Great theorems on a small screen

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Abstract

University of Gdańsk third year undergraduates take a course called "Elementary Number Theory". During the course we try to introduce some important problems such as The Circle Problem, Dirichlet's Divisor Problem, Prime Number Theorem (PNT) or Fermat's Last Theorem. Working with graphing calculators students can experiment with these theorems. Obviously during the course we do not prove Fermat's Last Theorem, but we use the visualizing power of calculators such as TI 92 Plus to help our students understand advanced mathematics better.

Introduction

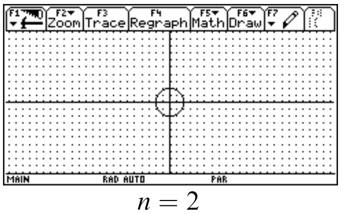
Definition, theorem, proof, definition, theorem, proof, ... It is rather a typical way of teaching mathematics at the undergraduate level at many Polish universities. We were taught in this way.

How far away have we gone from experiments in teaching and learning mathematics? Sometimes the phase of experimentation is not possible or difficult to arrange (theory of infinite sets, algebraic topology, algebraic geometry), although our experience confirms the thesis that with technology like TI 92 Plus it is easier to study some advanced number theory. We will give four examples taken from our course ''Elementary Theory of Numbers'' given for University of Gdańsk third year undergraduates.

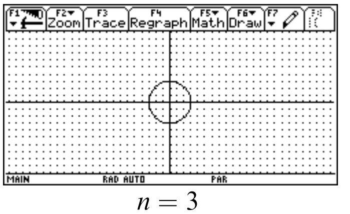
The Circle Problem

Recall that a lattice point is a point in the xy-plane with integral coordinates. We would like to count how many lattice points there are in the disk $x^2 + y^2 \leqslant n^2$.

The problem comes from Gauss, one of the greatest mathematicians of all times, the man who did many experiments in mathematics.



$n = 2$



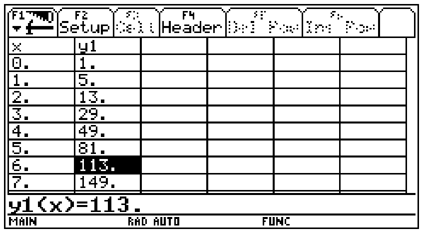
$n = 3$

What is very entertaining in the problem is its accessibility for students of all levels. There are a few methods to count the lattice points. We have chosen a method to get a formula in the TI 92 Plus language:

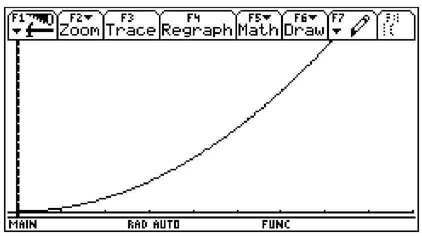
$$y1(x) = 2\sum\left(2floor\left(\sqrt{(x^2 - i^2)}\right) + 1, i, 1, floor(x)\right) + 2floor(x) + 1$$

where $y1(x)$ is the number of lattice points in the disk centered at the origin with radius x and $floor(x)$ is the integer part of a real number x.

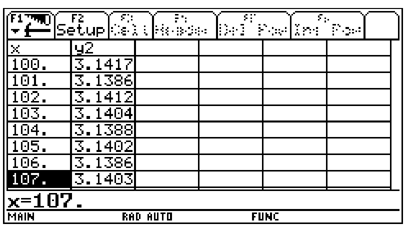
Let us have a look at the table of $y1(x)$.

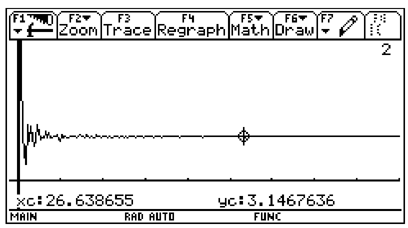


We could concentrate on investigating the function $y1(x)$ looking at its values only, but if we graph this function we will get something spectacular:



We have got a perfect parabola of the form $y = cx^2$. What is c? The value of c can be found easily if we look at the table and the graph of the function $y2(x) = y1(x)/x^2$.



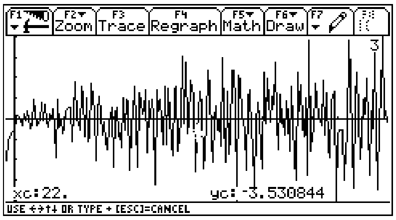


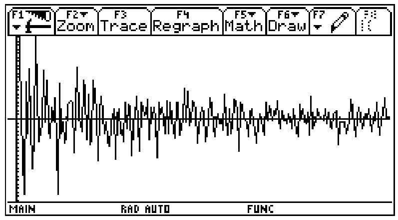
That gives us almost the same formula as the one discovered by Gauss

$$y1(x) = \pi x^2 + error\ term \tag{1}$$

The Circle Problem consists of finding the smallest possible error term in equation (1). During the course we tried to have a closer look at the Circle Problem studying the following two functions:

$$y3(x) = y1(x) - \pi x^2 \qquad\qquad y4(x) = y1(x)/x - \pi x$$



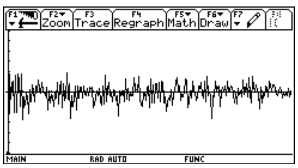


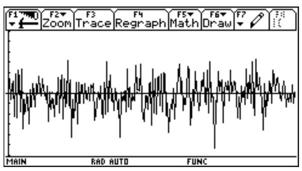
The function $y4(x)$ is oscillating around zero, which gives the classical Gauss result

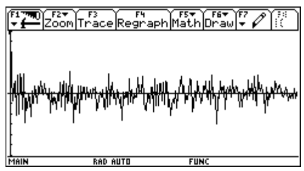
$$y1(x) = \pi x^2 + O(x)$$

where $O(x)$ means that the error term is less than Bx with an absolute constant B.

Many mathematicians succeeded in improving the Gauss result; we would like to illustrate three results showing three screens:







In 1906 Sierpiński proved that the error term is $O(x^{2/3})$.

In 1916 Hardy proved that the errorr term cannot be $O(x^{1/2})$.

In 1988 Iwaniec and Mozzochi proved that the error term is $O(x^{14/22})$.

Dirichlet's Divisor Problem

Our object now is to find an average value for the number $d(n)$ of divisors of n. When we presented this part of this paper to Polish teachers of mathematics its title was "Chaos and Order". It happens quite often with the mathematical objects that we study them noticing nothing but their chaotic behaviour. Then we change these objects very slightly and discover a perfect order, a conspicuous regularity. We believe that Dirichlet's Divisor Problem is a perfect example of this chaos to order transformation.

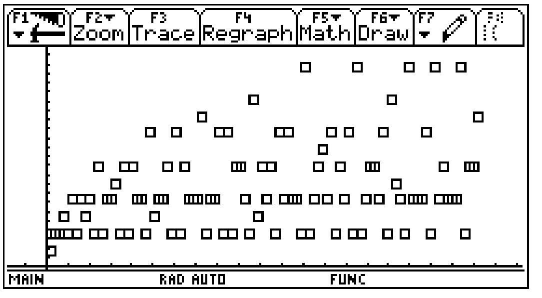
We define the function $d(n)$ in two ways:

$$d(n) = \sum (\mathit{floor}(\mathit{gcd}(n,i)/i), i, 1, n),$$

$$d1(n) = n + \sum (\mathit{floor}(n/i) + \mathit{floor}(-n/i), i, 1, n)$$

The second formula ($d1(n)$) is based on Lerch's beautiful result from 1887 (see Scholomiti, 1965[1]) concerning the $d(n)$ function.

Let us have a look at the graph of $d(n)$ (now we have to use lists, as $d(n)$ is defined on a discrete set). What we are going to see is a very chaotic picture.

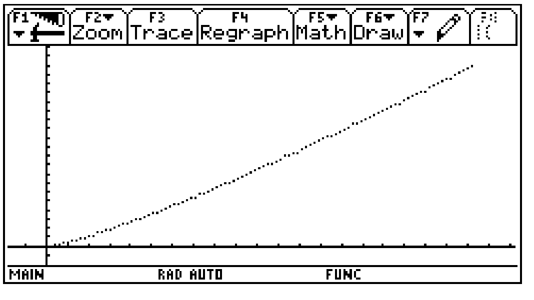


Instead of $d(n)$ we will study the function $d2(n)$ defined in the following way

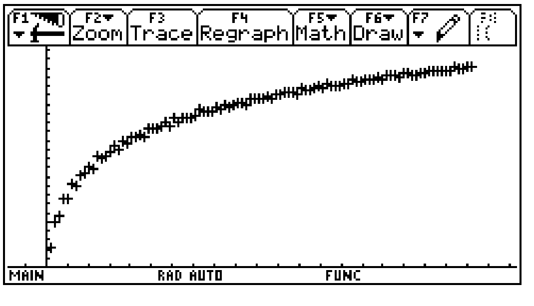
$$d2(n) = \sum (d(i), i, 1, n)$$

which in number theory is called a summatory function of an arithmetic function.

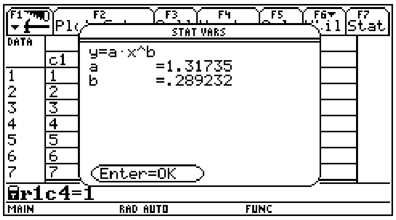
Looking at the graph we notice it looks like a line (that is the order we have mentioned before).

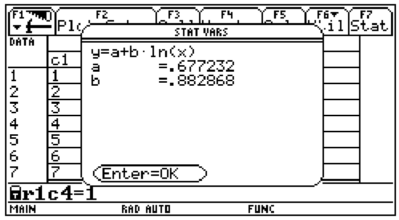


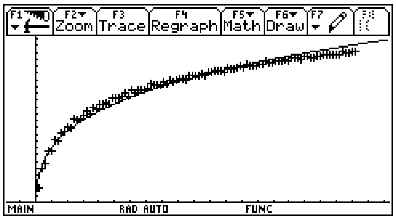
To check this we may consider another function $d3(n) = d2(n)/n$ and its graph.

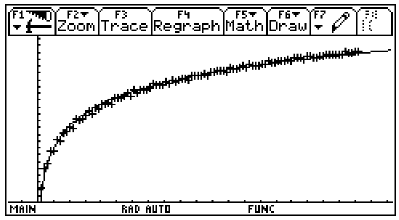


The graph definitely looks like a graph of the square root function or like the logarithmic function. It is possible to check this using regressions.



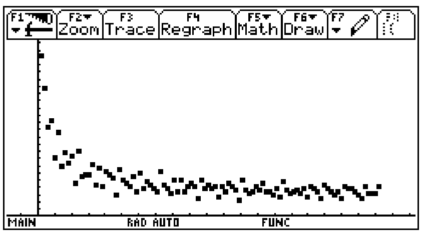






It follows from our considerations that $d3(n) \sim \ln n$, which means that $\lim_{n\to\infty} d3(n)/\ln n = 1$.

Let us check what $d3(n) - \ln n$ is.



We can write

$$d3(n) - \ln n \sim \text{constant}$$

$$\Downarrow$$

$$d2(n) = n \ln n + n \cdot constant + error\ term \tag{2}$$

The calculators helped our students to estimate the value of the constant, which in fact is $2\gamma - 1$, where γ is the Euler constant (the limit of $\sum_{i=1}^{n} 1/i - \ln n$ as $n \to \infty$; to four decimal places, its value is 0.5772). We did not seek the error term in (2). Finding the smallest possible term is another fascinating nad hard mathematical problem which is called Dirichlet's Divisor Problem.

Prime Number Theorem (PNT)

Gauss was the first to give careful attention to $\pi(x)$, the number of primes that do not exceed x. It is really astonishing that Gauss discovered PNT studying prime number tables contained in a book of logarithms that had been given to him as a present. Throughout his life Gauss was interested in the distribution of the prime numbers and he made extensive calculations. In a letter to Enke he wrote:

> "I used very often an idle quarter of an hour to count through another chiliad here and there".

Finally Gauss listed all the prime numbers up to 3 million and compared their distribution to the formula which he had conjectured.

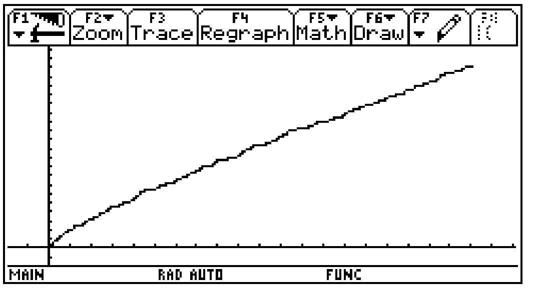
We are not going to state PNT now, as we should try to discover it with the aid of TI 92 Plus. We need to define one auxiliary function

$$charp(x) = when(is\Pr ime(x), 1, 0),$$

which is a characteristic function of the set of all primes. Then $\pi(x)$ (let us call it $pi(x)$) is easily found:

$$pi(x) = \sum (charp(i), i, 1, x)$$

Graphing the function $pi(x)$ in the $\boxed{Y=}$ mode is very slow, so we used lists and statistical plots to obtain the following picture:



All our students' attempts to get approximation of $\pi(x)$ failed—they used all types of regressions to receive rather exotic approximations of $\pi(x)$ $(pi(x))$.

We decided to attack this problem in a completely different way. Some probabilistic functions had been introduced:

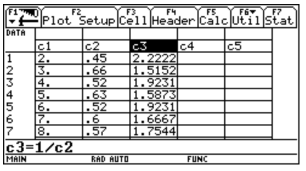
$$pr(x) = when(is\Pr ime(rand(x)), 1, 0)$$

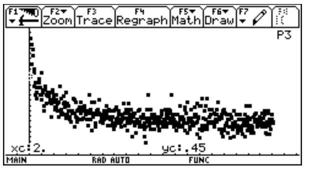
where $rand(x)$ returns a random integer in the interval $[1, x]$. We may assume that the probability to draw a prime from the set $\{1, 2, \ldots, x\}$ is approximately equal to $pi(x)/x$.

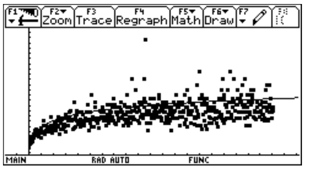
Now, we can start a simulation—drawing numbers; to simulate multiple drawing we use the function $pipr(x)$:

$$pipr(x) = \sum (pr(x), i, 1, y)/y$$

If we fix $y(y = 100)$ and take $x = 2, \ldots, 500$ the function $pipr(x)$ gives us a frequency of drawing primes from $\{2, \ldots, x\}$ in y attempts. Let us table and graph the results:







From the last picture we can risk to formulate PNT:

$$\lim_{n\to\infty} \frac{\pi(x)}{\left(\dfrac{x}{\ln x}\right)} = 1$$

The first important result in the direction of PNT was proved by Chebyshev in 1850; he showed that for sufficiently large x

$$0.89\,\frac{x}{\ln x} < \pi(x) < 1.11\,\frac{x}{\ln x}$$

Further, he proved that if $\lim_{n\to\infty} \pi(x)/(x/\ln x)$ exists, then it must be equal 1. PNT was proved in 1896 by Hadamard and de la Vallée Poussin.

Fermat's Last Theorem (FLT)

Undoubtely FLT is the most popular mathematical problem known not only to professionals, but to a general audience as well. Being happy with Wiles' solution of FLT, we should realize that still most of us can talk about FLT at the very popular level only.

We take this famous equation

$$x^n + y^n = z^n$$

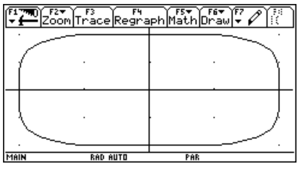
with $n \geqslant 3$, and transform it into a two-dimensional case

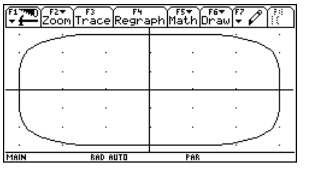
$$\left(\frac{x}{z}\right)^n + \left(\frac{y}{z}\right)^n = 1$$

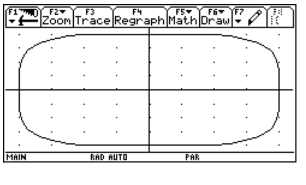
and

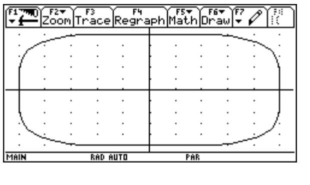
$$a^n + b^n = 1. \tag{3}$$

The curve representing equation (3) is closed for $n = 4, 6, 8, \ldots$. We will graph the curve for $n = 4$ switching on *Grid* and changing *xscl* and *yscl* in *WINDOW* $(xsl, yscl = 1/2, 1/3, 1/4, 1/5)$.









If we zoom in we can observe non-solvability of (3) in rational numbers with denominators equal to 2, 3, 4 and 5 as the curve representing equation (3) does not go through the grid points.

Teaching with technology (synopsis of Gdańsk's experiments)

We have decided to include experiments described in this paper to syllabuses of the number theory courses at our department. During classes on number theory our students are offered a list with many problems which illustrate famous theorems like Lagrange's four squares theorem. In this case the task for students is to write a program in the language of TI 92 for representing an integer as a sum of four squares of integers; they should also observe the number of such representations.

On the list there are problems from mathematical competitions as well. We have noticed that solving hard problems with the phase of experimentation is easier even for students with rather poor problem solving skills. In the students-evaluation of the course many of them stressed that with IT more complicated problems are accessible. We may think of such opinions as a confirmation of successful introduction of IT to teaching and learning advanced mathematics.

In the paper we have demonstrated with four examples that calculators can be excellent tools for teaching and learning mathematics. These tools allow students to deal with advanced topics, they can experiment with mathematics, rediscovering important results invented by masters.

References

1. Scholomiti, N. C., A property of the τ-function, *American Mathematical Monthly*, 1965, **72,** 745–747.

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