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Infinitely Many Positive Solutions of the Diophantine Equation $x^2 - kxy + y^2 + x = 0$

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Abstract— We prove that the equation $x^2 - kxy + y^2 + x = 0$ with $k \in N^+$ has an infinite number of positive integer solutions x and y if and only if $k = 3$. For $k = 3$ the quotient x/y is asymptotically equal to $(3 + \sqrt{5})/2$ or $(3 - \sqrt{5})/2$. Results of the paper are based on data obtained via Computer Algebra System (DERIVE 5). Some DERIVE procedures presented in the paper made it possible to discover interesting regularities concerning simple continued fractions of certain numbers.

Keywords— Diophantine equations, Computer Algebra System, Pell equation

1. INTRODUCTION

The first named author, working on solutions of problems presented by the Polish Mathematical Society to Polish students during the LI Mathematical Olympiad (1999), noticed that one of the problems leads to the equation

$$x^2 - kxy + y^2 + x = 0. \tag{1}$$

In [1] an analysis of equation (1) was carried out and the following theorem was proved.

THEOREM 1. *If positive integers x,y,k satisfy equation (1), then there exist positive integers c,e such that $x = c^2$, $y = ce$, and* $\mathrm{GCD}(c, e) = 1$.

SKETCH OF THE PROOF OF THEOREM 1. It follows from (1) that if p is a prime number, then

$$p \mid x \Rightarrow p \mid y.$$

Let $x = p^\mu x_1$, $y = p^\nu y_1$ with $\mathrm{GCD}(p, x_1) = \mathrm{GCD}(p, y_1) = 1$. Substituting these values of x and y to equation (1), we get $\mu = 2\nu$. It means that $x = p_1^{2\nu_1} \times \cdots \times p_r^{2\nu_r}$ and $y = p_1^{\nu_1} \times \cdots \times p_r^{\nu_r} \bar{y}$.

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Moreover, in [1] on the basis of some numerical data obtained from certain procedures (TURBO PASCAL 7.0, TI83-Basic, MAPLE V.4, and DERIVE 3.10) two observations were made.

- If x, y, k are positive integer solutions of (1), then $k = 3$.
- If $k = 3$ and $(x_1, y_1), (x_2, y_2), \dots$ are positive integer solutions of (1) with $x_1 \le y_1$, $x_2 \le y_2, \dots$, then the quotients y_n/x_n approach to the value $2.61083\dots$

The aim of the paper is to prove the validity of these observations. We will also give the exact value of the limit mentioned above. There is no doubt that a Computer Algebra System (CAS) like DERIVE can play a central role in observing, extracting, and checking out some mathematical regularities (see, for instance, formula (11)).

2. THE MAIN RESULT

We will seek integer solutions x and y of equation (1), which are positive; such a pair (x, y) we will call a positive solution.

A discussion concerning other nonzero solutions ($x \neq 0$ and $y \neq 0$) with an arbitrary k is given in the final chapter of the paper.

THEOREM 2. *Let k be a positive integer; equation (1) has an infinite number of positive integer solutions if and only if $k = 3$.*

The crucial argument in the proof of Theorem 2 comes from the theory of the Pell equation

$$x^2 - Dy^2 = M, \tag{2}$$

where D is a given positive integer not a perfect square and M is a given integer. Suppose that $\sqrt{D}$ is written as an infinite simple continued fraction $\sqrt{D} = \langle a_0, a_1, a_2, \dots \rangle$ (this notation is taken from [2]). The rational number $\langle a_0, a_1, a_2, \dots, a_n \rangle = h_n/k_n$ is called the n^{th} convergent to the infinite simple continued fraction of $\langle a_0, a_1, a_2, \dots \rangle$.

To prove Theorem 2, we will use the following three theorems.

THEOREM A. *Let $x_0^2 - Dy_0^2 = M$ be fulfilled for some integers x_0, y_0, and $a_0^2 - Db_0^2 = 1$ for some integers a_0,b_0. If $w = x_0 + y_0\sqrt{D}$, $j = a_0 + b_0\sqrt{D}$, then for any $n \in Z$ the pair (x_n, y_n) defined by*

$$x_n + y_n\sqrt{D} = wj^n \tag{3}$$

satisfies equation (2).

THEOREM B. *Let the integer M satisfy $|M| < \sqrt{D}$. Then, any positive integer solution $x = s$, $y = t$ of (2) with $\mathrm{GCD}(s,t) = 1$ satisfies $s = h_n$, $t = k_n$ for some positive integer n.*

THEOREM C. *Let $\{m_n\}_{n=0}^{\infty}$, $\{q_n\}_{n=0}^{\infty}$ be two sequences given by*

$$m_0 = 0, \qquad q_0 = 1, \qquad m_{n+1} = a_n q_n - m_n, \qquad q_{n+1} = \frac{D - m_{n+1}^2}{q_n}. \tag{4}$$

Then,

(i) m_n, q_n are integers for any positive integer n,
(ii) $h_n^2 - Dk_n^2 = (-1)^{n-1} q_{n+1}$ for any integer $n \ge -1$.

For the proofs of Theorems A–C, see [2] or [3].

PROOF OF THEOREM 2.

SUFFICIENCY ("$\Leftarrow$"). Let us have a closer look at solutions of the equation

$$x^2 - 3xy + y^2 + x = 0. \tag{5}$$

By Theorem 1, equation (5) takes the form

$$c^2 - 3ce + e^2 + 1 = 0. \tag{6}$$

We can consider (6) as the quadratic equation with respect to the unknown c. This equation has real solutions if and only if $5e^2 - 4 \geq 0$. As all considered equations are to be solved in integers, $5e^2 - 4$ should be a square, i.e., there exists an integer u such that

$$u^2 - 5e^2 = -4. \tag{7}$$

Now, we will use the well-known procedure for constructing infinitely many integer solutions to the Pell equation (Theorem A). The pair $(1, 1)$ satisfies the equation $u^2 - 5e^2 = -4$ and corresponds to $w = 1 + \sqrt{5}$ from Theorem A; the pair $(9, 4)$ corresponds to $j = 9 + 4\sqrt{5}$ and satisfies $u^2 - 5e^2 = 1$. So, the infinite sequence (u_n, e_n) of solutions to (5) can be obtained from the following matrix equation:

$$\begin{bmatrix} u_n \\ e_n \end{bmatrix} = \begin{bmatrix} 9 & 20 \\ 4 & 9 \end{bmatrix}^n \begin{bmatrix} u_0 \\ e_0 \end{bmatrix}, \qquad n = 0, 1, 2, \ldots. \tag{8}$$

In Table 1, we give the first eight solutions to (5) which are generated by the formulae from Theorem 1 and equations (6) and (8). Note that $c_i = (3e \pm u)/2$. These are not all solutions in the considered range (see final remarks).

Table 1.

u	e	c_1	c_2	$x_1 = c_1^2$	$y_1 = c_1 e$	$x_2 = c_2^2$	$y_2 = c_2 e$
1	1	1	2	1	1	4	2
29	13	5	34	25	65	1156	442
521	233	89	610	7921	20737	372100	142130
9349	4181	1597	10946	2550409	6677057	119814916	45765226

NECESSITY ("$\Rightarrow$"). For $k = 1$ there is only one nontrivial solution, $x = y = -1$; it follows from the following observation.

The curve representing $x^2 - xy + y^2 + x = 0$ is an ellipse within the rectangle $[-2, 0] \times [-1, 1]$ and the only lattice points on the ellipse are $(0, 0)$, $(-1, 0)$, $(-1, -1)$.

If $k = 2$, then the equation $x^2 - 2xy + y^2 + x = 0$ is equivalent to $(x - y)^2 + x = 0$, which obviously does not have any positive solution (x, y). Now, we may assume that $k \geq 3$ and solutions (x, y) of (1) are positive. From Theorem 1, we have $x = c^2$, $y = ce$ with $\mathrm{GCD}(c, e) = 1$. Again, as in the proof of "$\Leftarrow$", we obtain

$$c^2 - kce + e^2 + 1 = 0. \tag{9}$$

It follows from (9) and $\mathrm{GCD}(c, e) = 1$ that k must be odd.

On the contrary, suppose that k is even. As c, e cannot be both even ($\mathrm{GCD}(c, e) = 1$), there are two cases:

(i) exactly one of c or e is even;
(ii) c and e are both odd.

In Case (i) from (9), we get $4 \mid kce$ and $4 \mid c^2 + e^2 + 1$. In the second case, $2 \mid kce$ and $2 \mid c^2 + e^2 + 1$. In both cases, we get a contradiction.

Now, suppose k is odd and $k \geq 5$. As in the case $k = 3$, we deal with the equation

$$u^2 - \left(k^2 - 4\right) e^2 = -4, \tag{10}$$

to be solved in integers. We will show that there are no integer solutions of (10). It is easy to see that $k^2 - 4$ is not a square. For each irrational number of the form $\sqrt{k^2-4}$, we have the following infinite simple continued fraction, which is periodic:

$$\sqrt{k^2-4} = \left\langle k-1, \left(1, \frac{k-3}{2}, 2, \frac{k-3}{2}, 1, 2k-2\right)\right\rangle. \tag{11}$$

The proof of (11) is straightforward, but rather laborious (we omit it here, for details see [4]). In the last section, we presented a DERIVE-procedure, which helped to obtain this continued fraction expansion.

Now, applying the procedure defined in Theorem C with

$$\begin{array}{llll} a_0 = k-1, & a_{6n+1} = 1, & a_{6n+2} = \dfrac{k-3}{2}, & a_{6n+3} = 2, \\ a_{6n+4} = \dfrac{k-3}{2}, & a_{6n+5} = 1, & a_{6n} = 2k-2, & \end{array} \tag{12}$$

we get two eventually periodic sequences

$$\begin{aligned} \{m_n\}_{n=0}^{\infty} &= \{0, k-1, k-4, k-2, k-2, k-4, k-1, k-1, k-4, k-2, \\ &\quad k-2, k-4, k-1, \dots\}, \\ \left\{(-1)^{n-1} q_{n+1}\right\}_{n=-1}^{\infty} &= \{1, 5-2k, 4, 2-k, 4, 5-2k, 1, 5-2k, 4, 2-k, 4, 5-2k, 1, \dots\}. \end{aligned} \tag{13}$$

Suppose (u, e) is a solution of (10). If u and e have a common factor > 1 their GCD must be 2; dividing this out, we obtain a solution of $u^2 - (k^2-4)e^2 = -1$. In either case, we may apply the procedure in Theorem C with (12) to obtain (13). The second sequence in (13) gives a contradiction since it does not contain -4 or -1. Thus, (10) has no solution when $k > 3$.

3. ASYMPTOTIC OF QUOTIENTS FOR THE CASE $k = 3$

For equation (6) there are two solutions

$$c^-(e) = \frac{3e - \sqrt{5e^2-4}}{2}, \tag{14}$$

$$c^+(e) = \frac{3e + \sqrt{5e^2-4}}{2}. \tag{15}$$

We have established that there are an infinite number of integers c,e satisfying (6). By Theorem 1, if x,y are positive integer solutions of (3), then $x = c^2$, $y = ce$. It follows immediately that the quotient $x/y = c/e$ is asymptotically equal to $(3+\sqrt{5})/2$ or $(3-\sqrt{5})/2$.

4. SOME DERIVE PROCEDURES FOR THE EQUATION $x^2 - kxy + y^2 + x = 0$

We have already stressed that computer experiments played an essential role in searching regularities described in this paper and in [1,4] as well. To get some results from our paper, we apply DERIVE 5. For instance, searching solutions (x, y) to equation (5), we define the matrix

$$A := \begin{bmatrix} 9 & 20 \\ 4 & 9 \end{bmatrix}.$$

On the basis of equation (8), we can derive the following recursive formula for (u_n, e_n):

$$\begin{bmatrix} u_{n+1} \\ e_{n+1} \end{bmatrix} = A \begin{bmatrix} u_n \\ e_n \end{bmatrix}, \tag{16}$$

where n is any integer and the initial condition is as follows:

$$\begin{bmatrix} u_0 \\ e_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

From (16), we have

$$\begin{bmatrix} u_n \\ e_n \end{bmatrix} = A^n \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

where $n \in N$. Thus, to get integer values e_n of the unknown e satisfying equation (7) corresponding to n varying from nNeg to NPos, we may apply the following DERIVE 5 commands;

```
seqPower(nNeg, nPos) := VECTOR(A^n · [1, 1], n, nNeg, nPos),
    solE(nNeg, nPos) := SORT(ELEMENT(seqPower(nNeg, nPos)`, 2)).
```

Now, we can simplify the last expression with arbitrarily fixed values of nNeg and nPos. For instance,

$$\texttt{seqPower}(-5,4) = \begin{bmatrix} -1149851 & 514229 \\ -64079 & 28657 \\ -3571 & 1597 \\ -199 & 89 \\ -11 & 5 \\ 1 & 1 \\ 29 & 13 \\ 521 & 233 \\ 9349 & 4181 \\ 167761 & 75025 \end{bmatrix}$$

lists the values of the u_n and e_n in Columns 1 and 2, respectively, where $n = -5, -4, \ldots, 4$.

$$\texttt{solE}(-5,4) = [1, 5, 13, 89, 233, 1597, 4181, 28657, 75025, 514229],$$

gives all ten of the smallest values of e_n.

The essential moment in proving the main result of the paper (Theorem 2) was to get the expansion (11). In the DERIVE 5 unit named NUMBER.MTH, there is the function CONTINUED_FRACTION. The call to it must be completed with two parameters, which are the expression to be expanded in the convergent and the degree of this expansion. For instance,

```
CONTINUED_FRACTION(SQRT(5), 3) = [2, 4, 4, 4].
```

This result says that the third convergent of $\sqrt{5}$ is $\langle 2, 4, 4, 4 \rangle$. It suggests that $\sqrt{5} = \langle 2, (4) \rangle$, and one can check it.

To shorten the next expression, we define

```
cf(k, n) := CONTINUED_FRACTION(SQRT(k^2 - 4), n).
```

The simplification of the expression

```
VECTOR(APPEND([k], cf(k, 17)), k, 5, 15, 2)
```

returns the matrix with the following rows.

5	4	1	1	2	1	1	8	1	1	2	1	1	8	1	1	2	1	1
7	6	1	2	2	2	1	12	1	2	2	2	1	12	1	2	2	2	1
9	8	1	3	2	3	1	16	1	3	2	3	1	16	1	3	2	3	1
11	10	1	4	2	4	1	20	1	4	2	4	1	20	1	4	2	4	1
13	12	1	5	2	5	1	24	1	5	2	5	1	24	1	5	2	5	1
15	14	1	6	2	6	1	28	1	6	2	6	1	28	1	6	2	6	1

The left-most column lists the values of k $(= 5, 7, 9, \dots, 15)$, the next columns contain consecutive components c_j of the convergents $\langle c_0, c_1, c_2, \dots, c_{17}\rangle$. In every line one can detect at a glance the periodicity of the length 6. Columns listing c_j with odd js remain always the same values (1, 2, and 1 for $j = 1$, 3, and 5, respectively). Columns listing c_j with even js reveal the regularity which is also involved in formula (11).

Now, we are going to show how, on the basis of Theorem C, to display formulae (12) and (13). In DERIVE the detected periodicity may be coded as follows:

```
p(n, k) := IF(n = 0, k − 1, IF(n > 0 ∧ MOD(n, 6) = 1, 1, IF(MOD(n, 6) = 2, (k − 3)/2,
IF(MOD(n, 6) = 3, 2, IF(MOD(n, 6) = 4, (k − 3)/2, IF(MOD(n, 6) = 5, 1, 2 · k − 2)))))).
```

The simplification of the call

```
c(k) := VECTOR(p(n, k), n, 1, 13)
```

returns the vector

```
[1, (k−3)/2, 2, (k−3)/2, 1, 2·k−2, 1, (k−3)/2, 2, (k−3)/2, 1, 2·k−2, 1].
```

It lists the first thirteen partial quotients of the infinite simple continued fraction of $\sqrt{k^2-4}$. We complete with the code producing the sequences $\{m_n\}_{n=0}^{\infty}$, $\{q_n\}_{n=0}^{\infty}$ from Theorem C. In the construction given below, consecutive iterations of the vector v=[vsub1,vsub2,vsub3] yield the current index n and the values m_n and $(-1)^n q_n$, respectively.

```
t(n, k) := ITERATE([vsub1 + 1, p(vsub1 + 1, k) · vsub3 − vsub2, (k^2 − 4−
(p(vsub1 + 1, k) · vsub3 − vsub2)^2)/vsub3], v, [−1, 0, 1], n)
VECTOR(ELEMENT(t(n, k), 2), n, 0, 13)
[0, k − 1, k − 4, k − 2, k − 2, k − 4, k − 1, k − 1, k − 4, k − 2, k − 2, k − 4, k − 1, k − 1]
VECTOR((−1)^(n − 1) · ELEMENT(t(n + 1, k), 3), n, −1, 13)
[1, 5 − 2 · k, 4, 2 − k, 4, 5 − 2 · k, 1, 5 − 2 · k, 4, 2 − k, 4, 5 − 2 · k, 1, 5 − 2 · k, 4].
```

5. FINAL REMARKS

In the paper, for positive k we have considered the positive solution case only. It is easy to see that there are no positive solutions (x, y) of equation (1) for negative k.

When $k > 0$, then there are no solutions (x, y) of (1) such that $xy < 0$, although for instance in the $k = 3$ case positive solutions yield negative solutions and vice versa using the following.

(a) If $k = 3$ and (x, y) is a solution of (1), then so is $(1 - y, 1 - x)$.

There are also two produce for generating more solutions (x, y) with an arbitrary k.

(b) If (x, y) is a solution of (1), then so is $(x, kx - y)$.

(c) If (x, y) is a solution of (1), then so is $(ky - x - 1, y)$.

In the case $k = 3$, starting from the least positive solution and alternating the uses of (b) and (c), we can produce an infinite set of positive solutions to the titled equation which properly includes the set that equation (8) would produce. The first eight solutions obtained with this process are as follows:

$$(1,1) \to (1,2) \to (4,2) \to (4,10) \to (25,10) \to (25,65) \to (169,65) \to (169,442) \to \cdots.$$

The solutions (1,2), (4,10), (25,10), (169,65), and (169,442) are not in Table 1. It seems that in the case $k = 3$, all positive solutions of equation (1) are of the form

$$(x, y) = \left(f_{2n-1}^2, f_{2n-1} f_{2n-3}\right) \qquad \text{or} \qquad (x, y) = \left(f_{2n-1}^2, f_{2n-1} f_{2n+1}\right),$$

and all other solutions of (1) are given by

$$(x, y) = \left(-f_{2n}^2, -f_{2n}f_{2n-2}\right) \qquad \text{or} \qquad (x, y) = \left(-f_{2n}^2, -f_{2n}f_{2n+2}\right),$$

where $n = 1, 2, \ldots$ and $(f_m)_{m=-1}^{\infty}$ is the Fibonacci sequence starting from the index -1 ($f_{-1} = 1$, $f_0 = 0$, $f_1 = 1$, $f_2 = 1$, $f_{n+2} = f_{n+1} + f_n$ for $n = 1, 2, \ldots$). We hope that it is possible to characterize integer solutions of the title equation in the general case. Some computer experiments, which we have done recently, suggest that for many k there are infinitely many integer solutions.

REFERENCES

1. E. Bartz and A. Marlewski, A computer search of solutions of a certain Diophantine equation, *Pro Dialog* (in Polish, English summary) **10**, 47–57, (2000).
2. I. Niven and H.S. Zuckerman, *An Introduction to the Theory of Numbers*, Wiley, New York, (1980).
3. H.E. Rose, *A Course in Number Theory*, Oxford University Press, Oxford, (1996).
4. P. Zarzycki and A. Marlewski, Continued fractions with DERIVE, (submitted).