Configuration spaces of thick particles in a rectangle

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Let X be a topological space and X^k its k-fold cartesian product. Define the complete diagonal Δ of X^k as follows: $\Delta = \{(x_1, \dots, x_k) \in X^k | x_i = x_j \text{ for some } i \neq j\}$. The topological space $X^k \setminus \Delta$ is called the (ordered) configuration space of k particles in X and denoted by $F_k(X)$ the space. If X is a metric space, there are several possibilities to endow the cartesian product X^k with a metric. We can take on X^k the metric d_∞, for example. Given a number $\varepsilon > 0$, one can consider instead of the diagonal Δ in X^k its ε-neighborhood $U_\varepsilon(\Delta)$ and remove it from X^k. The obtained space is denoted by $F_k(X, \varepsilon)$ and called the (ordered) configuration space of k thick particles in X (see [4, 5]).

The topology of configuration spaces $F_k(X, \varepsilon)$ has been studied by several authors. In particular, K.Deeley [4, 5] considered $F_k(G, \varepsilon)$ for finite (multi)graphs G endowed with a standard graph metric. For this purpose, K.Deeley adopted PL Morse theory for affine polytope complexes and developed PL Morse-Bott theory (see also [2]). He studied how the topology of $F_k(G, \varepsilon)$ changes when ε varies.

If X is a compact manifold with the boundary ∂X embedded in Euclidean space $\mathbf{R}^n$ and the particles are disks D_i of the same radius ε centered at the points $x_i \in X$, we have an extra condition on positions of discs D_i in X that $d(x_i, \partial X) \geq \varepsilon, i = 1, \dots, k$. The corresponding configuration space of n discs in X is denoted $C_k(X, \varepsilon)$.

The case when X is a square Q in $\mathbf{R}^2$ was studied in [3]. The more general case, when X is a box B in Euclidean space, was treated in [1]. The topological properties of the configuration space $C_k(X, \varepsilon)$ were studied in [1] via the tautological function τ_k, defined on the space $F_k(X)$ (see Section 2 for precise definitions). The behavior of the critical values of the function τ_k gives a description of how the topology of the configurations spaces $C_k(B, \varepsilon)$ changes when ε varies. In [1], the authors develop the so called *min* type Morse theory and apply it to the study of the properties of the functions τ_k. Note that τ_k are not certainly smooth functions.

In [3], the authors also used approximations of τ_k by smooth functions. This approach gives, in general, a good description of critical values of τ_k for small numbers k and shows how changes the homotopy type of $C_k(Q, \varepsilon)$ when ε varies. If ε is small enough, the spaces $C_k(B, \varepsilon)$ and $F_k(B)$ are homotopy equivalent [1]. In [1] and [3], the description of critical points of the tautological function τ_k is also given in combinatorial terms (via existence of stress graphs). The homological methods turn out to be important and helpful in the study of critical values r of the function τ_k (see [1, 3]).

In this work, we study configuration spaces of n thick particles in a rectangle Q where the particles are represented by small (non overlapping) squares. The configuration space of n equal squares in Q is denoted by $F_n(Q, r)$.

In Section 1, we define for each $n \geq 2$ the tautological function $\theta_n \colon Q^n \to \mathbf{R}_+$. θ_n is an analogue of the tautological function τ_n, adopted to the study of the configuration space $F_n(Q, r)$. We adopt PL Morse-Bott theory to the study of the tautological function θ_n defined on the affine polytope Q^n. In Section 2, we define critical points of θ_n in geometric terms. The combinatorial description of critical points of θ_n is also given. We show that near the regular values r of the function θ_n, the homotopy type of sublevel spaces $\theta_n^{-1}[r, \infty)$ (i.e. the configuration spaces $F_n(Q,r))$ does not change. We also compare the critical values of both tautological functions, θ_n and τ_n.

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