2 = \{b\}\). Hence

\[0 = |J(U)S| = |UZ_1||UZ_2| = 1 \cdot 1 = 1.\]

This is a contradiction. \(\square\)

**Definition 4.** Define

\[\sigma_a^S := \sum_{i \in S} A_{a,i},\]

where \(a \in \{1, 2, \ldots, n\}, S \subset \{1, 2, \ldots, n\}\) and \(A \in \mathcal{D}^{n \times n}\).

Let us present relations between the above definition and the function \(J\).

**Proposition 5.** Let \(A\) be \((n \times n)\) HW-matrix, \(a, b \in \{1, 2, \ldots, n\}\) and \(S \in \mathcal{P}_n\). Then

1. \(|J(\{a, b\})S| = \sigma_a^S + \sigma_b^S\), where \(a, b \notin S\);
2. \(|J(\{a, b\})S| = \sigma_a^S + \sigma_b^S + A_{a,b} + 1\), where \(a \notin S, b \in S\).
3. \(|J(\{a, b\})S| = \sigma_a^S + \sigma_b^S + A_{a,b} + A_{b,a}\), where \(a, b \in S\).

**Proof.** 1. By a point 5. of Proposition 3 we know that \(J(\{a, b\}) \subset \{a, b\}'\). If \(J(\{a, b\})S = \emptyset\) we are done. On the contrary we shall consider the following cases.

(a) Assume \(|S| = 1\) and \(|J(\{a, b\})S| = 1\). We have two rows, which correspond to \(a\) and \(b\),

\[
\begin{array}{ccc}
2 & 2 & \ldots & 2 & 2 \\
2 & 3 & \ldots & 3 & 2 \\
\end{array}
\]  

(34)

with a number of columns equal to \(|S|\), and a number of columns with different coefficients equal to \(J(\{a, b\})\). Hence a sum of the upper row is equal to 2 and a sum of the down row is equal to 3. This finishes a proof in this case.

(a') Assume \(|S| = 1\) and \(|J(\{a, b\})S| = 0\). We also have (34) and a sum of the upper row is equal to 2 and a sum of the down row is also equal to 2. This finishes a proof in this case.

(b) Assume \(|S| = 0\). Then again we have two subcases \(|J(\{a, b\})S| = 1\), then a sum of the upper row of (34) is equal to 0 and a sum of the down row is equal to 1. The proof
of the case is complete. When \(|J(\{a, b\})S| = 0\) a sum of the upper row of (34) is 0 and a sum of the down row is also 0. This finished a proof of point 1. The proofs of other cases are similar and we put it as an exercise. \(\square\)

Using the above language we shall prove that for a \(HW\)-manifold there exists only a limited number of transgressive elements which are a product of degree one nontrivial polynomials.

**Proposition 6.** Let \(A\) be a \((n \times n)\) \(HW\)-matrix, \((n > 3)\) then there does not exist not empty set \(S \subset \{1, 2, \ldots, n\}\) such that

\[
\forall U \in \mathcal{P}_n |J(U)S| = |US|. \tag{35}
\]

**Proof.** It is the case \(S = Z_1 = Z_2\). Let us assume (35). We are going to divide the proof into four steps.

**Step 1.** We claim that, if \(a_1, a_2 \notin S\) and \(b \in S\) then \(A_{a_1, b} = A_{a_2, b}\). In fact, from (35) for \(U = \{a_1, a_2\}\), \(|J(\{a_1, a_2\})S| = |\{a_1, a_2\}S| = 0\). By Proposition 5 (1.), \(\sigma_{a_1}^S = \sigma_{a_2}^S := \sigma\). If \(a \notin S\) then from Proposition 5 (2.)

\[
1 = |J(\{a, b\})S| = |\{a, b\}S| = \sigma_a^S + \sigma_b^S + A_{a, b} + 1 = \sigma + \sigma_b^S + A_{a, b} + 1.
\]

Hence \(\forall a \notin S, A_{a, b} = \sigma + \sigma_b^S\) and Step 1 is proved.

**Step 2.** We claim that, if \(US = 0\) then \(J(U)S = 0\). In fact from Step 1, all elements (numbers of columns) of \(J(U)\) which are considered have not the first indexes from \(S\) and are equal each other. Then \(J(U)S = 0\).

**Step 3.** We claim that, if \(S \neq 0\) then \(#S = n - 1\). From Step 2, if \(0 \neq U \subset S'\) then \(J(U)S' \neq 0\). Let \(B\) be a diagonal submatrix of the matrix \(A\) related to the set \(S'\). Then \(B\) is a quadratic matrix with 1 on the diagonal and 2, 3 otherwise. Moreover all sums of rows of \(B\) have at some position an element 1. Hence, the only possible matrix \(B\) is \((1 \times 1)\) matrix.

**Step 4.** We claim that, if \(S \neq 0\) then \(n \leq 3\). For the proof, let us assume that \(n > 3\). From the Step 3, we can assume that \(S = \{2, 3, \ldots, n\}\). Let \(l_2\) denote a number of 2 at the first column of \(A\). We shall prove that \(|l_2| = 0\). In fact, we can assume that \(0 < l_2 < n - 1\) and at the first column, from the top we have first 2 then going down we have 3. On the contrary, suppose that \(l_2\) is odd and let \(v\) be a sum of the first \(2l_2 + 1\) rows. Since \(l_2 + 1\) is even \(v\) has not 1 on places \(1, 2, \ldots, l_2 + 1\). Then it has 1 on the position \(l_2 + 1\). Hence there exists \(k \geq l_2 + 1\) such that \(A_{1, r} \neq A_{k, r}\) or equivalently \(A_{1, r} + A_{k, r} = 1\). Let us consider a diagonal submatrix

\[
\begin{bmatrix}
1 & A_{1, r} \\
2 & A_{k, r} \\
3 & 1
\end{bmatrix}.
\]
A sum of elements at the first column and at the third column is 0, then it at the second column has to be \( \neq 0 \). Let \( U = \{1, k, r\} \). Since \( j(U) \subset U \) and \( n > 3 \), \( J(U) = \{k\} \). Finally

\[
1 = |\{k\} S| = |J(U) S| = |US| = |\{k, r\} S| = 0.
\]

That is a contradiction and \( l_2 \) is even. Moreover if \( l_3 \) is a number of 3 at the first column then \( l_3 = n - 1 - l_2 \) = 0 and a sum \( 1 + l_2 * 2 + l_3 * 2 = 1 \). But a sum of all rows is zero and we have a contradiction. This finishes a proof. \( \square \)

**Corollary 2.** At the space \( \text{Im}(d_2) \) we have not squares.

**Proof.** If \( l_Z \in \text{Im}(d_2) \), then \( S = Z = Z \). For \( n > 3 \) it is impossible. \( \square \)

**Proposition 7.** Let \( S, Z \subset \{1, 2, \ldots, n\} \) such that \( 0 \neq S \neq Z \). Let \( A, J \) be as in Proposition 6. Assume that

\[
\forall U \in \mathcal{P}_n \ | J(U) S | = |US| \cdot |UZ|
\]

then \# \( S = 2, |Z| = 0 \) and \( S \subset Z \).

**Proof.** On the beginning we claim that up to permutation and conjugation,

\[
A = \begin{bmatrix} \ast & 2 & \ast \\ \ast & B & \ast \\ \ast & 3 & \ast \end{bmatrix}, \quad (37)
\]

where \( B \) is a symmetric matrix with a column sums 3. Moreover a block 2 has rows indexed by numbers from the set \( Z \setminus S \) and a block 3 has rows indexed by numbers from the set \( 1 + Z = Z' \). In fact, from Proposition 4, \( S \subset Z \) and Corollary 1, \( S \subset Z \) and \( |S| = |Z| = 0 \). Let us change the indexes of \( A \) such that

\[
A = \begin{bmatrix} \ast & E & \ast \\ \ast & B & \ast \\ \ast & F & \ast \end{bmatrix}, \quad (38)
\]

and \( E \) has rows indexed by numbers from the set \( Z \setminus S \), \( B \) has rows indexed by numbers from \( S \) and \( F \) is indexed by \( 1 + Z = Z' \). From the point 1 of Proposition 5, for \( a, b \notin S \)

\[
\sigma^S_a + \sigma^S_b = |J(\{a,b\} S)| = |\{a,b\} S| \cdot |\{a,b\} Z| = 0.
\]

Hence \( \sigma^S_a = \sigma^S_b \). Let \( \sigma := \sigma^S_a \), for \( a \notin S \).

By the point 2 of Proposition 5 for \( b \in S \) and \( a \notin Z \),

\[
A_{a,b} = \sigma + \sigma^S_b. \quad (39)
\]

From the above all columns of the matrix \( F \) are constant. Again from the point 2 of Proposition 5 for \( b \in S, a \in Z \setminus S \),
\[ A_{a,b} = \sigma + \sigma_b^S + 1. \] 

(40)

It follows that also columns of the matrix \( E \) are constant. Let us conjugate columns of the matrix \( A \) such that \( E = 2 \). In that case \( \sigma = 0 \) since for \( a \in Z \setminus S \) we have 
\[ \sigma = \sigma_a^S = |S| \cdot 2 = 0. \]

From (40), for \( b \in S, 2 = 0 + \sigma_b^S + 1 \). Hence \( \sigma_b^S = 3 \) and \( F = 3 \), because from the formula (39) \( A_{a,b} = 0 + 3 \), for \( a \in Z' \) and \( b \in S \). Finally, from Proposition 5 for \( a, b \in S \) we have 
\[ A_{a,b} + A_{b,a} = 3 + 3 + A_{a,b} + A_{b,a} = \sigma_a^S + \sigma_b^S + A_{a,b} + A_{b,a} = |J(\{a,b\})S| = |\{a,b\}S| \cdot |\{a,b\}Z| = 0. \] 

(41)

To finish a proof it suffices to apply Lemma 6. \( \square \)

**Proposition 8.** We keep the notation from the previous propositions. Let us assume \( S, Z_1, Z_2 \in \mathcal{P}_n \) such that \( 0 \neq S, S \neq Z_1, S \neq Z_2 \) and

\[ \forall U \in \mathcal{P}_n |J(U)S| = |UZ_1| \cdot |UZ_2| \]

then \( \#S = 1, |Z_1| = |Z_2| = 0 \) and \( Z_1 + Z_2 = 1 \).

**Proof.** A proof is similar to the proof of Proposition 7. On the beginning we show that (up to permutation and conjugation)

\[ A = \begin{bmatrix} * & 2 & * \\ * & B & * \\ * & 3 & * \end{bmatrix}, \] 

(42)

where \( B \) is a symmetric matrix of odd dimension with sums of columns 1, a block 2 is indexed by the set \( Z_1 \setminus S \) and a block 3 is indexed by the set \( Z_2 \setminus S \). In fact, from assumption and Corollary 1, \( S = Z_1Z_2, |Z_1| = |Z_2| = 0 \) and \( Z_1 + Z_2 = 1 \). Hence \( |S| = 1 \).

Let us change the order of rows in the matrix \( A \) such that

\[ A = \begin{bmatrix} * & E & * \\ * & B & * \\ * & F & * \end{bmatrix}, \] 

(43)

and \( E \) is indexed by \( Z_1 \setminus S, B \) by \( S \) and \( F \) by \( Z_2 \setminus S \). From Proposition 5 we have \( \forall a, b \in Z_1 \setminus S, \sigma_a^S = \sigma_b^S := \sigma_E \). With similar consideration we have \( \forall a, b \in Z_2 \setminus S, \sigma_a^S = \sigma_b^S := \sigma_F \). Moreover, by Proposition 5 (2) for \( b \in S \) and \( a \in Z_1 \setminus S \),

\[ A_{a,b} = \sigma_E + \sigma_b^S + 1. \] 

(44)

From the above, all columns of the matrix \( E \) are the same. By analogy for \( b \in S \) and \( a \in Z_2 \setminus S \),

\[ A_{a,b} = \sigma_F + \sigma_b^S + 1 \] 

(45)
and columns of the matrix $F$ are also constant. Let us conjugate columns of $A$ such that $E = 2$. Then $\sigma_E = 2$, because for $a \in Z_1 \setminus S, \sigma_E = \sigma^S_a = |S| \cdot 2 = 2$ and for $b \in S, \sigma^S_b = 1$. The last equality follows from (44) because $2 = 2 + \sigma^S_b + 1$. Similarly, by (45) for $b \in S$ and $a \in Z_2 \setminus S$, we have $A_{a,b} = \sigma_F + 1 + 1 = \sigma_F$ and the matrix $F$ is constant and equal to $\sigma_F$. Finally, a matrix $B$ is symmetric since from Proposition 5

\[ \sigma^S_a + \sigma^S_b + A_{a,b} + A_{b,a} = |J(\{a,b\}S)| \]

what means,

\[ A_{a,b} + A_{b,a} = |\{a,b\}Z_1| \cdot |\{a,b\}Z_2| = 0. \]

We have still to show that $\sigma_F = 3$. In fact from assumption a column’s sums of $B$ are 1. Since $B$ is symmetric the same is true for rows. Let us calculate a sum of some column of $A$:

\[ (|Z_1| - |S|)2 + 1 + (|Z_2| - |S|)\sigma_F = 2 + 1 + \sigma_F = 3 + \sigma_F = 0. \]

To finish a proof of Proposition we have to apply Lemma 5. \hfill \Box

Summing up we have the following two possibilities:

I. $\#S = 1$ and $Z_1 + Z_2 = 1$;
II. $\#S = 2$ and $S = Z_1$, $S \neq Z_2$ or $S = Z_2, S \neq Z_1$.

Let us recall that $\text{Im}(d_2)$ is a $n$-dimensional $\mathbb{Z}_2$-space generated by $T_i, i = 1, 2, 3, \ldots, n$. We are interested in description of the set $D$ of elements in $\text{Im}d_2$ which are a product of two nontrivial linear polynomials, see (13). We claim that $D \leq n + 1$. In what follows, if it does not give a contradiction we shall write $T_i$ for $\bar{T}_i, i = 1, \ldots, n$.

Lemma 7. Let $w \in D$, then $w = T_i$ or $w = T_j + T_k$ for some $1 \leq i, j, k \leq n$.

Proof. On the beginning we shall prove that $T_i + T_j$ is a product of two nontrivial linear polynomials if and only if $T_i, T_j$ have a common component. It means there exists $p \neq 0$ s.t. $p|T_i$ and $p|T_j$. Let $T_i + T_j$ have a common component, then from the above case II we can assume that $j = i + 1$ and the matrix $A$ enclose:

\[
\begin{array}{c}
\begin{array}{c}
\ldots \\
2 & 2 \\
\end{array} \\
\begin{array}{c}
1 & 2 \\
2 & 1 \\
\end{array} \\
3 & 3 \\
\ldots
\end{array}
\]
By definition

\[ T_i = (x_1 + \cdots + x_i + x_{i+1})(x_i + x_{i+2} + \cdots + x_n) \]

\[ T_{i+1} = (x_1 + \cdots + x_i + x_{i+1})(x_{i+1} + \cdots + x_n). \]

For the proof of the opposite conclusion we shall need

**Definition 5.** Let \( X \) be a subset of some monoid. By \( \Gamma_X \) we define a graph with the vertex set \( X \) and two vertices \( a, b \) are connected by an edge \( a \rightarrow b \) if and only if \( f|a \) and \( f|b \). Put \( \Gamma := \Gamma_{T_1,T_2,...,T_n} \).

We claim that for \( n > 3 \) the graph

\[ i \rightarrow f \rightarrow j \rightarrow g \rightarrow k \] (46)

is not a subgraph of \( \Gamma \), where \( i := T_i, i = 1, 2, \ldots, n \). In fact we have two possibilities:

1. \( f = g \). Let \( i = 1, j = 2, k = 3 \) and let \( \mathcal{J} \) be an ideal generated by \( (f,T_4,\ldots,T_n) \) in the polynomial ring. Since there exists a nontrivial solution of system of \( (n-2) \) linear equation in \( (n-1) \) linear space an algebraic set \( V(\mathcal{J}) \) is not trivial. It means \( 0 \neq x \in V(\mathcal{J}) \). From definition \( x \in V(\mathcal{J}') \), where \( \mathcal{J}' \) is an ideal generated by \( (T_1,T_2,\ldots,T_n) \). But it is impossible.

2. \( f \neq g \). Using permutation of indexes and conjugation we can assume that in \( HW\)-matrix \( A, j = i+1, k = i+2 \). Recall that \( S = \{i,i+1\} \) and \( A \) is as in Lemma 6. Hence it has a diagonal block related to rows (columns) \( \{i,i+1,i+2\} \)

\[
\begin{bmatrix}
1 & 2 & b \\
2 & 1 & a \\
3 & 3 & 1 \\
\end{bmatrix},
\] (47)

and a matrix \( A \) has upper two first columns of (47) only elements 2, but lower only elements 3. Let us consider polynomials \( T_i, T_{i+1} \) and \( T_{i+2} \) for \( x_s = 0, s \notin \{i,i+1,i+2\} \) and denote it by \( \hat{T}_i \) respectively. We have

\[
\hat{T}_i = (x_i + x_{i+1})(x_i + x_{i+2})
\]

and

\[
\hat{T}_{i+1} = (x_i + x_{i+1})(x_{i+1} + x_{i+2}).
\]

The both polynomials are divided by \( (x_i + x_{i+1}) \). Hence \( \hat{T}_{i+1} \) and \( \hat{T}_{i+2} \) are divided by \( (x_{i+1} + x_{i+2}) \). From the above we can observe that

\[
\hat{T}_{i+2} = (x_{i+1} + x_{i+2})(x_{i+2} + x_i). \] (48)
By (48) and definition we get $a \neq b$. Hence a sum of all columns of the matrix (47) are equal to 0. But it is impossible, since $n > 3$. This finishes a proof of our claim and we have

**Corollary 3.** For $n > 3$ all connected components of a graph $\Gamma$ are points or edges $i \overrightarrow{f} j$. □

**Corollary 4.** For $n > 3$, $D = \{T_1, T_2, \ldots, T_n\}$ or $D = \{T_1, T_2, \ldots, T_n, T_i + T_j\}$ for some $1 \leq i, j \leq n$.

**Proof.** Conversely, suppose that edges

$$1 \overrightarrow{f} 2 \text{ and } 3 \overrightarrow{g} 4$$

are components of the graph $\Gamma$. Let us consider an ideal $\mathfrak{J} = (f, g, T_5, \ldots, T_n)$ in polynomial ring. Since there exists a nontrivial solution of system of $(n - 2)$ linear equation in $(n - 1)$ linear space an algebraic set $V(\mathfrak{J})$ is not trivial. It means $0 \neq x \in V(\mathfrak{J})$. But from definition $x \in V(\mathfrak{J}')$, where $\mathfrak{J}'$ is an ideal generated by $(T_1, T_2, \ldots, T_n)$. But it is impossible. This finishes a proof. □

Let us prove the Main Lemma.

**Main Lemma.** Let $n > 3$, then there are the following possibilities for the structure of the set $D$:

1. $D = \{T_1, T_2, \ldots, T_n\}$;
2. $D = \{T_1, T_2, \ldots, T_n, T_i + T_j\}$, and we can find a polynomial $p$ of degree one such that $p \mid T_i$ and $p \mid T_j$ for some $1 \leq i, j \leq n$. In the second case we can rediscover the set of generators $T_1, T_2, \ldots, T_n$.

**Proof.** We start from the simple observation. If $i \neq j$ and $T_i = p \cdot q, T_j = p \cdot r$ then $\forall i = 1, 2, \ldots, n$ $q + r$ is not divided $T_i$. In fact $q + r \neq p$ since in other case $T_i + T_j = p(q + r) = p^2$. By Corollary 2 it is impossible. Hence $T_i$ and $T_j$ are also not divided by $q + r$. Moreover, if $T_r = (q + r)s$ then $T_i + T_j + T_r = (q + r)(p + s)$. By Proposition 7, a decomposition for $\#S = 3$ is impossible. Let us prove the second point of the above lemma. From definition the graph $\Gamma_{T_1, \ldots, T_n}$ has connected components which are vertices for $r \notin \{i, j\}$, of the triangle with vertices $T_i, T_j, T_i + T_j$ and a constant label which is a component of $T_i$ and $T_j$. Let $T_i = p \cdot q$ and $T_j = p \cdot r$ then $T_i + T_j = p(q + r)$. The triangle is a connected component of a graph because by (46) for $r \notin \{i, j\}$ elements $p, q, r$ do not divide $T_r$. Also from the above simple observation, the element $(q + r)$ does not divide $T_r$.

We continue the proof of the Main Lemma. Let $w = \xi \eta$ where $\xi$ and $\eta$ are linear polynomials. Let us define $s(w) := \xi + \eta$. Since $HW$-manifolds are oriented $\sum_i s(T_i) = 0$. 

We claim that if $T_i + T_j \in D$, then $s(\xi) + s(\eta)$ recognizes subsets of order two of the set $\{T_i, T_j, T_i + T_j\}$. In fact, let $T_i = p \cdot q, T_j = p \cdot r$, then $T_i + T_j = p(q + r)$ and $s(T_i) + s(T_j) = q + r, s(T_i + T_j) = r, s(T_j) + s(T_i + T_j) = q$.

Let $n > 3$, then there are the following possibilities for the structure of the set $D$:

1. $D = \{T_1, T_2, \ldots, T_n\}$;
2. $D = \{T_1, T_2, \ldots, T_n, T_i + T_j\}$, for some $1 \leq i, j \leq n$. Let $n > 3$ if $D$ has $n$ elements we are done. If it has $(n + 1)$ elements then the graph $\Gamma_{T_1, T_2, \ldots, T_n}$ has $(n - 2)$ discrete connected components $D^c$ and a triangle. We proceed in two steps:

1. Put $s_{D^c} := \sum_{a \in D^c} s(a)$
2. From the triangle we take a unique pair $\xi, \eta$ such that

$$s(\xi) + s(\eta) + s_{D^c} = 0.$$ 

Hence $\{T_1, T_2, \ldots, T_n\} = \{\xi, \eta\} \cup D$. This finishes a proof of the Main Lemma. \(\square\)

For illustration of possibilities of the structure of the set $D$ we present two examples.

**Example 3.** Let $G \subset D^5$ correspond to $HW$-matrix

$$\begin{bmatrix}
1 & 2 & 2 & 2 & 2 \\
2 & 1 & 3 & 2 & 2 \\
3 & 2 & 1 & 3 & 2 \\
2 & 3 & 2 & 1 & 3 \\
3 & 3 & 3 & 2 & 1
\end{bmatrix}.$$ 

The set

$$D = \{T_1 = (x_1 + x_2)(x_1 + x_3 + x_4), T_2 = (x_1 + x_2 + x_3 + x_4)x_2, \}
T_3 = (x_1 + x_3)(x_2 + x_3 + x_4), T_4 = (x_1 + x_2 + x_4)(x_3 + x_4), \ (49)$$

$$T_5 = (x_1 + x_2 + x_3)x_4\}.$$ 

From Remark 1 the above group is isomorphic to the group $\Gamma_1$ of the Example 2. The next example illustrates the second case of the Main Lemma.

**Example 4.** Let a matrix

$$\begin{bmatrix}
1 & 2 & 2 & 2 & 2 \\
2 & 1 & 3 & 2 & 2 \\
2 & 2 & 1 & 3 & 2 \\
2 & 2 & 2 & 1 & 3 \\
3 & 3 & 2 & 2 & 1
\end{bmatrix} \in D^{5 \times 5}$$ be the second $HW$-matrix of dimension 5.

In this case we have

$$D = \{T_1 = (x_1 + x_2 + x_3 + x_4)x_1, T_2 = (x_1 + x_2 + x_3 + x_4)x_2, \}
T_3 = (x_1 + x_3 + x_4)(x_2 + x_3), T_4 = (x_1 + x_2 + x_4)(x_3 + x_4), \ (50)$$

$$T_5 = (x_1 + x_2 + x_3)x_4, T_1 + T_2\}.$$
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