ON SOME INVARIANTS OF LINKS IN THE SOLID TORUS AND THEIR ESTIMATES

JOZE MALEŠIĆ, MACIEJ MROCKIOWSKI

1. INTRODUCTION

We introduce a new invariant for links in the solid torus and find some estimates for it coming from skein modules.

The first author is supported by the Slovenian Research Agency grants P1-0292-0101 and J1-9643-0101. The second author is supported by the Polish ministry grant N200100831/0524. Both authors are supported by the bilateral Polish-Slovenian grant No. 10 for the years 2008/09.

2. INVARIANTS $J$ AND $J'$

We denote the solid torus $D \times S^1$ ($D$ a disk) by $T$. For $n \in \mathbb{N}$, $T$ is divided by $n$ meridional disks $D = D \times e^{2\pi i k/n}$, $k = 1, \ldots, n$, into $n$ chambers $C_1, \ldots, C_n$. Let $L$ be a (tame) link in $T$, transversal to each $D_i$. For $k \in \{1, \ldots, n\}$, $L \cap C_k$ consists of closed curves and arcs with boundary of two types: arcs with two boundary points on the same meridional disk on one side of $C_k$ (non traversing arcs) or arcs with boundary points on different meridional disks on each side of $C_k$ (traversing arcs).

Denote the number of traversing arcs in $C_k$ by $j'_k(L)$ and the number of all arcs (traversing and non traversing) in $C_k$ by $j_k(L)$. Let $j'(L) = \sum_{k=1}^{n} j'_k(L)$ and $j(L) = \sum_{k=1}^{n} j_k(L)$.

**Definition 1.** $J'_n(L) = \min_{L \sim L'} j'(L')$ and $J_n(L) = \min_{L \sim L'} j(L')$, where the minima are taken over all links $L'$ isotopic to $L$ and transversal to each $D_i$.

**Lemma 1.** For $n < m$, $J_n(L) \leq J_m(L)$ and $J'_n(L) \leq J'_m(L)$

$J_n(L) = nJ_1(L)$

$J'_n(L) = J'_n(L) + C$ for some $C \geq 0$, for $n$ large enough.

We denote by $v(L)$ the maximum $n$ for which $J'_n(L) = 0$ (it is $\infty$ if all $J'_n(L) = 0$). We say that $L$ is affine if it lies in a ball in $T$, which is equivalent to $J_1(L) = 0$.

**Lemma 2.** Let $L$ be a non affine link in $T$ and $k$ the number of its components. Then $v(L) \leq k$

**Proof.** Subdivide $T$ into $n$ chambers, $n \leq v(L)$, so that $L$ lies in these chambers with no traversing arcs. As $L$ is non affine it has to intersect each of the $n$ meridional disks. A component of $L$ cannot intersect two different $D_i$-s, otherwise there would be traversing arcs. So there are at least as many components in $L$ as the number
of $D_1$-s, which is $n$. Thus $n \leq k$. Now if $v(L) < \infty$, we can take $n = v(L)$. If $v(L) = \infty$ then $n$ and so $k$ would be arbitrarily large, which is impossible for a tame link which has a finite number of components. □

**Corollary 1.** $L$ is affine iff $v(L) = \infty$.

**Proof.** If $L$ is affine then clearly, for each $k$, $J_k(L) = 0$ so $v(L) = \infty$. It $L$ is non affine then $v(L) \neq \infty$ from the preceding Lemma. □

### 3. Conway-Alexander skein module

Let $t_k$ be the knot in $T$ which has a diagram presented below on the left for $k = 3$.

![Diagram of knot $t_3$](image)

There is a naturally defined multiplication of links in $T$. For instance on the right of the preceding Figure we have $t_3 t_{-1}$. Using this multiplication one can present the basis of the Conway-Alexander skein module:

$$\{ t_{k_1}^{i_1} \cdots t_{k_s}^{i_s} : s \in \mathbb{N} \cup \{0\}, k_1 < \cdots < k_s \in \mathbb{Z}^+, i_1 \cdots i_s \in \mathbb{N} \} \cup \{ t_0 \},$$

where $t_0$ is the trivial affine knot.

The element $t_1$ is a generator of $H_1(T) = \mathbb{Z}$ (we can take $[t_1] = 1$ so that

$$[t_{k_1}^{i_1} \cdots t_{k_s}^{i_s}] = \sum_{i=1}^{s} k_i i_i.$$

We define the degree of such elements taking $\deg(t_k) = |k|$ and $\deg(FG) = \deg(F)\deg(G)$. Then in each degree $d$ there is a finite number $b(d)$ of some monic monomials $T_{d,i}$, $1 \leq i \leq b(d)$, of degree $d$. For instance, for $d = 1$ there is $t_1$ and $t_{-1}$ and for $d = 2$ there are $t_2, t_{-2}, t_1^2, t_{-1}^2$ and $t_1 t_{-1}$.

For a link $L$, its Conway-Alexander polynomial is of the form:

$$\nabla(L) = P_0(z) t_0 + \sum_{i=1}^{b(2)} P_{2,i}(z) T_{2,i} + \sum_{i=1}^{b(4)} P_{4,i}(z) T_{4,i} + \cdots$$

or

$$\nabla(L) = \sum_{i=1}^{b(1)} P_{1,i}(z) T_{1,i} + \sum_{i=1}^{b(3)} P_{3,i}(z) T_{3,i} + \cdots$$

where all $T_{k,i}$-s have the same homology class as $L$.

For instance, if $L$ is 0-homologous then in degree 2 the only possible term is $t_1 t_{-1}$ and in degree 4 the only possible terms are $t_1^2 t_{-1}^2, t_2 t_{-2}, t_2 t_{-2}^2$ and $t_1^2 t_{-2}$.

Denote by $m_j(L)$ the minimum degree of $z$ appearing in all $P_{j,i}$-s (equal $\infty$ if all these polynomials are 0).

**Theorem 1.** Let $L$ be a non affine link in $T$. Then $m_j(L) \geq v(L) j / 2$. 
**Proof.** From the corollary, \( v(L) < \infty \). If \( v(L) = 0 \) the inequality is trivial. So assume \( v(L) > 0 \). By assumption \( L \) can lie in \( v(L) \) chambers with no traversing arcs. To obtain an element of degree \( j \) there has to be \( j \) traversing arcs in each chamber. Switching a crossing does not change the type of any arc, whereas smoothing a crossing can at most create two traversing arcs. Thus, to get the \( v(L)j \) traversing arcs needed, one has to do at least \( v(L)j = 2 \) smoothings, each corresponding to a multiplication by \( z \). So the lowest possible degree of \( z \) is \( v(L)j = 2 \).  

**Example 1** For a chain with \( n \) components \( L_n \), \( \nabla(L_n) \) will have only terms of degree 0 and 2. The degree 2 part will be of the form \( z^n t^1 t^{-1} \). Clearly, from the Figure, \( v(L_n) \geq n \). From the Proposition \( v(L) \leq n \). Thus \( v(L) = n \).

### 4. HOMFLYPT skein module

For a link \( L \), its HOMFLYPT polynomial is of the form:

\[
H(L) = Q_0(x, z)t_0 + \sum_{i=1}^{b(2)} Q_2,i(x, z)T_{2,i} + \sum_{i=1}^{b(4)} Q_4,i(x, z)T_{4,i} + ..
\]

or

\[
H(L) = \sum_{i=1}^{b(1)} Q_1,i(x, z)T_{1,i} + \sum_{i=1}^{b(3)} Q_3,i(x, z)T_{3,i} + ..
\]

where all \( T_{k,i} \)-s have the same homology class as \( L \) and the \( Q_{k,i} \)-s are polynomials in \( z \) and Laurent polynomials in \( x \).

For the HOMFLYPT skein module, smoothing corresponds to multiplying by \( z \). On the other hand some \( z \) are cancelled when there are trivial components at the cost of multiplying by \( x^{-1} - x \). For \( Q \), a polynomial in \( z \) and Laurent polynomial in \( x \) let \( Q = (x^{-1} - x)^a Q' \), where \( x^{-1} - x \) does not divide \( Q' \), and let \( d(Q) \) be the sum of the degree in \( z \) and \( a \).

Denote by \( m_j'(L) \) the minimum of \( d(Q_{j,i,i}) \) for all \( i,j \)-s.

The following theorem is proved similarly to the Conway-Alexander polynomial case:

**Theorem 2.** Let \( L \) be a non affine link in \( T \). Then \( m_j'(L) \geq v(L)j/2 \).