AN INTRODUCTION TO
MALLIAVIN CALCULUS
WITH APPLICATIONS TO ECONOMICS

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Preface

These are unpolished lecture notes from the course BF 05 "Malliavin calculus with applications to economics", which I gave at the Norwegian School of Economics and Business Administration (NHH), Bergen, in the Spring semester 1996. The application I had in mind was mainly the use of the Clark-Ocone formula and its generalization to finance, especially portfolio analysis, option pricing and hedging. This and other applications are described in the impressive paper by Karatzas and Ocone [KO] (see reference list in the end of Chapter 5). To be able to understand these applications, we had to work through the theory and methods of the underlying mathematical machinery, usually called the Malliavin calculus. The main literature we used for this part of the course are the books by Ustunel [U] and Nualart [N] regarding the analysis on the Wiener space, and the forthcoming book by Holden, Øksendal, Ubøe and Zhang [HØUZ] regarding the related white noise analysis (Chapter 3). The prerequisites for the course are some basic knowledge of stochastic analysis, including Ito integrals, the Ito representation theorem and the Girsanov theorem, which can be found in e.g. [Ø1].

The course was followed by an inspiring group of (about a dozen) students and employees at HNN. I am indebted to them all for their active participation and useful comments. In particular, I would like to thank Knut Aase for his help in getting the course started and his constant encouragement. I am also grateful to Kerry Back, Darrell Duffie, Yaozhong Hu, Monique Jeanblanc-Picque and Dan Ocone for their useful comments and to Dina Haraldsson for her proficient typing.

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Contents

1 The Wiener-Ito chaos expansion

The celebrated Wiener-Ito chaos expansion is fundamental in stochastic analysis. In particular, it plays a crucial role in the Malliavin calculus. We therefore give a detailed proof.

The first version of this theorem was proved by Wiener in 1938. Later Ito (1951) showed that in the Wiener space setting the expansion could be expressed in terms of *iterated Ito integrals* (see below).

Before we state the theorem we introduce some useful notation and give some auxiliary results.

Let $W(t) = W(t,\omega)$; $t \geq 0$, $\omega \in \Omega$ be a 1-dimensional Wiener process (Brownian motion) on the probability space $(\Omega, \mathcal{F}, P)$ such that $W(0,\omega) = 0$ a.s. P.

For $t \geq 0$ let $\mathcal{F}_t$ be the σ-algebra generated by $W(s,\cdot)$; $0 \leq s \leq t$. Fix $T > 0$ (constant).

A real function $g : [0,T]^n \to \mathbf{R}$ is called *symmetric* if

$$g(x_{\sigma_1}, \ldots, x_{\sigma_n}) = g(x_1, \ldots, x_n) \tag{1.1}$$

for all permutations σ of $(1, 2, \ldots, n)$. If in addition

$$\|g\|^2_{L^2([0,T]^n)} := \int_{[0,T]^n} g^2(x_1, \ldots, x_n) dx_1 \cdots dx_n < \infty \tag{1.2}$$

we say that $g \in \widehat{L}^2([0,T]^n)$, the space of *symmetric square integrable functions* on $[0,T]^n$.

Let

$$S_n = \{(x_1, \ldots, x_n) \in [0,T]^n; \;\; 0 \leq x_1 \leq x_2 \leq \cdots \leq x_n \leq T\}. \tag{1.3}$$

The set S_n occupies the fraction $\frac{1}{n!}$ of the whole n-dimensional box $[0,T]^n$. Therefore, if $g \in \widehat{L}^2([0,T]^n)$ then

$$\|g\|^2_{L^2([0,T]^n)} = n! \int_{S_n} g^2(x_1, \ldots, x_n) dx_1 \ldots dx_n = n! \|g\|^2_{L^2(S_n)} \tag{1.4}$$

If f is any real function defined on $[0,T]^n$, then the *symmetrization* $\tilde{f}$ of f is defined by

$$\widetilde{f}(x_1, \ldots, x_n) = \frac{1}{n!} \sum_{\sigma} f(x_{\sigma_1}, \ldots, x_{\sigma_n}) \tag{1.5}$$

where the sum is taken over all permutations σ of $(1, \ldots, n)$. Note that $\widetilde{f} = f$ if and only if f is symmetric. For example if

$$f(x_1, x_2) = x_1^2 + x_2 \sin x_1$$

then

$$\widetilde{f}(x_1, x_2) = \frac{1}{2}[x_1^2 + x_2^2 + x_2 \sin x_1 + x_1 \sin x_2].$$

Note that if f is a deterministic function defined on S_n $(n \geq 1)$ such that

$$\|f\|^2_{L^2(S_n)} := \int_{S_n} f^2(t_1, \ldots, t_n) dt_1 \cdots dt_n < \infty,$$

then we can form *the (n-fold) iterated Ito integral*

$$J_n(f) := \int_0^T \int_0^{t_n} \cdots \int_0^{t_3} (\int_0^{t_2} f(t_1, \ldots, t_n) dW(t_1)) dW(t_2) \cdots dW(t_{n-1}) dW(t_n), \tag{1.6}$$

because at each Ito integration with respect to $dW(t_i)$ the integrand is $\mathcal{F}_t$-adapted and square integrable with respect to $dP \times dt_i$, $1 \leq i \leq n$.

Moreover, applying the Ito isometry iteratively we get

$$\begin{aligned} E[J_n^2(h)] &= E[\{\int_0^T (\int_0^{t_n} \cdots \int_0^{t_2} h(t_1, \ldots, t_n) dW(t_1) \cdots) dW(t_n)\}^2] \\ &= \int_0^T E[(\int_0^{t_n} \cdots \int_0^{t_2} h(t_1, \ldots, t_n) dW(t_1) \cdots dW(t_{n-1}))^2] dt_n \\ &= \cdots = \int_0^T \int_0^{t_n} \cdots \int_0^{t_2} h^2(t_1, \ldots, t_n) dt_1 \cdots dt_n = \|h\|^2_{L^2(S_n)}. \qquad (1.7) \end{aligned}$$

Similarly, if $g \in L^2(S_m)$ and $h \in L^2(S_n)$ with $m < n$, then by the Ito isometry applied iteratively we see that

$$\begin{aligned} &E[J_m(g) J_n(h)] \\ &= E[\{\int_0^T (\int_0^{s_m} \cdots \int_0^{s_2} g(s_1, \ldots, s_m) dW(s_1) \cdots dW(s_m)\} \\ &\quad \{\int_0^T (\int_0^{s_m} \cdots \int_0^{t_2} h(t_1, \ldots, t_{n-m}, s_1, \ldots, s_m) dW(t_1) \cdots) dW(s_m)\}] \\ &= \int_0^T E[\{\int_0^{s_m} \cdots \int_0^{s_2} g(s_1, \ldots, s_{m-1}, s_m) dW(s_1) \cdots dW(s_{m-1})\} \\ &\quad \{\int_0^{s_m} \cdots \int_0^{t_2} h(t_1, \ldots, s_{m-1}, s_m) dW(t_1) \cdots dW(s_{m-1})\}] ds_m \\ &= \int_0^T \int_0^{s_m} \cdots \int_0^{s_2} E[g(s_1, s_2, \ldots, s_m) \int_0^{s_1} \cdots \int_0^{t_2} h(t_1, \ldots, t_{n-m}, s_1, \ldots, s_m) \\ &\quad dW(t_1) \cdots dW(t_{n-m})] ds_1, \cdots ds_m \\ &= 0 \qquad (1.8) \end{aligned}$$

because the expected value of an Ito integral is zero.

We summarize these results as follows:

$$E[J_m(g)J_n(h)] = \begin{cases} 0 & \text{if } n \neq m \\ (g,h)_{L^2(S_n)} & \text{if } n = m \end{cases} \tag{1.9}$$

where

$$(g,h)_{L^2(S_n)} = \int\limits_{S_n} g(x_1,\dots,x_n)h(x_1,\dots,x_n)dx_1\cdots dx_n \tag{1.10}$$

is the inner product of $L^2(S_n)$.

Note that (1.9) also holds for $n=0$ or $m=0$ if we define

$$J_0(g) = g \qquad \text{if } g \text{ is a constant}$$

and

$$(g,h)_{L^2(S_0)} = gh \qquad \text{if } g,h \text{ are constants.}$$

If $g \in \widehat{L}^2([0,T]^n)$ we define

$$I_n(g) := \int\limits_{[0,T]^n} g(t_1,\dots,t_n)dW^{\otimes n}(t) := n!J_n(g) \tag{1.11}$$

Note that from (1.7) and (1.11) we have

$$E[I_n^2(g)] = E[(n!)^2 J_n^2(g)] = (n!)^2\|g\|^2_{L^2(S_n)} = n!\|g\|^2_{L^2([0,T]^n)} \tag{1.12}$$

for all $g \in \widehat{L}^2([0,T]^n)$.

Recall that the *Hermite polynomials* $h_n(x);\ n=0,1,2,\dots$ are defined by

$$h_n(x) = (-1)^n e^{\frac{1}{2}x^2}\frac{d^n}{dx^n}(e^{-\frac{1}{2}x^2}); \qquad n=0,1,2,\dots \tag{1.13}$$

Thus the first Hermite polynomials are

$$\begin{aligned} h_0(x) &= 1,\ h_1(x)=x,\ h_2(x)=x^2-1,\ h_3(x)=x^3-3x, \\ h_4(x) &= x^4-6x^2+3,\ h_5(x)=x^5-10x^3+15x,\dots \end{aligned}$$

There is a useful formula due to Ito [I] for the iterated Ito integral in the special case when the integrand is the tensor power of a function $g \in L^2([0,T])$:

$$n!\int_0^T\int_0^{t_n}\cdots\int_0^{t_2} g(t_1)g(t_2)\cdots g(t_n)dW(t_1)\cdots dW(t_n) = \|g\|^n h_n(\frac{\theta}{\|g\|}), \tag{1.14}$$

where

$$\|g\| = \|g\|_{L^2([0,T])} \qquad \text{and} \qquad \theta = \int_0^T g(t)dW(t).$$

For example, choosing $g \equiv 1$ and $n = 3$ we get

$$6 \cdot \int_0^T \int_0^{t_3} \int_0^{t_2} dW(t_1)dW(t_2)dW(t_3) = T^{3/2} h_3\left(\frac{W(T)}{T^{1/2}}\right) = W^3(T) - 3T\,W(T).$$

THEOREM 1.1. (The Wiener-Ito chaos expansion) Let φ be an $\mathcal{F}_T$-measurable random variable such that

$$\|\varphi\|^2_{L^2(\Omega)} := \|\varphi\|^2_{L^2(P)} := E_P[\varphi^2] < \infty\,.$$

Then there exists a (unique) sequence $\{f_n\}_{n=0}^{\infty}$ of (deterministic) functions $f_n \in \widehat{L}^2([0,T]^n)$ such that

$$(1.15) \qquad \varphi(\omega) = \sum_{n=0}^{\infty} I_n(f_n) \qquad \text{(convergence in } L^2(P)\text{)}.$$

Moreover, we have the isometry

$$(1.16) \qquad \|\varphi\|^2_{L^2(P)} = \sum_{n=0}^{\infty} n!\|f_n\|^2_{L^2([0,T]^n)}$$

Proof. By the Ito representation theorem there exists an $\mathcal{F}_t$-adapted process $\varphi_1(s_1,\omega)$, $0 \le s_1 \le T$ such that

$$(1.17) \qquad E\Big[\int_0^T \varphi_1^2(s_1,\omega)ds_1\Big] \le \|\varphi\|^2_{L^2(P)}$$

and

$$(1.18) \qquad \varphi(\omega) = E[\varphi] + \int_0^T \varphi_1(s_1,\omega)dW(s_1)$$

Define

$$(1.19) \qquad g_0 = E[\varphi] \quad \text{(constant)}.$$

For a.a. $s_1 \le T$ we apply the Ito representation theorem to $\varphi_1(s_1,\omega)$ to conclude that there exists an $\mathcal{F}_t$-adapted process $\varphi_2(s_2,s_1,\omega)$; $0 \le s_2 \le s_1$ such that

$$(1.20) \qquad E\Big[\int_0^{s_1} \varphi_2^2(s_2,s_1,\omega)ds_2\Big] \le E[\varphi_1^2(s_1)] < \infty$$

and

$$(1.21) \qquad \varphi_1(s_1,\omega) = E[\varphi_1(s_1)] + \int_0^{s_1} \varphi_2(s_2,s_1,\omega)dW(s_2).$$

Substituting (1.21) in (1.18) we get

$$\varphi(\omega)=g_0+\int_0^T g_1(s_1)dW(s_1)+\int_0^T(\int_0^{s_1}\varphi_2(s_2,s_1,\omega)dW(s_2)dW(s_1) \tag{1.22}$$

where

$$g_1(s_1)=E[\varphi_1(s_1)]. \tag{1.23}$$

Note that by the Ito isometry, (1.17) and (1.20) we have

$$E[\{\int_0^T(\int_0^{s_1}\varphi_2(s_1,s_2,\omega)dW(s_2))dW(s_1)\}^2]=\int_0^T(\int_0^{s_1}E[\varphi_2^2(s_1,s_2,\omega)]ds_2)ds_1\leq\|\varphi\|^2_{L^2(P)}. \tag{1.24}$$

Similarly, for a.a. $s_2\leq s_1\leq T$ we apply the Ito representation theorem to $\varphi_2(s_2,s_1,\omega)$ to get an $\mathcal{F}_t$-adapted process $\varphi_3(s_3,s_2,s_1,\omega)$; $0\leq s_3\leq s_2$ such that

$$E[\int_0^{s_2}\varphi_3^2(s_3,s_2,s_1,\omega)ds_3]\leq E[\varphi_2^2(s_2,s_1)]<\infty \tag{1.25}$$

and

$$\varphi_2(s_2,s_1,\omega)=E[\varphi_2(s_2,s_1,\omega)]+\int_0^{s_2}\varphi_3(s_3,s_2,s_1,\omega)dW(s_3). \tag{1.26}$$

Substituting (1.26) in (1.22) we get

$$\begin{aligned}\varphi(\omega)&=g_0+\int_0^T g_1(s_1)dW(s_1)+\int_0^T(\int_0^{s_1}g_2(s_2,s_1)dW(s_2))dW(s_1)\\&+\int_0^T(\int_0^{s_1}(\int_0^{s_2}\varphi_3(s_3,s_2,s_1,\omega)dW(s_3))dW(s_2))dW(s_1),\end{aligned} \tag{1.27}$$

where

$$g_2(s_2,s_1)=E[\varphi_2(s_2,s_1)];\qquad 0\leq s_2\leq s_1\leq T. \tag{1.28}$$

By the Ito isometry, (1.17), (1.20) and (1.25) we have

$$E[\{\int_0^T\int_0^{s_1}\int_0^{s_2}\varphi_3(s_3,s_2,s_1,\omega)dW(s_3)dW(s_2)dW(s_3)\}^2]\leq\|\varphi\|^2_{L^2(P)}\,. \tag{1.29}$$

By iterating this procedure we obtain by induction after n steps a process $\varphi_{n+1}(t_1,t_2,\ldots,t_{n+1},\omega)$; $0\leq t_1\leq t_2\leq\cdots\leq t_{n+1}\leq T$ and $n+1$ deterministic functions $g_0,g_1,\ldots,g_n$ with g_0 constant and g_k defined on S_k for $1\leq k\leq n$, such that

$$\varphi(\omega)=\sum_{k=0}^n J_k(g_k)+\int_{S_{n+1}}\varphi_{n+1}dW^{\otimes(n+1)}, \tag{1.30}$$

where

$$(1.31) \qquad \int_{S_{n+1}} \varphi_{n+1} dW^{\otimes(n+1)} = \int_0^T \int_0^{t_{n+1}} \cdots \int_0^{t_2} \varphi_{n+1}(t_1, \dots, t_{n+1}, \omega) dW(t_1) \cdots dW(t_{n+1})$$

is the $(n+1)$-fold iterated integral of φ_{n+1}. Moreover,

$$(1.32) \qquad E[\{\int_{S_{n+1}} \varphi_{n+1} dW^{\otimes(n+1)}\}^2] \leq \|\varphi\|^2_{L^2(\Omega)}.$$

In particular, the family

$$\psi_{n+1} := \int_{S_{n+1}} \varphi_{n+1} dW^{\otimes(n+1)}; \qquad n = 1, 2, \dots$$

is bounded in $L^2(P)$. Moreover

$$(1.33) \qquad (\psi_{n+1}, J_k(f_k))_{L^2(\Omega)} = 0 \qquad \text{for } k \leq n, \ f_k \in L^2([0,T]^k).$$

Hence by the Pythagorean theorem

$$(1.34) \qquad \|\varphi\|^2_{L^2(\Omega)} = \sum_{k=0}^{n} \|J_k(g_k)\|^2_{L^2(\Omega)} + \|\psi_{n+1}\|^2_{L^2(\Omega)}$$

In particular,

$$\sum_{k=0}^{n} \|J_k(g_k)\|^2_{L^2(\Omega)} < \infty$$

and therefore $\sum_{k=0}^{\infty} J_k(g_k)$ is strongly convergent in $L^2(\Omega)$. Hence

$$\lim_{n\to\infty} \psi_{n+1} =: \psi \quad \text{exists (limit in } L^2(\Omega))$$

But by (1.33) we have

$$(1.35) \qquad (J_k(f_k), \psi)_{L^2(\Omega)} = 0 \qquad \text{for all } k \text{ and all } f_k \in L^2([0,T]^k)$$

In particular, by (1.14) this implies that

$$E[h_k(\frac{\theta}{\|g\|}) \cdot \psi] = 0 \qquad \text{for all } g \in L^2([0,T]), \text{ all } k \geq 0$$

where $\theta = \int_0^T g(t) dW(t)$.

But then, from the definition of the Hermite polynomials,

$$E[\theta^k \cdot \psi] = 0 \qquad \text{for all } k \geq 0$$

which again implies that

$$E[\exp\theta\cdot\psi]=\sum_{k=0}^{\infty}\frac{1}{k!}E[\theta^k\cdot\psi]=0.$$

Since the family

$$\{\exp\theta;\quad g\in L^2([0,T])\}$$

is dense in $L^2(\Omega)$ (see [Ø1], Lemma 4.9), we conclude that

$$\psi=0.$$

Hence

$$\varphi(\omega)=\sum_{k=0}^{\infty}J_k(g_k)\qquad(\text{convergence in }\ L^2(\Omega)) \tag{1.36}$$

and

$$\|\varphi\|^2_{L^2(\Omega)}=\sum_{k=0}^{n}\|J_k(g_k)\|^2_{L^2(\Omega)}. \tag{1.37}$$

Finally, to obtain (1.15)–(1.16) we proceed as follows:

The function g_n is only defined on S_n, but we can extend g_n to $[0,T]^n$ by putting

$$g_n(t_1,\dots,t_n)=0\qquad\text{if }(t_1,\dots,t_n)\in[0,T]^n\setminus S_n. \tag{1.38}$$

Now define

$$f_n=\widetilde{g}_n,\qquad\text{the symmetrization of }g.$$

Then

$$I_n(f_n)=n!J_n(f_n)=n!J_n(\widetilde{g}_n)=J_n(g_n)$$

and (1.15)–(1.16) follow from (1.36) and (1.37). □

Examples

1) What is the Wiener-Ito expansion of

$$\varphi(\omega)=W^2(T,\omega)\,?$$

From (1.14) we get

$$2\int_0^T\Big(\int_0^{t_2}dW(t_1)\Big)dW(t_2)=Th_2\Big(\frac{W(T)}{T^{1/2}}\Big)=W^2(T)-T,$$

and therefore

$$W^2(T)=T+I_2(1).$$

2) Note that for $t \in (0,T)$ we have

$$\int_0^T (\int_0^{t_2} \mathcal{X}_{\{t_1<t<t_2\}}(t_1,t_2)dW(t_1))dW(t_2)$$

$$= \int_t^T W(t)dW(t_2) = W(t)(W(T)-W(t)).$$

Hence, if we put

$$\varphi(\omega) = W(t)(W(T)-W(t)), \qquad g(t_1,t_2) = \mathcal{X}_{\{t_1<t<t_2\}}$$

we see that

$$\varphi(\omega) = J_2(g) = 2J_2(\tilde{g}) = I_2(f_2),$$

where

$$f_2(t_1,t_2) = \tilde{g}(t_1,t_2) = \frac{1}{2}(\mathcal{X}_{\{t_1<t<t_2\}} + \mathcal{X}_{\{t_2<t<t_1\}}).$$

Exercises

1.1 a) Let $h_n(x)$; $n = 0,1,2,\dots$ be the Hermite polynomials, defined in (1.13). Prove that

$$\exp(tx - \frac{t^2}{2}) = \sum_{n=0}^{\infty} \frac{t^n}{n!} h_n(x) \qquad \text{for all } t, x.$$

(Hint: Write

$$\exp(tx - \frac{t^2}{2}) = \exp(\frac{1}{2}x^2) \cdot \exp(-\frac{1}{2}(x-t)^2)$$

and apply the Taylor formula on the last factor.)

b) Show that if $\lambda > 0$ then

$$\exp(tx - \frac{t^2\lambda}{2}) = \sum_{n=0}^{\infty} \frac{t^n \lambda^{\frac{n}{2}}}{n!} h_n(\frac{x}{\sqrt{\lambda}}).$$

c) Let $g \in L^2([0,T])$ be deterministic. Put

$$\theta = \theta(\omega) = \int_0^T g(s)dW(s)$$

and

$$\|g\| = \|g\|_{L^2([0,T])} = (\int_0^T g^2(s)ds)^{1/2}.$$

Show that

$$\exp(\int_0^T g(s)dW(s) - \frac{1}{2}\|g\|^2) = \sum_{n=0}^{\infty} \frac{\|g\|^n}{n!} h_n(\frac{\theta}{\|g\|})$$

d) Let $t \in [0,T]$. Show that

$$\exp(W(t) - \frac{1}{2}t) = \sum_{n=0}^{\infty} \frac{t^{n/2}}{n!} h_n(\frac{W(t)}{\sqrt{t}}).$$

1.2 Find the Wiener-Ito chaos expansion of the following random variables:

a) $\varphi(\omega) = W(t,\omega)$ $\quad$ ($t \in [0,T]$ fixed)

b) $\varphi(\omega) = \int_0^T g(s)dW(s)$ $\quad$ ($g \in L^2([0,T])$ deterministic)

c) $\varphi(\omega) = W^2(t,\omega)$ $\quad$ ($t \in [0,T]$ fixed)

d) $\varphi(\omega) = \exp(\int_0^T g(s)dW(s))$ $\quad$ ($g \in L^2([0,T])$ deterministic)

(Hint: Use (1.14).)

1.3 The *Ito representation theorem* states that if $F \in L^2(\Omega)$ is $\mathcal{F}_T$-measurable, then there exists a unique $\mathcal{F}_t$-adapted process $\varphi(t,\omega)$ such that

$$F(\omega) = E[F] + \int_0^T \varphi(t,\omega)dW(t). \tag{1.40}$$

(See e.g. [Ø1], Theorem 4.10.)

As we will show in Chapter 5, this result is important in mathematical finance. Moreover, it is important to be able to find more explicitly the integrand $\varphi(t,\omega)$. This is achieved by the *Clark-Ocone formula*, which says that (under some extra conditions)

$$\varphi(t,\omega) = E[D_tF|\mathcal{F}_t](\omega), \tag{1.41}$$

where D_tF is the (Malliavin) derivative of F. We will return to this in Chapters 4 and 5.

For special functions $F(\omega)$ it is possible to find $\varphi(t,\omega)$ directly, by using the Ito formula. For example, find $\varphi(t,\omega)$ when

a) $F(\omega) = W^2(T)$

b) $F(\omega) = \exp W(T)$

c) $F(\omega) = \int_0^T W(t)dt$

d) $F(\omega) = W^3(T)$

e) $F(\omega) = \cos W(T)$

(Hint: Check that $N(t) := e^{\frac{1}{2}t}\cos W(t)$ is a martingale.)

1.4. [Hu] Suppose the function F of Exercise 1.3 has the form

$$F(\omega) = f(X(T)) \tag{1.42}$$

where $X(t) = X(t,\omega)$ is an Ito diffusion given by

$$dX(t) = b(X(t))dt + \sigma(X(t))dW(t); \qquad X(0) = x \in \mathbf{R}. \tag{1.43}$$

Here $b\colon \mathbf{R} \to \mathbf{R}$ and $\sigma\colon \mathbf{R} \to \mathbf{R}$ are given Lipschitz continuous functions of at most linear growth, so (1.43) has a unique solution $X(t)$; $t \geq 0$. Then there is a useful formula for the process $\varphi(t,\omega)$ in (1.40). This is described as follows:

If g is a real function with the property

$$E^x[|g(X(t))|] < \infty \qquad \text{for all } t \geq 0,\ x \in \mathbf{R} \tag{1.44}$$

(where E^x denotes expectation w.r.t. the law of $X(t)$ when $X(0) = x$) then we define

$$u(t,x) := P_t g(x) := E^x[g(X(t))]; \qquad t \geq 0,\ x \in \mathbf{R} \tag{1.45}$$

Suppose that there exists $\delta > 0$ such that

$$|\sigma(x)| \geq \delta \qquad \text{for all } x \in \mathbf{R} \tag{1.46}$$

Then $u(t,x) \in C^{1,2}(\mathbf{R}^+ \times \mathbf{R})$ and

$$\frac{\partial u}{\partial t} = b(x)\frac{\partial u}{\partial x} + \frac{1}{2}\sigma^2(x)\frac{\partial^2 u}{\partial x^2} \qquad \text{for all } t \geq 0,\ x \in \mathbf{R} \tag{1.47}$$

(Kolmogorov's backward equation).

See e.g. [D2], Theorem 13.18 p. 53 and [D1], Theorem 5.11 p. 162 and [Ø1], Theorem 8.1.

a) Use Ito's formula for the process

$$Y(t) = g(t, X(t)) \quad \text{with} \quad g(t,x) = P_{T-t}f(x)$$

to show that

$$f(X(T)) = P_T f(x) + \int_0^T [\sigma(\xi)\frac{\partial}{\partial \xi}P_{T-t}f(\xi)]_{\xi = X(t)}dW(t) \tag{1.48}$$

for all $f \in C^2(\mathbf{R})$.

In other words, with the notation of Exercise 1.3 we have shown that if $F(\omega) = f(X(T))$ then

$$E[F] = P_T f(x) \quad \text{and} \quad \varphi(t,\omega) = [\sigma(\xi)\frac{\partial}{\partial \xi}P_{T-t}f(\xi)]_{\xi = X(t)}. \tag{1.49}$$

Use (1.49) to find $E[F]$ and $\varphi(t,\omega)$ when

b) $F(\omega) = W^2(T)$

c) $F(\omega) = W^3(T)$

d) $F(\omega) = X(T, \omega)$

where

$$dX(t) = \rho X(t)dt + \alpha X(t)dW(t) \qquad (\rho, \alpha \text{ constants})$$

i.e. $X(t)$ is geometric Brownian motion.

e) Extend formula (1.48) to the case when $X(t) \in \mathbf{R}^n$ and $f\colon \mathbf{R}^n \to \mathbf{R}$. In this case condition (1.46) must be replaced by the condition

$$\eta^T \sigma^T(x)\sigma(x)\eta \geq \delta|\eta|^2 \qquad \text{for all } x, \eta \in \mathbf{R}^n \tag{1.50}$$

where $\sigma^T(x)$ denotes the transposed of the $m \times n$-matrix $\sigma(x)$.

2 The Skorohod integral

The Wiener-Ito chaos expansion is a convenient starting point for the introduction of several important stochastic concepts, including the *Skorohod integral.* This integral may be regarded as an extenstion of the Ito integral to integrands which are not necessarily $\mathcal{F}_t$-adapted. It is also connected to the Malliavin derivative. We first introduce some convenient notation.

Let $u(t,\omega)$, $\omega \in \Omega$, $t \in [0,T]$ be a stochastic process (always assumed to be (t,ω)-measurable), such that

$$u(t,\cdot) \qquad \text{is } \mathcal{F}_T\text{-measurable for all } t \in [0,T] \tag{2.1}$$

and

$$E[u^2(t,\omega)] < \infty \qquad \text{for all } t \in [0,T]. \tag{2.2}$$

Then for each $t \in [0,T]$ we can apply the Wiener-Ito chaos expansion to the random variable $\omega \to u(t,\omega)$ and obtain functions $f_{n,t}(t_1,\dots,t_n) \in \widehat{L}^2(\mathbf{R}^n)$ such that

$$u(t,\omega) = \sum_{n=0}^{\infty} I_n(f_{n,t}(\cdot)). \tag{2.3}$$

The functions $f_{n,t}(\cdot)$ depend on the parameter t, so we can write

$$f_{n,t}(t_1,\dots,t_n) = f_n(t_1,\dots,t_n,t) \tag{2.4}$$

Hence we may regard f_n as a function of $n+1$ variables $t_1,\dots,t_n,t$. Since this function is symmetric with respect to its first n variables, its *symmetrization* $\widetilde{f}_n$ as a function of $n+1$ variables $t_1,\dots,t_n,t$ is given by, with $t_{n+1} = t$,

$$\widetilde{f}_n(t_1,\dots,t_{n+1}) = \tag{2.5}$$

$$\frac{1}{n+1}[f_n(t_1,\dots,t_{n+1}) + \cdots + f_n(t_1,\dots,t_{i-1},t_{i+1},\dots,t_{n+1},t_i) + \cdots + f_n(t_2,\dots,t_{n+1},t_1)],$$

where we only sum over those permutations σ of the indices $(1,\dots,n+1)$ which interchange the *last* component with one of the others and leave the rest in place.

EXAMPLE 2.1. Suppose

$$f_{2,t}(t_1,t_2) = f_2(t_1,t_2,t) = \frac{1}{2}[\mathcal{X}_{\{t_1<t<t_2\}} + \mathcal{X}_{\{t_2<t<t_1\}}].$$

Then the symmetrization $\widetilde{f}_2(t_1,t_2,t_3)$ of f_2 as a function of 3 variables is given by

$$\widetilde{f}_2(t_1,t_2,t_3) = \frac{1}{3}[\frac{1}{2}(\mathcal{X}_{\{t_1<t_3<t_2\}} + \mathcal{X}_{\{t_2<t_3<t_1\}}) + \frac{1}{2}(\mathcal{X}_{\{t_1<t_2<t_3\}}$$
$$+\mathcal{X}_{\{t_3<t_2<t_1\}}) + \frac{1}{2}(\mathcal{X}_{\{t_2<t_1<t_3\}} + \mathcal{X}_{\{t_3<t_1<t_2\}})]$$

This sum is $\frac{1}{6}$ except on the set where some of the variables coincide, but this set has measure zero, so we have

$$\widetilde{f}_2(t_1,t_2,t_3) = \frac{1}{6} \quad \text{a.e.} \tag{2.6}$$

DEFINITION 2.2. Suppose $u(t,\omega)$ is a stochastic process satisfying (2.1), (2.2) and with Wiener-Ito chaos expansion

$$u(t,\omega) = \sum_{n=0}^{\infty} I_n(f_n(\cdot,t)). \tag{2.7}$$

Then we define *the Skorohod integral* of u by

$$\delta(u) := \int_0^T u(t,\omega)\delta W(t) := \sum_{n=0}^{\infty} I_{n+1}(\widetilde{f}_n) \qquad \text{(when convergent)} \tag{2.8}$$

where $\widetilde{f}_n$ is the symmetrization of $f_n(t_1,\dots,t_n,t)$ as a function of $n+1$ variables $t_1,\dots,t_n,t$.

We say u is *Skorohod-integrable* and write $u \in \mathrm{Dom}(\delta)$ if the series in (2.8) converges in $L^2(P)$. By (1.16) this occurs iff

$$E[\delta(u)^2] = \sum_{n=0}^{\infty}(n+1)!\|\widetilde{f}_n\|^2_{L^2([0,T]^{n+1})} < \infty\,. \tag{2.9}$$

EXAMPLE 2.3. Let us compute the Skorohod integral

$$\int_0^T W(T,\omega)\delta W(t).$$

Here $u(t,\omega) = W(T,\omega) = \int_0^T 1\,dW(t)$, so

$$f_0 = 0, \quad f_1 = 1 \quad \text{and} \quad f_n = 0 \quad \text{for all } n \geq 2.$$

Hence

$$\delta(u) = I_2(\widetilde{f}_1) = I_2(1) = 2\int_0^T\Big(\int_0^{t_2} dW(t_1)\Big)dW(t_2) = W^2(T,\omega) - T.$$

Note that even if $W(T,\omega)$ does not depend on t, we have

$$\int_0^T W(T,\omega)\delta W(t) \neq W(T,\omega)\int_0^T \delta W(t) \qquad \text{(but see (3.64)).}$$

EXAMPLE 2.4. What is $\int_0^T W(t,\omega)[W(T,\omega) - W(t,\omega)]\delta W(t)$?

Note that

$$\int_0^T \Big(\int_0^{t_2} \mathcal{X}_{\{t_1<t<t_2\}}(t_1,t_2)dW(t_1)\Big)dW(t_2)$$

$$= \int_0^T W(t,\omega)\mathcal{X}_{\{t<t_2\}}(t_2)dW(t_2)$$

$$= W(t,\omega)\int_t^T dW(t_2) = W(t,\omega)[W(T,\omega)-W(t,\omega)].$$

Hence

$$\begin{aligned} u(t,\omega) := {} & W(t,\omega)[W(T,\omega)-W(t,\omega)] = J_2(\mathcal{X}_{\{t_1<t<t_2\}}(t_1,t_2)) \\ = {} & I_2(f_2(\cdot,t)), \end{aligned}$$

where

$$f_2(t_1,t_2,t) = \frac{1}{2}(\mathcal{X}_{\{t_1<t<t_2\}} + \mathcal{X}_{\{t_2<t<t_1\}}).$$

Hence by Example 2.1 and (1.14)

$$\begin{aligned} \delta(u) &= I_3(\widetilde{f_2}) = I_3(\frac{1}{6}) = (\frac{1}{6})I_3(1) \\ &= \frac{1}{6}[W^3(T,\omega) - 3T\,W(T,\omega)]. \end{aligned}$$

As mentioned earlier the Skorohod integral is an extension of the Ito integral. More precisely, if the integrand $u(t,\omega)$ is $\mathcal{F}_t$-adapted, then the two integrals coincide. To prove this, we need a characterization of $\mathcal{F}_t$-adaptedness in terms of the functions $f_n(\cdot,t)$ in the chaos expansion:

LEMMA 2.5. Let $u(t,\omega)$ be a stochastic process satisfying (2.1), (2.2) and let

$$u(t,\omega) = \sum_{n=0}^{\infty} I_n(f_n(\cdot,t))$$

be the Wiener-Ito chaos expansion of $u(t,\cdot)$, for each $t\in[0,T]$. Then $u(t,\omega)$ is $\mathcal{F}_t$-adapted if and only if

(2.10) $$f_n(t_1,\dots,t_n,t) = 0 \qquad \text{if} \quad t < \max_{1\le i\le n} t_i\,.$$

REMARK The statement (2.10) should – as most statements about L^2-functions – be regarded as an almost everywhere (a.e.) statement. More precisely, (2.10) means that for each $t\in[0,T]$ we have

$$f_n(t_1,\dots,t_n,t) = 0 \qquad \text{for a.a. } (t_1,\dots,t_n)\in H,$$

where $H = \{(t_1,\dots,t_n)\in[0,T]^n;\ t<\max_{1\le i\le n} t_i\}$.

Proof of Lemma 2.5. First note that for any $g \in \widehat{L}^2([0,T]^n)$ we have

$$\begin{aligned}
E[I_n(g)|\mathcal{F}_t] &= n! E[J_n(g)|\mathcal{F}_t] \\
&= n! E[\int_0^T \{\int_0^{t_n} \cdots \int_0^{t_2} g(t_1,\dots,t_n) dW(t_1)\cdots\} dW(t_n)|\mathcal{F}_t] \\
&= n! \int_0^t \{\int_0^{t_n} \cdots \int_0^{t_2} g(t_1,\dots,t_n) dW(t_1)\cdots\} dW(t_n) \\
&= n! J_n(g(t_1,\dots,t_n)\cdot \mathcal{X}_{\{\max t_i<t\}}) \\
&= I_n(g(t_1,\dots,t_n)\cdot \mathcal{X}_{\{\max t_i<t\}}). \qquad (2.11)
\end{aligned}$$

Hence

$$\begin{aligned}
&u(t,\omega) \quad \text{is } \mathcal{F}_t\text{-adpted} \\
&\Leftrightarrow E[u(t,\omega)|\mathcal{F}_t] = u(t,\omega) \\
&\Leftrightarrow \sum_{n=0}^{\infty} E[I_n(f_n(\cdot,t))|\mathcal{F}_t] = \sum_{n=0}^{\infty} I_n(f_n(\cdot,t)) \\
&\Leftrightarrow \sum_{n=0}^{\infty} I_n(f_n(\cdot,t)\cdot \mathcal{X}_{\{\max t_i<t\}}) = \sum_{n=0}^{\infty} I_n(f_n(\cdot,t)) \\
&\Leftrightarrow f_n(t_1,\dots,t_n,t)\cdot \mathcal{X}_{\{\max t_i<t\}} = f_n(t_1,\dots,t_n,t) \quad \text{a.e.,}
\end{aligned}$$

by uniqueness of the Wiener-Ito expansion. Since the last identity is equivalent to (2.10), the Lemma is proved. □

THEOREM 2.6. (The Skorohod integral is an extension of the Ito integral) Let $u(t,\omega)$ be a stochastic process such that

$$E[\int_0^T u^2(t,\omega)dt] < \infty \qquad (2.12)$$

and suppose that

$$u(t,\omega) \quad \text{is } \mathcal{F}_t\text{-adapted for } t \in [0,T]. \qquad (2.13)$$

Then $u \in \mathrm{Dom}(\delta)$ and

$$\int_0^T u(t,\omega)\delta W(t) = \int_0^T u(t,\omega)dW(t) \qquad (2.14)$$

Proof. First note that by (2.5) and Lemma 2.5 we have

$$\widetilde{f}_n(t_1,\dots,t_n,t_{n+1}) = \frac{1}{n+1} f_n(\cdots,t_{j-1},t_{j+1},\dots,t_j) \qquad (2.15)$$

where

$$t_j = \max_{1\le i\le n+1} t_i\,.$$

Hence

$$\begin{aligned}
\|\widetilde{f}_n\|^2_{L^2([0,T]^{n+1})} &= (n+1)! \int\limits_{S_{n+1}} \widetilde{f}_n^2(x_1,\dots,x_{n+1})dx_1\cdots dx_{n+1} \\
&= \frac{(n+1)!}{(n+1)^2} \int\limits_{S_{n+1}} f_n^2(x_1,\dots,x_{n+1})dx_1\cdots dx_n \\
&= \frac{n!}{n+1} \int_0^T (\int_0^t \int_0^{x_n} \cdots \int_0^{x_2} f_n^2(x_1,\dots,x_n,t)dx_1\cdots dx_n)dt \\
&= \frac{n!}{n+1} \int_0^T (\int_0^T \int_0^{x_n} \cdots \int_0^{x_2} f_n^2(x_1,\dots,x_n,t)dx_1\cdots dx_n)dt \\
&= \frac{1}{n+1} \int_0^T \|f_n(\cdot,t)\|^2_{L^2([0,T]^n)}dt,
\end{aligned}$$

again by using Lemma 2.5.

Hence, by (1.16),

$$\begin{aligned}
\sum_{n=0}^{\infty}(n+1)!\|\widetilde{f}_n\|^2_{L^2([0,T]^{n+1})} &= \sum_{n=0}^{\infty} n! \int_0^T \|f_n(\cdot,t)\|^2_{L^2([0,T]^n)}dt \\
&= \int_0^T (\sum_{n=0}^{\infty} n!\|f_n(\cdot,t)\|^2_{L^2([0,T]^n)})dt \\
&= E[\int_0^T u^2(t,\omega)dt] < \infty \qquad \text{by assumption.} \qquad (2.16)
\end{aligned}$$

This proves that $u \in \mathrm{Dom}(\delta)$.

Finally, to prove (2.14) we again apply (2.15):

$$\begin{aligned}
\int_0^T u(t,\omega)dW(t) &= \sum_{n=0}^{\infty} \int_0^T I_n(f_n(\cdot,t))dW(t) \\
&= \sum_{n=0}^{\infty} \int_0^T \{n! \int\limits_{0\le t_1\le\cdots\le t_n\le t} f_n(t_1,\dots,t_n,t)dW(t_1)\cdots dW(t_n)\}dW(t) \\
&= \sum_{n=0}^{\infty} \int_0^T n!(n+1) \int\limits_{0\le t_1\le\cdots\le t_n\le t_{n+1}} \widetilde{f}_n(t_1,\dots,t_n,t_{n+1})dW(t_1)\cdots dW(t_n)dW(t_{n+1}) \\
&= \sum_{n=0}^{\infty}(n+1)!J_{n+1}(\widetilde{f}_n) = \sum_{n=0}^{\infty} I_{n+1}(\widetilde{f}_n) = \int_0^T u(t,\omega)\delta W(t).
\end{aligned}$$

□

Exercises

2.1 Compute the following Skorohod integrals:

a) $\int_0^T W(t)\delta W(t)$

b) $\int_0^T (\int_0^T g(s)dW(s))\delta W(t) \quad (g \in L^2([0,T])$ deterministic$)$

c) $\int_0^T W^2(t_0)\delta W(t) \quad (t_0 \in [0,T]$ fixed$)$

d) $\int_0^T \exp(W(T))\delta W(t)$

(Hint: Use Exercise 1.2.)

3 White noise, the Wick product and stochastic integration

This chapter gives an introduction to the white noise analysis and its relation to the analysis on Wiener spaces discussed in the first two chapters. Although it is not strictly necessary for the following chapters, it gives a useful alternative approach. Moreover, it provides a natural platform for the *Wick product*, which is closely related to Skorohod integration (see (3.22)). For example, we shall see that the Wick calculus can be used to simplify the computation of these integrals considerably.

The Wick product was introduced by C. G. Wick in 1950 as a renormalization technique in quantum physics. This concept (or rather a relative of it) was introduced by T. Hida and N. Ikeda in 1965. In 1989 P. A. Meyer and J. A. Yan extended the construction to cover Wick products of stochastic distributions (Hida distributions), including the white noise.

The Wick product has turned out to be a very useful tool in stochastic analysis in general. For example, it can be used to facilitate both the theory and the explicit calculations in stochastic integration and stochastic differential equations. For this reason we include a brief introduction in this course. It remains to be seen if the Wick product also has more direct applications in economics.

General references for this section are [H], [HKPS], [HØUZ], [HP], [LØU 1-3], [Ø1], [Ø2] and [GHLØUZ].

We start with the construction of the *white noise probability space* $(\mathcal{S}', \mathcal{B}, \mu)$:

Let $\mathcal{S} = \mathcal{S}(\mathbf{R})$ be *the Schwartz space of rapidly decreasing smooth functions* on $\mathbf{R}$ with the usual topology and let $\mathcal{S}' = \mathcal{S}'(\mathbf{R})$ be its dual (the space of *tempered distributions*). Let $\mathcal{B}$ denote the family of all Borel subsets of $\mathcal{S}'(\mathbf{R})$ (equipped with the weak-star topology). If $\omega \in \mathcal{S}'$ and $\phi \in \mathcal{S}$ we let

$$\omega(\phi) = \langle \omega, \phi \rangle \tag{3.1}$$

denote the action of ω on ϕ. (For example, if ω is a measure m on $\mathbf{R}$ then

$$\langle \omega, \phi \rangle = \int_{\mathbf{R}} \phi(x) dm(x)$$

and if ω is evaluation at $x_0 \in \mathbf{R}$ then

$$\langle \omega, \phi \rangle = \phi(x_0) \quad \text{etc.)}$$

By the Minlos theorem [GV] there exists a probability meaure μ on $\mathcal{S}'$ such that

$$\int_{\mathcal{S}'} e^{i\langle \omega, \phi \rangle} d\mu(\omega) = e^{-\frac{1}{2}\|\phi\|^2} \quad ; \ \phi \in \mathcal{S} \tag{3.2}$$

where

$$\|\phi\|^2 = \int_{\mathbf{R}} |\phi(x)|^2 dx = \|\phi\|^2_{L^2(\mathbf{R})}. \tag{3.3}$$

μ is called the *white noise probability measure* and $(\mathcal{S}', \mathcal{B}, \mu)$ is called *the white noise probability space.*

DEFINITION 3.1 *The (smoothed) white noise process* is the map

$$w : \mathcal{S} \times \mathcal{S}' \to \mathbf{R}$$

given by

$$w(\phi, \omega) = w_\phi(\omega) = \langle \omega, \phi \rangle \quad ; \ \phi \in \mathcal{S}, \omega \in \mathcal{S}' \tag{3.4}$$

From w_ϕ we can construct a Wiener process (Brownian motion) W_t as follows:

STEP 1. (The Ito isometry)

$$E_\mu[\langle \cdot, \phi \rangle^2] = \|\phi\|^2 \quad ; \ \phi \in \mathcal{S} \tag{3.5}$$

where E_μ denotes expectation w.r.t. μ, so that

$$E_\mu[\langle \cdot, \phi \rangle^2] = \int_{\mathcal{S}'} \langle \omega, \phi \rangle^2 d\mu(\omega).$$

STEP 2. Use Step 1 to define, for arbitrary $\psi \in L^2(\mathbf{R})$,

$$\begin{aligned} &\langle \omega, \psi \rangle := \lim \langle \omega, \phi_n \rangle, \\ &\qquad \text{where} \quad \phi_n \in \mathcal{S} \quad \text{and} \quad \phi_n \to \psi \text{ in } L^2(\mathbf{R}) \end{aligned} \tag{3.6}$$

STEP 3. Use Step 2 to define

$$\widetilde{W}_t(\omega) = \widetilde{W}(t, \omega) := \langle \omega, \chi_{[0,t]}(\cdot) \rangle \quad \text{for } t \geq 0 \tag{3.7}$$

by choosing

$$\psi(s) = \chi_{[0,t]}(s) = \begin{cases} 1 & \text{if } s \in [0, t] \\ 0 & \text{if } s \notin [0, t] \end{cases}$$

which belongs to $L^2(\mathbf{R})$ for all $t \geq 0$.

STEP 4. Prove that $\widetilde{W}_t$ has a *continuous modification* $W_t = W_t(\omega)$, i.e.

$$P[\widetilde{W}_t(\cdot) = W_t(\cdot)] = 1 \quad \text{for all } t.$$

This continuous process $W_t = W_t(\omega) = W(t, \omega) = W(t)$ is a Wiener process.

Note that when the Wiener process $W_t(\omega)$ is constructed this way, then each ω is interpreted as an element of $\Omega := \mathcal{S}'(\mathbf{R})$, i.e. as a tempered distribution.

From the above it follows that the relation between smoothed white noise $w_\phi(\omega)$ and the Wiener process $W_t(\omega)$ is

$$w_\phi(\omega) = \int_{\mathbf{R}} \phi(t) dW_t(\omega) \quad ; \ \phi \in \mathcal{S} \tag{3.8}$$

where the integral on the right is the Wiener-Itô integral.

The Wiener-Itô chaos expansion revisited

As before let the *Hermite polynomials* $h_n(x)$ be defined by

(3.9) $$h_n(x) = (-1)^n e^{\frac{x^2}{2}} \frac{d^n}{dx^n}(e^{-\frac{x^2}{2}}) \ ; \ n = 0, 1, 2, \cdots$$

This gives for example

$$h_0(x) = 1, h_1(x) = x, h_2(x) = x^2 - 1, h_3(x) = x^3 - 3x$$
$$h_4(x) = x^4 - 6x^2 + 3, h_5(x) = x^5 - 10x^3 + 15x, \cdots$$

Let e_k be the k'th *Hermite function* defined by

(3.10) $$e_k(x) = \pi^{-\frac{1}{4}}((k-1)!)^{-\frac{1}{2}} \cdot e^{-\frac{x^2}{2}} h_{k-1}(\sqrt{2}x); \quad k = 1, 2, \cdots$$

Then $\{e_k\}_{k \geq 1}$ constitutes an orthonormal basis for $L^2(\mathbf{R})$ and $e_k \in \mathcal{S}$ for all k.

Define

(3.11) $$\theta_k(\omega) = \langle \omega, e_k \rangle = W_{e_k}(\omega) = \int_{\mathbf{R}} e_k(x) dW_x(\omega)$$

Let $\mathcal{J}$ denote the set of all finite multi-indices $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ $(m = 1, 2, \ldots)$ of non-negative integers α_i. If $\alpha = (\alpha_1, \cdots, \alpha_m) \in \mathcal{J}$ we put

(3.12) $$H_\alpha(\omega) = \prod_{j=1}^{m} h_{\alpha_j}(\theta_j)$$

For example, if $\alpha = \epsilon_k = (0, 0, \cdots, 1)$ with 1 on k'th place, then

$$H_{\epsilon_k}(\omega) = h_1(\theta_k) = \langle \omega, e_k \rangle,$$

while

$$H_{3,0,2}(\omega) = h_3(\theta_1) h_0(\theta_2) h_2(\theta_3) = (\theta_1^3 - 3\theta_1) \cdot (\theta_3^2 - 1).$$

The family $\{H_\alpha(\cdot)\}_{\alpha \in \mathcal{J}}$ is an orthogonal basis for the Hilbert space

(3.13) $$L^2(\mu) = \{X : \mathcal{S}' \to \mathbf{R} \quad \text{such that} \quad \|X\|^2_{L^2(\mu)} := \int_{\mathcal{S}'} X(\omega)^2 d\mu(\omega) < \infty\}.$$

In fact, we have

THEOREM 3.2 (The Wiener-Ito chaos expansion theorem II)

For all $X \in L^2(\mu)$ there exist (uniquely determined) numbers $c_\alpha \in \mathbf{R}$ such that

(3.14) $$X(\omega) = \sum_\alpha c_\alpha H_\alpha(\omega).$$

Moreover, we have

(3.15) $$\|X\|^2_{L^2(\mu)} = \sum_\alpha \alpha! c_\alpha^2$$

where $\alpha! = \alpha_1!\alpha_2!\cdots\alpha_m!$ if $\alpha = (\alpha_1, \alpha_2, \cdots \alpha_m)$.

Let us compare with the equivalent formulation of this theorem in terms of *multiple Ito integrals*: (See Chapter 1)

If $\psi(t_1, t_2, \cdots, t_n)$ is a real symmetric function in its n (real) variables $t_1, \cdots, t_n$ and $\psi \in L^2(\mathbf{R}^n)$, i.e.

$$\|\psi\|_{L^2(\mathbf{R}^n)} := [\int_{\mathbf{R}^n} |\psi(t_1, t_2, \cdots, t_n)|^2 dt_1 dt_2 \cdots dt_n]^{1/2} < \infty \tag{3.16}$$

then its *n-tuple Ito integral* is defined by

$$\begin{aligned} I_n(\psi) :=& \int_{\mathbf{R}^n} \psi dW^{\otimes n} := \\ & n! \int_{-\infty}^{\infty} (\int_{-\infty}^{t_n} (\int_{-\infty}^{t_{n-1}} \cdots (\int_{-\infty}^{t_2} \psi(t_1, t_2, \cdots, t_n) dW_{t_1}) dW_{t_2} \cdots) dW_{t_n} \end{aligned} \tag{3.17}$$

where the integral on the right consists of n iterated Ito integrals (note that in each step the corresponding integrand is adapted because of the upper limits of the preceding integrals). Applying the Ito isometry n times we see that

$$E[(\int_{\mathbf{R}^n} \psi dW^{\otimes n})^2] = n! \|\psi\|^2_{L^2(\mathbf{R}^n)}; \quad n \geq 1 \tag{3.18}$$

For $n = 0$ we adopt the convention that

$$I_0(\psi) := \int_{\mathbf{R}^0} \psi dW^{\otimes 0} = \psi = \|\psi\|_{L^2(\mathbf{R}^0)} \quad \text{when } \psi \text{ is constant} \tag{3.19}$$

Let $\widehat{L}^2(\mathbf{R}^n)$ denote the set of symmetric real functions (on $\mathbf{R}^n$) which are square integrable with respect to Lebesque measure. Then we have (see Theorem 1.1):

THEOREM 3.3 (The Wiener-Ito chaos expansion theorem I)

For all $X \in L^2(\mu)$ there exist (uniquely determined) functions $f_n \in \widehat{L}^2(\mathbf{R}^n)$ such that

$$X(\omega) = \sum_{n=0}^{\infty} \int_{\mathbf{R}^n} f_n dW^{\otimes n}(\omega) = \sum_{n=0}^{\infty} I_n(f_n) \tag{3.20}$$

Moreover, we have

$$\|X\|^2_{L^2(\mu)} = \sum_{n=0}^{\infty} n! \|f_n\|^2_{L^2(\mathbf{R}^n)} \tag{3.21}$$

REMARK The connection between these two expansions in Theorem 3.2 and Theorem 3.3 is given by

$$f_n = \sum_{\substack{\alpha \in \mathcal{J} \\ |\alpha| = n}} c_\alpha e_1^{\otimes \alpha_1} \hat{\otimes} e_2^{\otimes \alpha_2} \hat{\otimes} \cdots \hat{\otimes} e_m^{\otimes \alpha_m} \quad ; \ n = 0, 1, 2, \cdots \tag{3.22}$$

where $|\alpha| = \alpha_1 + \cdots + \alpha_m$ if $\alpha = (\alpha_1, \cdots, \alpha_m) \in \mathcal{J}$ $(m = 1, 2, \cdots)$. The functions $e_1, e_2, \cdots$ are defined in (3.10) and $\otimes$ and $\hat{\otimes}$ denote tensor product and symmetrized tensor product, respectively. For example, if f and g are real functions on $\mathbf{R}$ then

$$(f \otimes g)(x_1, x_2) = f(x_1)g(x_2)$$

and

$$(f \hat{\otimes} g)(x_1, x_2) = \frac{1}{2}[f(x_1)g(x_2) + f(x_2)g(x_1)] \; ; \; (x_1, x_2) \in \mathbf{R}^2.$$

Analogous to the test functions $\mathcal{S}(\mathbf{R})$ and the tempered distributions $\mathcal{S}'(\mathbf{R})$ on the real line $\mathbf{R}$, there is a useful space of *stochastic test functions* $(\mathcal{S})$ and a space of *stochastic distributions* $(\mathcal{S})^*$ on the white noise probability space:

DEFINITION 3.4 ([Z])

a) We say that $f = \sum_{\alpha \in \mathcal{J}} a_\alpha H_\alpha \in L^2(\mu)$ belongs to the *Hida test function space* $(\mathcal{S})$ if

(3.23) $$\sum_{\alpha \in \mathcal{J}} \alpha! a_\alpha^2 \{\prod_{j=1}^{\infty} (2j)^{\alpha_j}\}^k < \infty \quad \text{for all} \quad k < \infty$$

b) A formal sum $F = \sum_{\alpha \in \mathcal{J}} b_\alpha H_\alpha$ belongs to the *Hida distribution space* $(\mathcal{S})^*$ if

(3.24) $$\text{there exists} \quad q < \infty \text{ s.t. } \sum_{\alpha \in \mathcal{J}} \alpha! c_\alpha^2 \{\prod_{j=1}^{\infty} (2j)^{\alpha_j}\}^{-q} < \infty$$

$(\mathcal{S})^*$ is the dual of $(\mathcal{S})$. The action of $F = \sum_\alpha b_\alpha H_\alpha \in (\mathcal{S})^*$ on $f = \sum_\alpha a_\alpha H_\alpha \in (\mathcal{S})$ is given by

$$\langle F, f \rangle = \sum_\alpha \alpha! a_\alpha b_\alpha$$

We have the inclusions

$$(\mathcal{S}) \subset L^2(\mu) \subset (\mathcal{S})^*.$$

EXAMPLE 3.5

a) The *smoothed white noise* $w_\phi(\cdot)$ belongs to $(\mathcal{S})$ if $\phi \in \mathcal{S}$, because if $\phi = \sum_j c_j e_j$ we have

(3.25) $$w_\phi = \sum_j c_j H_{\epsilon_j}$$

so $w_\phi \in (\mathcal{S})$ if and only if (using (3.23))

$$\sum_j c_j^2 (2j)^k < \infty \quad \text{for all} \quad k,$$

which holds because $\phi \in \mathcal{S}$. (See e.g. [RS]).

b) The *singular (pointwise) white noise* $\overset{\bullet}{W}_t(\cdot)$ is defined as follows:

$$\overset{\bullet}{W}_t(\omega) = \sum_k e_k(t) H_{\epsilon_k}(\omega) \tag{3.26}$$

Using (3.24) one can verify that $\overset{\bullet}{W}_t(\cdot) \in (\mathcal{S})^*$ for all t. **This is the precise definition of singular/pointwise white noise!**

The Wick product

In addition to a canonical vector space structure, the spaces $(\mathcal{S})$ and $(\mathcal{S})^*$ also have a natural multiplication:

DEFINITION 3.6 If $X = \sum_\alpha a_\alpha H_\alpha \in (\mathcal{S})^*$, $Y = \sum_\beta b_\beta H_\beta \in (\mathcal{S})^*$ then the *Wick product*, $X \diamond Y$, of X and Y is defined by

$$X \diamond Y = \sum_{\alpha,\beta} a_\alpha b_\beta H_{\alpha+\beta} = \sum_\gamma \Big(\sum_{\alpha+\beta=\gamma} a_\alpha b_\beta \Big) H_\gamma \tag{3.27}$$

Using (3.24) and (3.23) one can now verify the following:

$$X, Y \in (\mathcal{S})^* \Rightarrow X \diamond Y \in (\mathcal{S})^* \tag{3.28}$$

$$X, Y \in (\mathcal{S}) \Rightarrow X \diamond Y \in (\mathcal{S}) \tag{3.29}$$

(Note, however, that $X, Y \in L^2(\mu) \not\Rightarrow X \diamond Y \in L^2(\mu)$ in general)

EXAMPLE 3.7

(i) *The Wick square of white noise* is

(singular case) $$\overset{\bullet}{W}_t^{\diamond 2} = (\overset{\bullet}{W}_t)^{\diamond 2} = \sum_{k,m} e_k(t) e_m(t) H_{\epsilon_k + \epsilon_m}$$

(smoothed case) $$w_\phi^{\diamond 2} = \sum_{k,m} c_k c_m H_{\epsilon_k + \epsilon_m} \quad \text{if} \quad \phi = \sum c_k e_k \in \mathcal{S}$$

Since

$$H_{\epsilon_k + \epsilon_m} = \begin{cases} H_{\epsilon_k} \cdot H_{\epsilon_m} & \text{if } k \neq m \\ H_{\epsilon_k}^2 - 1 & \text{if } k = m \end{cases}$$

we see that

$$w_\phi^{\diamond 2} = w_\phi^2 - \sum_k c_k^2 = w_\phi^2 - \|\phi\|^2.$$

Note, in particular, that $w_\phi^{\diamond 2}$ is not positive. In fact, $E[w_\phi^{\diamond 2}] = 0$ by (2.5).

(ii) *The Wick exponential of smoothed white noise* is defined by

$$\exp^{\diamond} w_\phi = \sum_{n=0}^{\infty} \frac{1}{n!} w_\phi^{\diamond n} \quad ; \ \phi \in \mathcal{S}.$$

It can be shown that (see Exercise 1.1)

$$\exp^{\diamond} w_\phi = \exp(w_\phi - \frac{1}{2}\|\phi\|^2) \tag{3.30}$$

so $\exp^{\diamond} w_\phi$ *is* positive. Moreover, we have

$$E_\mu[\exp^{\diamond} w_\phi] = 1 \qquad \text{for all } \phi \in \mathcal{S}. \tag{3.31}$$

Why the Wick product?

We list some reasons that the Wick product is natural to use in stochastic calculus:

1) First, note that if (at least) one of the factors X, Y is deterministic, then

$$X \diamond Y = X \cdot Y$$

Therefore the two types of products, the Wick product and the ordinary (ω-pointwise) product, coincide in the deterministic calculus. So when one extends a deterministic model to a stochastic model by introducing noise, it is not obvious which interpretation to choose for the products involved. The choice should be based on additional modelling and mathematical considerations.

2) The Wick product is the only product which is defined for singular white noise $\overset{\bullet}{W}_t$. Pointwise product $X \cdot Y$ does not make sense in $(\mathcal{S})^*$!

3) The Wick product has been used for 40 years already in quantum physics as a renormalization procedure.

4) Last, but not least: There is a fundamental relation between Ito/Skorohod integrals and Wick products, given by

$$\int Y_t(\omega)\delta W_t(\omega) = \int Y_t \diamond \overset{\bullet}{W}_t \, dt \tag{3.32}$$

(see [LØU 2], [B]).

Here the integral on the right is interpreted as a Pettis integral with values in $(\mathcal{S})^*$.

In view of (3.32) one could say that the Wick product is the core of Ito integration, hence it is natural to use in stochastic calculus in general.

Finally we recall the definition of a pair of dual spaces, $\mathcal{G}$ and $\mathcal{G}^*$, which are sometimes useful. See [PT] and the references therein for more information.

DEFINITION 3.8

a) Let $\lambda \in \mathbf{R}$. Then the space $\mathcal{G}_\lambda$ consists of all formal expansions

$$X = \sum_{n=0}^{\infty} \int_{\mathbf{R}^n} f_n dW^{\otimes n} \tag{3.33}$$

such that

$$\|X\|_\lambda := [\sum_{n=0}^{\infty} n! e^{2\lambda n} \|f_n\|^2_{L^2(\mathbf{R}^n)}]^{\frac{1}{2}} < \infty \tag{3.34}$$

For each $\lambda \in \mathbf{R}$ the space $\mathcal{G}_\lambda$ is a Hilbert space with inner product

$$(X, Y)_{\mathcal{G}_\lambda} = \sum_{n=0}^{\infty} n! e^{2\lambda n} (f_n, g_n)_{L^2(\mathbf{R}^n)}$$

$$\text{if } X = \sum_{n=0}^{\infty} \int_{\mathbf{R}^n} f_n dW^{\otimes n}, \ Y = \sum_{m=0}^{\infty} \int_{\mathbf{R}^m} g_m dW^{\otimes m} \tag{3.35}$$

Note that $\lambda_1 \leq \lambda_2 \Rightarrow \mathcal{G}_{\lambda_2} \subseteq \mathcal{G}_{\lambda_1}$. Define

$$\mathcal{G} = \bigcap_{\lambda \in \mathbf{R}} \mathcal{G}_\lambda, \quad \text{with projective limit topology.} \tag{3.36}$$

b) $\mathcal{G}^*$ is defined to be the dual of $\mathcal{G}$. Hence

$$\mathcal{G}^* = \bigcup_{\lambda \in \mathbf{R}} \mathcal{G}_\lambda, \quad \text{with inductive limit topology.} \tag{3.37}$$

REMARK. Note that an element $Y \in \mathcal{G}^*$ can be represented as a formal sum

$$Y = \sum_{n=0}^{\infty} \int_{\mathbf{R}^n} g_n dW^{\otimes n} \tag{3.38}$$

where $g_n \in \hat{L}^2(\mathbf{R}^n)$ and $\|Y\|_\lambda < \infty$ for *some* $\lambda \in \mathbf{R}$, while an $X \in \mathcal{G}$ satisfies $\|X\|_\lambda < \infty$ for *all* $\lambda \in \mathbf{R}$.

If $X \in \mathcal{G}$ and $Y \in \mathcal{G}^*$ have the representations (3.33), (3.38), respectively, then the action of Y on X, $\langle Y, X \rangle$, is given by

$$\langle Y, X \rangle = \sum_{n=0}^{\infty} n! (f_n, g_n)_{L^2(\mathbf{R}^n)} \tag{3.39}$$

where

$$(f_n, g_n)_{L^2(\mathbf{R}^n)} = \int_{\mathbf{R}^n} f(x) g(x) dx \tag{3.40}$$

One can show that

$$(\mathcal{S}) \subset \mathcal{G} \subset L^2(\mu) \subset \mathcal{G}^* \subset (\mathcal{S})^*. \tag{3.41}$$

The space $\mathcal{G}^*$ is not big enough to contain the singular white noise W_t. However, it does often contain the solution X_t of stochastic differential equations. This fact allows one to deduce some useful properties of X_t.

Like $(\mathcal{S})$ and $(\mathcal{S})^*$ the spaces $\mathcal{G}$ and $\mathcal{G}^*$ are closed under Wick product ([PT, Theorem 2.7]):

$$X_1, X_2 \in \mathcal{G} \Rightarrow X_1 \diamond X_2 \in \mathcal{G} \tag{3.42}$$

$$Y_1, Y_2 \in \mathcal{G}^* \Rightarrow Y_1 \diamond Y_2 \in \mathcal{G}^* \tag{3.43}$$

The Wick product in terms of iterated Ito integrals

The definition we have given of the Ito product is based on the chaos expansion II, because only this is general enough to include the singular white noise. However, it is useful to know how the Wick product is expressed in terms of chaos expansion I for $L^2(\mu)$-functions or, more generally, for elements of $\mathcal{G}^*$:

THEOREM 3.9 Suppose $X = \sum\limits_{n=0}^{\infty} I_n(f_n) \in \mathcal{G}^*$, $Y = \sum\limits_{m=0}^{\infty} I_m(g_m) \in \mathcal{G}^*$. Then the Wick product of X and Y can be expressed by

$$X \diamond Y = \sum_{n,m=0}^{\infty} I_{n+m}(f_n \widehat{\otimes} g_m) = \sum_{k=0}^{\infty} \Big(\sum_{n+m=k} I_k(f_n \widehat{\otimes} g_m)\Big). \tag{3.44}$$

For example, integration by parts gives that

$$\Big(\int_{\mathbf{R}} f(x)dW_x\Big) \diamond \Big(\int_{\mathbf{R}} g(y)dW_y\Big) = \int_{\mathbf{R}^2} (f \widehat{\otimes} g)(x,y) dW^{\otimes 2}$$

$$= \int_{\mathbf{R}} \Big(\int_{-\infty}^{y} (f(x)g(y) + f(y)g(x)) dW_x \Big) dW_y$$

$$= \int_{\mathbf{R}} g(y)\Big(\int_{-\infty}^{y} f(x)dW_x\Big)dW_y + \int_{\mathbf{R}} f(y)\Big(\int_{-\infty}^{y} g(x)dW_x\Big)dW_y$$

$$= \Big(\int_{\mathbf{R}} g(y)dW_y\Big)\Big(\int_{\mathbf{R}} f(x)dW_x\Big) - \int_{\mathbf{R}} f(t)g(t)dt\,. \tag{3.45}$$

Some properties of the Wick product

We list below some useful properties of the Wick product. Some are easy to prove, others harder. For complete proofs see [HØUZ].

For arbitrary $X, Y, Z \in \mathcal{G}^*$ we have

(3.46) $X \diamond Y = Y \diamond X$ (commutative law)

(3.47) $X \diamond (Y \diamond Z) = (X \diamond Y) \diamond Z$ (associative law)

(3.48) $X \diamond (Y + Z) = (X \diamond Y) + (X \diamond Z)$ (distributive law)

Thus the Wick algebra obeys the same rules as the ordinary algebra. For example,

$$(X+Y)^{\diamond 2} = X^{\diamond 2} + 2X \diamond Y + Y^{\diamond 2} \qquad \text{(no Ito formula!)} \tag{3.49}$$

and

$$\exp^{\diamond}(X+Y) = \exp^{\diamond}(X) \diamond \exp^{\diamond}(Y). \tag{3.50}$$

Note, however, that combinations of ordinary products and Wick products requires caution. For example, in general we have

$$X \cdot (Y \diamond Z) \neq (X \cdot Y) \diamond Z\,.$$

A remarkable property of the Wick product is that

$$E_\mu[X \diamond Y] = E_\mu[X] \cdot E_\mu[Y] \tag{3.51}$$

whenever X, Y and $X \diamond Y$ are μ-integrable. (Note that it is *not* required that X and Y are independent!)

A reformulation of (3.45) is that

$$w_\phi \diamond w_\psi = w_\phi \cdot w_\psi - \frac{1}{2}\int_{\mathbf{R}} \phi(t)\psi(t)dt; \qquad \phi, \psi \in \delta\,.$$

(See also Example 3.7(i))

In particular,

$$W_t^{\diamond 2} = W_t^2 - t; \qquad t \geq 0 \tag{3.52}$$

and

$$\text{if} \quad \operatorname{supp}\phi \cap \operatorname{supp}\psi = \emptyset, \quad \text{then} \qquad w_\phi \diamond w_\psi = w_\phi \cdot w_\psi \tag{3.53}$$

Hence

$$\begin{aligned} &\text{If} \quad 0 \leq t_1 \leq t_2 \leq t_3 \leq t_4 \quad \text{then} \\ &(W_{t_4} - W_{t_3}) \diamond (W_{t_2} - W_{t_1}) = (W_{t_4} - W_{t_3}) \cdot (W_{t_2} - W_{t_1}) \end{aligned} \tag{3.54}$$

More generally, it can be proved that if $F(\omega)$ is $\mathcal{F}_t$-measurable and $h > 0$, then

$$F \diamond (W_{t+h} - W_t) = F \cdot (W_{t+h} - W_t) \tag{3.55}$$

(For a proof see e.g. Exercise 2.22 in [HØUZ].)

Note that from (3.44) we have that

$$\Big(\int_{\mathbf{R}} g(t)dW_t\Big)^{\diamond n} = n! \int_{\mathbf{R}^n} g^{\otimes n}(x_1, \ldots, x_n)dW^{\otimes n}; \qquad g \in L^2(\mathbf{R}). \tag{3.56}$$

Combining this with (1.14) we get, with $\theta = \int_{\mathbf{R}} g dW$,

$$\theta^{\diamond n} = \|g\|^n h_n\Big(\frac{\theta}{\|g\|}\Big). \tag{3.57}$$

In particular,

$$W_t^{\diamond n} = t^{n/2} h_n\Big(\frac{W_t}{\sqrt{t}}\Big), \qquad n = 0, 1, 2, \ldots \tag{3.58}$$

Moreover, combining (3.57) with *the generating formula* for Hermite polynomials

$$\exp\Big(tx - \frac{t^2}{2}\Big) = \sum_{n=0}^{\infty} \frac{t^n}{n!} h_n(x) \qquad \text{(see Exercise 1.1)} \tag{3.59}$$

we get (see Example 3.7 (ii))

$$\exp\Big(\int_{\mathbf{R}} g(t) dW_t - \frac{1}{2}\|g\|^2\Big) = \sum_{n=0}^{\infty} \frac{\|g\|^n}{n!} h_n\Big(\frac{\theta}{\|g\|}\Big)$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \theta^{\diamond n} = \exp^{\diamond} \theta\,. \tag{3.60}$$

Hence

$$\exp^{\diamond}\Big(\int_{\mathbf{R}} g dW\Big) = \exp\Big(\int_{\mathbf{R}} g dW - \frac{1}{2}\|g\|^2\Big); \qquad g \in L^2(\mathbf{R}). \tag{3.61}$$

In particular,

$$\exp^{\diamond}(W_t) = \exp\Big(W_t - \frac{1}{2}t\Big); \qquad t \geq 0. \tag{3.62}$$

Combining the properties above with the fundamental relation (3.32) for Skorohod integration, we get a powerful calculation technique for stochastic integration. First of all, note that, by (3.32),

$$\int_0^T \overset{\bullet}{W}_t \, dt = W_T\,. \tag{3.63}$$

Moreover, using (3.48) one can deduce that

$$\int_0^T X \diamond Y_t \diamond \overset{\bullet}{W}_t \, dt = X \diamond \int_0^T Y_t \diamond \overset{\bullet}{W}_t \, dt \tag{3.64}$$

if X does not depend on t.

(Compare this with the fact that for Skorohod integrals we generally have

$$\int_0^T X \cdot Y_t \delta W_t \neq X \cdot \int_0^T Y_t \delta W_t\,, \tag{3.65}$$

even if X does not depend on t.)

To illustrate the use of Wick calculus, let us again consider Example 2.4:

$$\begin{aligned}\int_0^T W_t \cdot [W_T - W_t]\delta W_t &= \int_0^T W_t \diamond (W_T - W_t) \diamond \overset{\bullet}{W}_t\, dt \\ &= \int_0^T W_t \diamond W_T \diamond \overset{\bullet}{W}_t\, dt - \int_0^T W_t^{\diamond 2} \diamond \overset{\bullet}{W}_t\, dt \\ &= W_T \diamond \int_0^T W_t \diamond \overset{\bullet}{W}_t\, dt - \frac{1}{3} W_T^{\diamond 3} = \frac{1}{6} W_T^{\diamond 3} = \frac{1}{6}[W_T^3 - 3TW_T],\end{aligned}$$

where we have used (3.54), (3.48), (3.64) and (3.58).

Exercises

3.1 Use the identity (3.32) and Wick calculus to compute the following Skorohod integrals

a) $\int_0^T W(T)\delta W(t) = \int_0^T W(T) \diamond \overset{\bullet}{W}(t) dt$

b) $\int_0^T (\int_0^T g(s)dW(s))\delta W(t) \quad (g \in L^2([0,T])$ deterministic)

c) $\int_0^T W^2(t_0)\delta W(t) \quad (t_0 \in [0,T]$ fixed)

d) $\int_0^T \exp(W(T))\delta W(t)$

Compare with your calculations in Exercise 2.1!

4 Differentiation

Let us first recall some basic concepts from classical analysis:

DEFINITION 4.1. Let U be an open subset of $\mathbf{R}^n$ and let f be a function from U into $\mathbf{R}^m$.

a) We say that f has a *directional derivative* at the point $x \in U$ in the direction $y \in \mathbf{R}^n$ if

$$D_y f(x) := \lim_{\varepsilon \to 0} \frac{f(x+\varepsilon y) - f(x)}{\varepsilon} = \frac{d}{d\varepsilon}[f(x+\varepsilon y)]_{\varepsilon=0} \tag{4.1}$$

exists. If this is the case we call the vector $D_y f(x) \in \mathbf{R}^m$ the directional derivative (at x in direction y). In particular, if we choose y to be the j'th unit vector $e_j = (0,\dots,1,\dots,0)$, with 1 on j'th place, we get

$$D_{\varepsilon_j} f(x) = \frac{\partial f}{\partial x_j},$$

the j'th partial derivative of f.

b) We say that f is *differentiable* at $x \in U$ if there exists a matrix $A \in \mathbf{R}^{m\times n}$ such that

$$\lim_{\substack{h\to 0\\ h\in\mathbf{R}^n}} \frac{1}{|h|}\cdot |f(x+h) - f(x) - Ah| = 0 \tag{4.2}$$

If this is the case we call A *the derivative* of f at x and we write

$$A = f'(x).$$

The following relations between the two concepts are well-known:

PROPOSITION 4.2.

(i) If f is differentiable at $x \in U$ then f has a directional derivative in all directions $y \in \mathbf{R}^n$ and

$$D_y f(x) = f'(x)y \tag{4.3}$$

(ii) Conversely, if f has a directional derivative at all $x \in U$ in all the directions $y = e_j$; $1 \le j \le n$ and all the partial derivatives

$$D_{e_j} f(x) = \frac{\partial f}{\partial x_j}(x)$$

are continuous functions of x, then f is differentiable at all $x \in U$ and

$$f'(x) = [\frac{\partial f_i}{\partial x_j}]_{\substack{1\le i\le m\\ 1\le j\le n}} \in \mathbf{R}^{m\times n}, \tag{4.4}$$

where f_i is component number i of f, i.e.

$$f = \begin{bmatrix} f_1 \\ \vdots \\ f_m \end{bmatrix}$$

We will now define similar operations in a more general context. First let us recall some basic concepts from functional analysis:

DEFINITION 4.3. Let X be a Banach space, i.e. a complete, normed vector space (over $\mathbf{R}$), and let $\|x\|$ denote the norm of the element $x \in X$. A *linear functional* on X is a linear map

$$T: X \to \mathbf{R}$$

(T is called linear if $T(ax+y) = aT(x)+T(y)$ for all $a \in \mathbf{R}$, $x, y \in X$). A linear functional T is called *bounded* (or *continuous*) if

$$|||T|||:= \sup_{\|x\|\leq 1} |T(x)| < \infty$$

Sometimes we write $\langle T, x\rangle$ or Tx instead of $T(x)$ and call $\langle T, x\rangle$ "the action of T on x". The set of all bounded linear functionals is called *the dual* of X and is denoted by X^*. Equipped with the norm $|||\cdot|||$ the space X^* becomes a Banach space also.

EXAMPLE 4.4.

(i) $X = \mathbf{R}^n$ with the usual Euclidean norm $|x| = \sqrt{x_1^2 + \cdots + x_n^2}$ is a Banach space. In this case it is easy to see that we can identify X^* with $\mathbf{R}^n$.

(ii) Let $X = C_0([0, T])$, the space of continuous, real functions ω on $[0, T]$ such that $\omega(0) = 0$. Then

$$\|\omega\|_\infty := \sup_{t\in[0,T]} |\omega(t)|$$

is a norm on X called the uniform norm. This norm makes X into a Banach space and its dual X^* can be identified with the space $\mathcal{M}([0, T])$ of all signed measures ν on $[0, T]$, with norm

$$|||\nu||| = \sup_{|f|\leq 1} \int_0^T f(t)d\nu(t) = |\nu|([0, T])$$

(iii) If $X = L^p([0, T]) = \{f: [0, T] \to \mathbf{R}; \int_0^T |f(t)|^p dt < \infty\}$ equipped with the norm

$$\|f\|_p = [\int_0^T |f(t)|^p dt]^{1/p} \qquad (1 \leq p < \infty)$$

then X is a Banach space, whose dual can be identified with $L^q([0, T])$, where

$$\frac{1}{p} + \frac{1}{q} = 1\,.$$

In particular, if $p = 2$ then $q = 2$ so $L^2([0, T])$ is its own dual.

We now extend the definitions we had for $\mathbf{R}^n$ to arbitrary Banach spaces:

DEFINITION 4.5. Let U be an open subset of a Banach space X and let f be a function from U into $\mathbf{R}^m$.

a) We say that f has a *directional derivative* (or *Gateaux derivative*) at $x \in U$ in the direction $y \in X$ if

$$D_y f(x) := \frac{d}{d\varepsilon}[f(x+\varepsilon y)]_{\varepsilon=0} \in \mathbf{R}^m \tag{4.5}$$

exists. If this is the case we call $D_y f(x)$ the *directional* (or *Gateaux*) derivative of f (at x in the direction y).

b) We say that f is *Frechet-differentiable* at $x \in U$ if there exists a bounded linear map

$$A \colon X \to \mathbf{R}^m$$

(i.e. $A = \begin{bmatrix} A_1 \\ \vdots \\ A_m \end{bmatrix}$ with $A_i \in X^*$ for $1 \le i \le m$) such that

$$\lim_{\substack{h\to 0\\ h\in X}} \frac{1}{\|h\|} \cdot |f(x+h) - f(x) - A(h)| = 0 \tag{4.6}$$

If this is the case we call A the *Frechet derivative* of f at x and we write

$$A = f'(x) = \begin{bmatrix} f'(x)_1 \\ \vdots \\ f'(x)_m \end{bmatrix} \in (X^*)^m. \tag{4.7}$$

Similar to the Euclidean case (Proposition 4.2) we have

PROPOSITION 4.6.

(i) If f is Frechet-differentiable at $x \in U \subset X$ then f has a directional derivative at x in all directions $y \in X$ and

$$D_y f(x) = \langle f'(x), y\rangle \in \mathbf{R}^m \tag{4.8}$$

where

$$\langle f'(x), y\rangle = (\langle f'(x)_1, y\rangle, \dots, \langle f'(x)_m, y\rangle)$$

is the m-vector whose i'th component is the action of the i'th component $f'(x)_i$ of $f'(x)$ on y.

(ii) Conversely, if f has a directional derivative at all $x \in U$ in all directions $y \in X$ and the (linear) map

$$y \to D_y f(x)\,; \qquad y \in X$$

is continuous for all $x \in U$, then there exists an element $\nabla f(x) \in (X^*)^m$ such that

$$D_y f(x) = \langle \nabla f(x), y \rangle .$$

If this map $x \to \nabla f(x) \in (X^*)^m$ is continuous on U, then f is Frechet differentiable and

$$f'(x) = \nabla f(x) . \tag{4.9}$$

We now apply these operations to the Banach space $\Omega = C_0([0,T])$ considered in Example 4.4 (ii) above. This space is called *the Wiener space*, because we can regard each path

$$t \to W(t,\omega)$$

of the Wiener process starting at 0 as an element ω of $C_0([0,1])$. Thus we may identify $W(t,\omega)$ with the value $\omega(t)$ at time t of an element $\omega \in C_0([0,T])$:

$$W(t,\omega) = \omega(t)$$

With this identification the Wiener process simply becomes the space $\Omega = C_0([0,T])$ and the probability law P of the Wiener process becomes the measure μ defined on the cylinder sets of Ω by

$$\begin{aligned}\mu(\{\omega; \omega(t_1) \in F_1, \dots, \omega(t_k) \in F_k\}) &= P[W(t_1) \in F_1, \dots, W(t_k) \in F_k] \\ &= \int\limits_{F_1 \times \cdots \times F_k} \rho(t_1, x, x_1)\rho(t_1 - t_1, x, x_2) \cdots \rho(t_k - t_{k-1}, x_{k-1}, x_k) dx_1, \cdots dx_k\end{aligned}$$

where $F_i \subset \mathbf{R}$; $0 \le t_1 < t_2 < \cdots < t_k$ and

$$\rho(t,x,y) = (2\pi t)^{-1/2} \exp(-\frac{1}{2}|x-y|^2); \qquad t > 0;\ x, y \in \mathbf{R} .$$

The measure μ is called *the Wiener measure* on Ω. In the following we will write $L^2(\Omega)$ for $L^2(\mu)$ and $L^2([0,T] \times \Omega)$ for $L^2(\lambda \times \mu)$ etc., where λ is the Lebesgue measure on $[0,T]$.

Just as for Banach spaces in general we now define

DEFINITION 4.6. As before let $L^2([0,T])$ be the space of (deterministic) square integrable functions with respect to Lebesgue measure $\lambda(dt) = dt$ on $[0,T]$. Let $F\colon \Omega \to \mathbf{R}$ be a random variable, choose $g \in L^2([0,T])$ and put

$$\gamma(t) = \int_0^t g(s) ds \in \Omega . \tag{4.10}$$

Then we define *the directional derivative* of F at the point $\omega \in \Omega$ in direction $\gamma \in \Omega$ by

$$D_\gamma F(\omega) = \frac{d}{d\varepsilon}[F(\omega + \varepsilon\gamma)]_{\varepsilon=0} , \tag{4.11}$$

if the derivative exists in some sense (to be made precise below).

Note that we only consider the derivative in special directions, namely in the directions of elements γ of the form (4.10). The set of $\gamma \in \Omega$ which can be written on the form (4.10) for some $g \in L^2([0,T])$ is called *the Cameron-Martin* space and denoted by H. It turns out that it is difficult to obtain a tractable theory involving derivatives in all directions. However, the derivatives in the directions $\gamma \in H$ are sufficient for our purposes.

DEFINITION 4.7. Assume that $F\colon \Omega \to \mathbf{R}$ has a directional derivative in all directions γ of the form

$$\gamma(t) = \int_0^t g(s)ds \qquad \text{with } g \in L^2([0,T])$$

in *the strong sense* that

$$\mathbf{D}_\gamma F(\omega) := \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}[F(\omega + \varepsilon\gamma) - F(\omega)] \tag{4.12}$$

exists in $L^2(\Omega)$. Assume in addition that there exists $\psi(t,\omega) \in L^2([0,T] \times \Omega)$ such that

$$\mathbf{D}_\gamma F(\omega) = \int_0^T \psi(t,\omega) g(t) dt\,. \tag{4.13}$$

Then we say that F is *differentiable* and we set

$$\mathbf{D}_t F(\omega) := \psi(t,\omega). \tag{4.14}$$

We call $\mathbf{D}_t F(\omega) \in L^2([0,T] \times \Omega)$ *the derivative of* F.

The set of all differentiable random variables is denoted by $\mathcal{D}_{1,2}$.

EXAMPLE 4.8. Suppose $F(\omega) = \int_0^T f(s)dW(s) = \int_0^T f(s)d\omega(s)$, where $f(s) \in L^2([0,T])$. Then if $\gamma(t) = \int_0^t g(s)ds$ we have

$$\begin{aligned} F(\omega + \varepsilon\gamma) &= \int_0^T f(s)(d\omega(s) + \varepsilon d\gamma(s)) \\ &= \int_0^T f(s)d\omega(s) + \varepsilon \int_0^T f(s)g(s)ds, \end{aligned}$$

and hence $\frac{1}{\varepsilon}[F(\omega + \varepsilon\gamma) - F(\omega)] = \int_0^T f(s)g(s)ds$ for all $\varepsilon > 0$.

Comparing with (4.13) we see that $F \in \mathcal{D}_{1,2}$ and

$$\mathbf{D}_t F(\omega) = f(t); \qquad t \in [0,T],\ \omega \in \Omega. \tag{4.15}$$

In particular, choose

$$f(t) = \mathcal{X}_{[0,t_1]}(t)$$

we get

$$F(\omega) = \int_0^T \mathcal{X}_{[0,t_1]}(s)dW(s) = W(t_1,\omega)$$

and hence

$$\mathbf{D}_t(W(t_1,\omega)) = \mathcal{X}_{[0,t_1]}(t). \tag{4.16}$$

Let $\mathbb{P}$ denote the family of all random variables $F\colon \Omega \to \mathbf{R}$ of the form

$$F(\omega) = \varphi(\theta_1,\dots,\theta_n)$$

where $\varphi(x_1,\dots,x_n) = \sum\limits_{\alpha} a_\alpha x^\alpha$ is a polynomial in n variables $x_1,\dots,x_n$ and $\theta_i = \int\limits_0^T f_i(t)dW(t)$ for some $f_i \in L^2([0,T])$ (deterministic).

Such random variables are called *Wiener polynomials*. Note that $\mathbb{P}$ is dense in $L^2(\Omega)$.

By combining (4.16) with the chain rule we get that $\mathbb{P} \subset \mathcal{D}_{1,2}$:

LEMMA 4.9. Let $F(\omega) = \varphi(\theta_1,\dots,\theta_n) \in \mathbb{P}$. Then $F \in \mathcal{D}_{1,2}$ and

$$\mathbf{D}_t F(\omega) = \sum_{i=1}^n \frac{\partial\varphi}{\partial x_i}(\theta_1,\dots,\theta_n)\cdot f_i(t). \tag{4.17}$$

Proof. Let $\psi(t,\omega)$ denote the right hand side of (4.17). Since

$$\sup_{s\in[0,T]} E[|W(s)|^N] < \infty \qquad \text{for all } N \in \mathbf{N},$$

we see that

$$\begin{aligned}\frac{1}{\varepsilon}[F(\omega+\varepsilon\gamma) - F(\omega)] &= \frac{1}{\varepsilon}[\varphi(\theta_1+\varepsilon(f_1,g),\dots,\theta_n+\varepsilon(f_n,g) - \varphi(\theta_1,\dots,\theta_n)]\\ &\to \sum_{i=1}^n \frac{\partial\varphi}{\partial x_i}(\theta_1,\dots,\theta_n)\cdot \mathbf{D}_\gamma(\theta_i) \qquad \text{in } L^2(\Omega) \text{ as } \varepsilon \to 0\end{aligned}$$

Hence F has a directional derivative in direction γ (in the strong sense) and by (4.15) we have

$$\mathbf{D}_\gamma F(\omega) = \int_0^T \psi(t,\omega)g(t)dt.$$

□

We now introduce the following norm, $\|\cdot\|_{1,2}$, on $\mathcal{D}_{1,2}$:

$$\|F\|_{1,2} = \|F\|_{L^2(\Omega)} + \|\mathbf{D}_t F\|_{L^2([0,T]\times\Omega)}; \qquad F \in \mathcal{D}_{1,2}\,. \tag{4.18}$$

Unfortunately, it is not clear if $\mathcal{D}_{1,2}$ is closed under this norm, i.e. if any $\|\cdot\|_{1,2}$-Cauchy sequence in $\mathcal{D}_{1,2}$ converges to an element of $\mathcal{D}_{1,2}$. To avoid this difficulty we work with the following family:

DEFINITION 4.10. We define $\mathbb{D}_{1,2}$ to be the closure of the family $\mathbb{P}$ with respect to the norm $\|\cdot\|_{1,2}$.

Thus $\mathbb{D}_{1,2}$ consists of all $F \in L^2(\Omega)$ such that there exists $F_n \in \mathbb{P}$ with the property that

$$F_n \to F \qquad \text{in } L^2(\Omega) \text{ as } n \to \infty \tag{4.19}$$

and

$$\{\mathbf{D}_t F_n\}_{n=1}^{\infty} \qquad \text{is convergent in } L^2([0,T] \times \Omega). \tag{4.20}$$

If this is the case, it is tempting to *define*

$$D_t F := \lim_{n\to\infty} \mathbf{D}_t F_n\,.$$

However, for this to work we need to know that this defines $D_t F$ uniquely. In other words, if there is another sequence $G_n \in \mathbb{P}$ such that

$$G_n \to F \qquad \text{in } L^2(\Omega) \text{ as } n \to \infty \tag{4.21}$$

and

$$\{\mathbf{D}_t G_n\}_{n=1}^{\infty} \qquad \text{is convergent in } L^2([0,T] \times \Omega), \tag{4.22}$$

does it follow that

$$\lim_{n\to\infty} \mathbf{D}_t F_n = \lim_{n\to\infty} \mathbf{D}_t G_n\ ? \tag{4.23}$$

By considering the difference $H_n = F_n - G_n$ we see that the answer to this question is *yes*, in virtue of the following theorem:

THEOREM 4.11. (Closability of the operator $\mathbf{D}_t$)
Suppose $\{H_n\}_{n=1}^{\infty} \subset \mathbb{P}$ has the properties

$$H_n \to 0 \qquad \text{in } L^2(\Omega) \text{ as } n \to \infty \tag{4.26}$$

and

$$\{\mathbf{D}_t H_n\}_{n=1}^{\infty} \qquad \text{converges in } L^2([0,T] \times \Omega) \text{ as } n \to \infty \tag{4.27}$$

Then

$$\lim_{n\to\infty} \mathbf{D}_t H_n = 0.$$

The proof is based on the following useful result:

LEMMA 4.12. (Integration by parts)
Suppose $F \in \mathcal{D}_{1,2}$, $\varphi \in \mathcal{D}_{1,2}$ and $\gamma(t) = \int_0^t g(s)ds$ with $g \in L^2([0,T])$. Then

$$E[\mathbf{D}_\gamma F \cdot \varphi] = E[F \cdot \varphi \cdot \int_0^T g dW] - E[F \cdot \mathbf{D}_\gamma \varphi] \tag{4.28}$$

Proof. By the Girsanov theorem we have

$$E[F(\omega + \varepsilon\eta) \cdot \varphi(\omega)] = E[F(\omega)\varphi(\omega - \varepsilon\gamma) \cdot \exp(\varepsilon \int_0^T g dW - \frac{1}{2}\varepsilon^2 \int_0^T g^2 ds)]$$

and this gives

$$\begin{aligned} E[\mathbf{D}_\gamma F(\omega) \cdot \varphi(\omega) &= E[\lim_{\varepsilon \to 0} \frac{1}{\varepsilon}[F(\omega + \varepsilon\gamma) - F(\omega)] \cdot \varphi(\omega)] \\ &= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} E[F(\omega + \varepsilon\gamma)\varphi(\omega) - F(\omega)\varphi(\omega)] \\ &= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} E[F(\omega)[\varphi(\omega - \varepsilon\gamma) \exp(\varepsilon \int_0^T g dW - \frac{1}{2}\varepsilon^2 \int_0^T g^2 ds) - \varphi(\omega)]] \\ &= E[F(\omega) \cdot \frac{d}{d\varepsilon}[\varphi(\omega - \varepsilon\gamma) \exp(\varepsilon \int_0^T g dW - \frac{1}{2}\varepsilon^2 \int_0^T g^2 ds)]_{\varepsilon=0}] \\ &= E[F(\omega)\varphi(\omega) \cdot \int_0^T g dW] - E[F(\omega)\mathbf{D}_\gamma \varphi(\omega)]. \end{aligned}$$

□

Proof of Theorem 4.11. By Lemma 4.12 we get

$$\begin{aligned} E[\mathbf{D}_\gamma H_n \cdot \varphi] &= E[H_n \varphi \cdot \int_0^T g dW] - E[H_n \cdot \mathbf{D}_\gamma \varphi] \\ &\to 0 \qquad \text{as } n \to \infty \text{ for all } \varphi \in \mathbb{P}. \end{aligned}$$

Since $\{\mathbf{D}_\gamma H_n\}_{n=1}^\infty$ converges in $L^2(\Omega)$ and $\mathbb{P}$ is dense in $L^2(\Omega)$ we conclude that $\mathbf{D}_\gamma H_n \to 0$ in $L^2(\Omega)$ as $n \to \infty$. Since this holds for all $\gamma = \int_0^\cdot g ds$, we obtain that $\mathbf{D}_t H_n \to 0$ in $L^2([0,T] \times \Omega)$. □

In view of Theorem 4.11 and the discussion preceding it, we can now make the following (unambiguous) definition:

DEFINITION 4.13. Let $F \in \mathbb{D}_{1,2}$, so that there exists $\{F_n\} \subset \mathbb{P}$ such that

$$F_n \to F \qquad \text{in } L^2(\Omega)$$

and

$$\{\mathbf{D}_t F_n\}_{n=1}^\infty \qquad \text{is convergent in } L^2([0,T] \times \Omega).$$

Then we *define*

(4.29)
$$D_tF = \lim_{n\to\infty} \mathbf{D}_tF_n$$

and

$$D_\gamma F = \int_0^T \mathrm{D}_tF \cdot g(t)dt$$
$$\text{for all} \quad \gamma(t) = \int_0^t g(s)ds \qquad \text{with } g \in L^2([0,T]).$$

We will call D_tF *the Malliavin derivative* of F.

REMARK Strictly speaking we now have two apparently different definitions of the derivative of F:

1) The derivative $\mathbf{D}_tF$ of $F \in \mathcal{D}_{1,2}$ given by Definition 4.7.
2) The Malliavin derivative D_tF of $F \in \mathbb{D}_{1,2}$ given by Definition 4.13.

However, the next result shows that if $F \in \mathcal{D}_{1,2} \cap \mathbb{D}_{1,2}$ then the two derivatives coincide:

LEMMA 4.14. Let $F \in \mathcal{D}_{1,2} \cap \mathbb{D}_{1,2}$ and suppose that $\{F_n\} \subset \mathbb{P}$ has the properties

(4.30)
$$F_n \to F \quad \text{in } L^2(\Omega) \qquad \text{and} \qquad \mathbf{D}_tF_n \quad \text{converges in } L^2([0,T]\times\Omega).$$

Then

(4.31)
$$\mathbf{D}_tF = \lim_{n\to\infty} \mathbf{D}_tF_n\,.$$

Hence

(4.32)
$$D_tF = \mathbf{D}_tF \qquad \text{for} \quad F \in \mathcal{D}_{1,2} \cap \mathbb{D}_{1,2}.$$

Proof. By (4.30) we get that $\mathbf{D}_\gamma F_n$ converges in $L^2(\Omega)$ for each $\gamma(t) = \int_0^t g(s)ds$; $g \in L^2([0,T])$. By Lemma 4.12 we get

$$\begin{aligned} &E[(\mathbf{D}_\gamma F_n - \mathbf{D}_\gamma F)\cdot\varphi] \\ &\qquad = E[(F_n - F)\cdot\varphi\cdot\int_0^t gdW] - E[(F_n - F)\cdot\mathbf{D}_\gamma\varphi] \\ &\qquad \to 0 \qquad \text{for all } \varphi \in \mathbb{P} \text{ by (4.30).} \end{aligned}$$

Hence $\mathbf{D}_\gamma F_n \to \mathbf{D}_\gamma F$ in $L^2(\Omega)$ and (4.31) follows. □

In view of Lemma 4.14 we will now use the same symbol D_tF and $D_\gamma F$ for the derivative and directional derivative, respectively, of elements $F \in \mathcal{D}_{1,2} \cup \mathbb{D}_{1,2}$.

REMARK 4.15. Note that it follows from the definition of $\mathbb{D}_{1,2}$ that if $F_n \in \mathbb{D}_{1,2}$ for $n = 1, 2, \ldots$ and $F_n \to F$ in $L^2(\Omega)$ and

$$\{D_tF_n\}_n \quad \text{is convergent in } \; L^2([0,T] \times \Omega)$$

then

$$F \in \mathbb{D}_{1,2} \quad \text{and} \quad D_tF = \lim_{n\to\infty} D_tF_n\,.$$

Since an arbitrary $F \in L^2(\Omega)$ can be represented by its chaos expansion

$$F(\omega) = \sum_{n=0}^{\infty} I_n(f_n); \qquad f_n \in \widehat{L}^2([0,T]^n)$$

it is natural to ask if we can express the derivative of F (if it exists) by means of this. We first consider a special case:

LEMMA 4.16. Suppose $F(\omega) = I_n(f_n)$ for some $f_n \in \widehat{L}^2([0,T]^n)$. Then $F \in \mathbb{D}_{1,2}$ and

$$D_tF(\omega) = nI_{n-1}(f_n(\cdot,t)), \tag{4.33}$$

where the notation $I_{n-1}(f_n(\cdot,t))$ means that the $(n-1)$-iterated Ito integral is taken with respect to the $n-1$ first variables $t_1, \ldots, t_{n-1}$ of $f_n(t_1, \ldots, t_{n-1}, t)$ (i.e. t is fixed and kept outside the integration).

Proof. First consider the special case when

$$f_n = f^{\otimes n}$$

for some $f \in L^2([0,T])$, i.e. when

$$f_n(t_1, \ldots, t_n) = f(t_1) \ldots f(t_n); \qquad (t_1, \ldots, t_n) \in [0,T]^n.$$

Then by (1.14)

$$I_n(f_n) = \|f\|^n h_n\Big(\frac{\theta}{\|f\|}\Big), \tag{4.34}$$

where $\theta = \int_0^T f dW$ and h_n is the Hermite polynomial of order n.

Hence by the chain rule

$$D_tI_n(f_n) = \|f\|^n h_n'\Big(\frac{\theta}{\|f\|}\Big) \cdot \frac{f(t)}{\|f\|}$$

A basic property of the Hermite polynomials is that

$$h_n'(x) = nh_{n-1}(x). \tag{4.35}$$

This gives, again by (1.14),

$$D_t I_n(f_n) = n\|f\|^{n-1} h_{n-1}\Big(\frac{\theta}{\|f\|}\Big) f(t) = n I_{n-1}(f^{\otimes(n-1)}) f(t) = n I_{n-1}(f_n(\cdot, t)).$$

Next, suppose f_n has the form

$$f_n = \eta_1^{\widehat{\otimes} n_1} \widehat{\otimes}\, \eta_2^{\widehat{\otimes} n_2} \widehat{\otimes} \cdots \widehat{\otimes}\, \eta_k^{\widehat{\otimes} n_k}; \qquad n_1 + \cdots + n_k = n \tag{4.36}$$

where $\widehat{\otimes}$ denotes symmetrized tensor product and $\{\eta_j\}$ is an orthonormal basis for $L^2([0,T])$. Then by an extension of (1.14) we have (see [I])

$$I_n(f_n) = h_{n_1}(\theta_1) \cdots h_{n_k}(\theta_k) \tag{4.37}$$

with

$$\theta_j = \int_0^T \eta_j dW$$

and again (4.33) follows by the chain rule. Since any $f_n \in \widehat{L}^2([0,T]^n)$ can be approximated in $L^2([0,T]^n)$ by linear combinations of functions of the form given by the right hand side of (4.36), the general result follows. □

LEMMA 4.17 Let $\mathbb{P}_0$ denote the set of Wiener polynomials of the form

$$p_k\Big(\int_0^T e_1 dW, \ldots, \int_0^T e_k dW\Big)$$

where $p_k(x_1, \ldots, x_k)$ is an arbitrary polynomial in k variables and $\{e_1, e_2, \ldots\}$ is a given orthonormal basis for $L^2([0,T])$. Then $\mathbb{P}_0$ is dense in $\mathbb{P}$ in the norm $\|\cdot\|_{1,2}$.

Proof. If $q := p(\int_0^T f_1 dW, \ldots, \int_0^T f_k dW) \in \mathbb{P}$ we approximate q by

$$q^{(m)} := p\Big(\int_0^T \sum_{j=0}^m (f_1, e_j) e_j dW, \ldots, \int_0^T \sum_{j=0}^m (f_k, e_j) e_j dW\Big)$$

Then $q^{(m)} \to q$ in $L^2(\Omega)$ and

$$D_t q^{(m)} = \sum_{i=1}^k \frac{\partial p}{\partial x_i} \cdot \sum_{j=1}^m (f_i, e_j) e_j(t) \to \sum_{i=1}^k \frac{\partial p}{\partial x_i} \cdot f_i(t)$$

in $L^2([0,T] \times \Omega)$ as $m \to \infty$. □

THEOREM 4.18. Let $F = \sum_{n=0}^{\infty} I_n(f_n) \in L^2(\Omega)$. Then $F \in \mathbb{D}_{1,2}$ if and only if

$$\sum_{n=1}^{\infty} n\, n! \|f_n\|^2_{L^2([0,T]^n)} < \infty \tag{4.38}$$

and if this is the case we have

$$D_tF = \sum_{n=0}^{\infty} nI_{n-1}(f_n(\cdot,t)). \tag{4.39}$$

Proof. Define $F_m = \sum\limits_{n=0}^{m} I_n(f_n)$. Then $F_m \in \mathbb{D}_{1,2}$ and $F_m \to F$ in $L^2(\Omega)$. Moreover, if $m > k$ we have by Lemma 4.15,

$$\begin{aligned}
\|D_tF_m - D_tF_k\|^2_{L^2([0,T]\times\Omega)} &= \|\sum_{n=k+1}^{m} nI_{n-1}(f_n(\cdot,t))\|^2_{L^2([0,T]\times\Omega)} \\
&= \int_0^T E[\{\sum_{n=k+1}^{m} nI_{n-1}(f_n(\cdot,t))\}^2]dt \\
&= \int_0^T \sum_{n=k+1}^{m} n^2(n-1)!\|f_n(\cdot,t)\|^2_{L^2([0,T]^{n-1})}dt \\
&= \sum_{n=k+1}^{m} n\,n!\|f_n\|^2_{L^2([0,T]^n)}\,. \qquad (4.40)
\end{aligned}$$

Hence if (4.38) holds then $\{D_tF_n\}_{n=1}^{\infty}$ is convergent in $L^2([0,T]\times\Omega)$ and hence $F \in \mathbb{D}_{1,2}$ and

$$D_tF = \lim_{m\to\infty} D_tF_m = \sum_{n=0}^{\infty} nI_{n-1}(f_n(\cdot,t)).$$

Conversely, if $F \in \mathbb{D}_{1,2}$ then there exist polynomials $p_k(x_1,\dots,x_{n_k})$ of degree k and $\eta_1,\dots,\eta_{n_k} \geq 0$ as in (4.36) such that if we put $F_k = p_k(\theta_1,\dots,\theta_{n_k}) = \sum\limits_{m_i:\sum m_i \leq k} a_{m_1,\dots,m_{n_k}} \prod\limits_{i=1}^{n_k} h_{m_i}(\theta_i)$ (for some $a_{m_1,\dots,m_{n_k}} \in \mathbf{R}$) then $F_k \in \mathbb{P}$ and $F_k \to F$ in $L^2(\Omega)$ and

$$D_tF_k \to D_tF \qquad \text{in } L^2([0,T]\times\Omega), \text{ as } k\to\infty.$$

By applying (4.37) we see that there exist $f_j^{(k)} \in \widehat{L}^2([0,T]^j)$; $1 \leq j \leq k$ such that

$$F_k = \sum_{j=0}^{k} I_j(f_j^{(k)}).$$

Since $F_k \to F$ in $L^2(\Omega)$ we have

$$\sum_{j=0}^{k} j!\|f_j^{(k)} - f_j\|^2_{L^2([0,T]^j)} \leq \|F_k - F\|^2_{L^2(\Omega)} \to 0 \qquad \text{as } k\to\infty.$$

Therefore $\|f_j^{(k)} - f_j\|_{L^2([0,T]^j)} \to 0$ as $k\to\infty$, for all j. This implies that

$$\|f_j^{(k)}\|_{L^2([0,T]^j)} \to \|f_j\|_{L^2([0,T]^j)} \qquad \text{as } k\to\infty, \text{ for all } j. \tag{4.41}$$

Similarly, since $D_tF_k \to D_tF$ in $L^2([0,T]\times\Omega)$ we get by the Fatou lemma combined with the calculation leading to (4.40) that

$$\sum_{j=0}^{\infty} j\cdot j!\|f_j\|^2_{L^2([0,T]^j)} = \sum_{j=0}^{\infty} \lim_{k\to\infty}(j\cdot j!\|f_j^{(k)}\|^2_{L^2([0,T]^j)})$$

$$
\begin{aligned}
&\leq \varliminf_{k\to\infty} \sum_{j=0}^{\infty} j \cdot j! \|f_j^{(k)}\|^2_{L^2([0,T]^j)} \\
&= \varliminf_{k\to\infty} \|D_t F_k\|^2_{L^2([0,T]\times\Omega)} = \|D_t F\|^2_{L^2([0,T]\times\Omega)} < \infty\,,
\end{aligned}
$$

where we have put $f_j^{(k)} = 0$ for $j > k$. Hence (4.38) holds and the proof is complete. □

Exercises

4.1 Find the Malliavin derivative $D_t F(\omega)$ of the following random variables:

a) $F(\omega) = W(T)$

b) $F(\omega) = \exp(W(t_0)) \quad (t_0 \in [0,T])$

c) $F(\omega) = \int_0^T s^2 dW(s)$

d) $F(\omega) = \int_0^T (\int_0^{t_2} \cos(t_1 + t_2) dW(t_1)) dW(t_2)$

e) $F(\omega) = 3W(s_0)W^2(t_0) + \ln(1 + W^2(s_0)) \quad (s_0, t_0 \in [0,T])$

f) $F(\omega) = \int_0^T W(t_0) \delta W(t) \quad (t_0 \in [0,T])$

(Hint: Use Exercise 2.1b).)

4.2 **a)** Find the Malliavin derivative $D_t F(\omega)$ when

$$
F(\omega) = \exp(\int_0^T g(s) dW(s)) \qquad (g \in L^2([0,T]))
$$

by using that (see Exercise 1.2d))

$$
F(\omega) = \sum_{n=0}^{\infty} I_n[f_n],
$$

with

$$
f_n(t_1, \dots, t_n) = \frac{1}{n!} \exp(\frac{1}{2} \|g\|^2_{L^2([0,T])}) g(t_1) \dots g(t_n)
$$

b) Verify the result by using the chain rule.

4.3 Verify the integration by parts formula (4.28) in the following case:

$$
F(\omega) = \int_0^T \psi(s) dW(s) \qquad \text{with } \psi \in L^2([0,T]) \text{ deterministic},
$$

$\varphi \equiv 1$.

5 The Clark-Ocone formula and its generalization. Application to finance

In this section we look at the connection between differentiation D and Skorohod integration δ and apply this to prove the Clark-Ocone formula and its generalization needed for e.g. portfolio applications.

First we establish some useful results about conditional expectation:

DEFINITION 5.1. Let G be a Borel subset of $[0,T]$. Then we define $\mathcal{F}_G$ to be the σ-algebra generated by all random variables of the form

$$\int_A dW(t) := \int_0^T \mathcal{X}_A(t)dW(t); \qquad A \subset G \quad \text{Borel set} \tag{5.1}$$

Thus if $G = [0,t]$ we have, with this notation

$$\mathcal{F}_{[0,t]} = \mathcal{F}_t \qquad \text{for } t \geq 0.$$

LEMMA 5.2. Let $g \in L^2([0,T])$ be deterministic. Then

$$E[\int_0^T g(t)dW(t)|\mathcal{F}_G] = \int_0^T \mathcal{X}_G(t)g(t)dW(t).$$

Proof. By definition of conditional expectation, it suffices to verify that

$$\int_0^T \mathcal{X}_G(t)g(t)dW(t) \qquad \text{is } \mathcal{F}_G\text{-measurable} \tag{5.2}$$

and

$$E[F(\omega)\int_0^T g(t)dW(t) = E[F(\omega)\int_0^T \mathcal{X}_G(t)g(t)dW(t)] \tag{5.3}$$

for all bounded $\mathcal{F}_G$-measurable random variables F.

To prove (5.2) we may assume that g is continuous, because the continuous functions are dense in $L^2([0,T])$. If g is continuous, then

$$\int_0^T \mathcal{X}_G(t)g(t)dW(t) = \lim_{\Delta t_i \to 0} \sum_i g(t_i) \int_{t_i}^{t_i+1} \mathcal{X}_G(t)dW(t)$$

(limit in $L^2(\Omega)$) and since each term in the sum is $\mathcal{F}_G$-measurable the sum and its limit is.

To prove (5.2) we may assume that

$$F(\omega) = \int_0^T \mathcal{X}_A(t)dW(t) \qquad \text{for some } A \subset G.$$

This gives that the left hand side of (5.3) becomes

$$E[\int_0^T \mathcal{X}_A(t)dW(t) \cdot \int_0^T g(t)dW(t)] = E[\int_0^T \mathcal{X}_A(t)g(t)dt],$$

by the Ito isometry. Similarly, the right hand side becomes

$$E[\int_0^T \mathcal{X}_A(t)dW(t) \cdot \int_0^T \mathcal{X}_G(t)g(t)dW(t)] = E[\int_0^T \mathcal{X}_A(t)\mathcal{X}_G(t)g(t)dt]$$

$$= E[\int_0^T \mathcal{X}_A(t)g(t)dt] \qquad \text{since } A \subset G.$$

□

LEMMA 5.3. Let $v(t,\omega) \in \mathbf{R}$ be a stochastic process such that

(i) $v(t,\cdot)$ is $\mathcal{F}_t \cap \mathcal{F}_G$-measurable for all t and

(ii) $E[\int_0^T v^2(t,\omega)dt] < \infty$.

Then

$$\int_G v(t,\omega)dW(t) \qquad \text{is } \mathcal{F}_G\text{-measurable.}$$

Proof. By a standard approximation procedure we see that we may assume that $v(t,\omega)$ is an elementary process of the form

$$v(t,\omega) = \sum_i v_i(\omega)\mathcal{X}_{[t_i,t_{i+1})}(t)$$

where $0 = t_0 < t_1 < \cdots < t_n = T$ and $v_i(\cdot)$ is $\mathcal{F}_{t_i} \cap \mathcal{F}_G$-measurable. For such v we have

$$\int_G v(t,\omega)dW(t) = \sum_i v_i(\omega) \int_{G\cap[t_i,t_{i+1})} dW(t),$$

which is a sum of products of $\mathcal{F}_G$-measurable functions and hence $\mathcal{F}_G$-measurable. □

LEMMA 5.4. Let $u(t,\omega)$ be an $\mathcal{F}_t$-adapted process such that

$$E[\int_0^T u^2(t,\omega)dt] < \infty.$$

Then

$$E[\int_0^T u(t,\omega)dW(t)|\mathcal{F}_G] = \int_G E[u(t,\omega)|\mathcal{F}_G]dW(t)$$

Proof. By Lemma 5.3 it suffices to verify that

$$E[f(\omega)\int_0^T u(t,\omega)dW(t)] = E[f(\omega)\int_G E[u(t,\omega)|\mathcal{F}_G]dW(t)] \tag{5.4}$$

for all $f(\omega)$ of the form

$$f(\omega) = \int_A dW(t); \qquad A \subset G$$

For such f we obtain by the Ito isometry that the left hand side of (5.4) is equal to

$$E[\int_0^T \mathcal{X}_A(t)u(t,\omega)dt] = \int_A E[u(t,\omega)]dt$$

while the right hand side is equal to

$$E[\int_0^T \mathcal{X}_A(t)\mathcal{X}_G(t)E[u(t,\omega)|\mathcal{F}_G]dt] = \int_0^T \mathcal{X}_A(t)E[E[u(t,\omega)|\mathcal{F}_G]]dt$$
$$= \int_A E[u(t,\omega)]dt\,.$$

□

PROPOSITION 5.5. Let $f_n \in \widehat{L}^2([0,T]^n)$. Then

$$E[I_n(f_n)|\mathcal{F}_G] = I_n[f_n\mathcal{X}_G^{\otimes n}], \tag{5.5}$$

where

$$(f_n\mathcal{X}_G^{\otimes n})(t_1,\dots,t_n) = f_n(t_1,\dots,t_n)\mathcal{X}_G(t_1)\cdots\mathcal{X}_G(t_n).$$

Proof. We proceed by induction on n. For $n=1$ we have by Lemma 5.4

$$E[I_1(f_1)|\mathcal{F}_G] = E[\int_0^T f_1(t_1)dW(t_1)|\mathcal{F}_G] = \int_0^T f_1(t_1)\mathcal{X}_G(t_1)dW(t).$$

Assume that (5.5) holds for $n=k$. Then by Lemma 5.4

$$E[I_{k+1}(f_{k+1})|\mathcal{F}_G]$$
$$= (k+1)!E[\int_0^T\{\int_0^{t_k}\cdots\int_0^{t_2} f_{k+1}(t_1,\dots,t_{k+1})dW(t_1)\cdots\}dW(t_{k+1})|\mathcal{F}_G]$$
$$= (k+1)!\int_0^T E[\int_0^{t_k}\cdots\int_0^{t_2} f_{k+1}(t_1,\dots,t_{k+1})dW(t_1)\cdots dW(t_k)|\mathcal{F}_G]\cdot\mathcal{X}_G(t_{k+1})dW(t_{k+1})$$

$$= (k+1)! \int_0^T \int_0^{t_k} \cdots \int_0^{t_2} f_{k+1}(t_1, \ldots, t_{k+1}) \mathcal{X}_G(t_1) \cdots \mathcal{X}_G(t_{k+1}) dW(t_1) \cdots dW(t_{k+1})$$
$$= I_{k+1}[f_{k+1} \mathcal{X}_G^{\otimes(k+1)}],$$

and the proof is complete. □

PROPOSITION 5.6. If $F \in \mathbb{D}_{1,2}$ then $E[F|\mathcal{F}_G] \in \mathbb{D}_{1,2}$ and

$$D_t(E[F|\mathcal{F}_G]) = E[D_t F|\mathcal{F}_G] \cdot \mathcal{X}_G(t).$$

Proof. First asume that $F = I_n(f_n)$ for some $f_n \in \widehat{L}^2([0,T]^n)$. Then by Proposition 5.5

$$\begin{aligned} D_t(E[F|\mathcal{F}_G]) &= D_t E[I_n(f_n)|\mathcal{F}_G] \\ &= D_t[I_n(f_n \cdot \mathcal{X}_G^{\otimes n})] = n I_{n-1}[f_n(\cdot, t)\mathcal{X}_G^{\otimes(n-1)}(\cdot) \cdot \mathcal{X}_G(t)] \\ &= n I_{n-1}[f_n(\cdot, t)\mathcal{X}_G^{\otimes(n-1)}(\cdot)] \cdot \mathcal{X}_G(t) \\ &= E[D_t F|\mathcal{F}_G] \cdot \mathcal{X}_G(t). \end{aligned} \tag{5.6}$$

Next, suppose $F \in \mathbb{D}_{1,2}$ is arbitrary. Then as in the proof of Theorem 4.16 we see that we can find $F_k \in \mathbb{P}$ such that

$$F_k \to F \quad \text{in} \quad L^2(\Omega) \quad \text{and} \quad D_t F_k \to D_t F \quad \text{in} \quad L^2(\Omega \times [0,T])$$

as $k \to \infty$, and there exists $f_j^{(k)} \in \widehat{L}^2([0,T]^j)$ such that

$$F_k = \sum_{j=0}^{k} I_j(f_j^{(k)}) \qquad \text{for all } k.$$

By (5.6) we have

$$D_t(E[F_k|\mathcal{F}_G]) = E[D_t F_k|\mathcal{F}_G] \cdot \mathcal{X}_G(t) \qquad \text{for all } k$$

and taking the limit of this as $k \to \infty$ we obtain the result. □

COROLLARY 5.7. Let $u(s,\omega)$ be an $\mathcal{F}_s$-adapted stochastic process and assume that $u(s,\cdot) \in \mathbb{D}_{1,2}$ for all s. Then

(i) $$D_t u(s,\omega) \qquad \text{is} \quad \mathcal{F}_s\text{-adapted for all } t$$

and

(ii) $$D_t u(s,\omega) = 0 \qquad \text{for } t > s.$$

Proof. By Proposition 5.6 we have that

$$D_t u(s,\omega) = D_t(E[u(s,\omega)|\mathcal{F}_s]) = E[D_t u(s,\omega)|\mathcal{F}_s] \cdot \mathcal{X}_{[0,s]}(t),$$

from which (i) and (ii) follow immediately. □

We now have all the necessary ingredients for our first main result in this section:

THEOREM 5.8. (The Clark-Ocone formula) Let $F \in \mathbb{D}_{1,2}$ be $\mathcal{F}_T$-measurable. Then

$$F(\omega) = E[F] + \int_0^T E[D_t F|\mathcal{F}_t](\omega) dW(t). \tag{5.7}$$

Proof. Write $F = \sum_{n=0}^{\infty} I_n(f_n)$ with $f_n \in \widehat{L}^2([0,T]^n)$. Then by Theorem 4.16, Proposition 5.5 and Definition 2.2

$$\int_0^T E[D_t F|\mathcal{F}_t] dW(t) = \int_0^T E[\sum_{n=1}^{\infty} nI_{n-1}(f_n(\cdot,t))|\mathcal{F}_t] dW(t) = \int_0^T \sum_{n=1}^{\infty} nE[I_{n-1}(f_n(\cdot,t))|\mathcal{F}_t] dW(t)$$

$$= \int_0^T \sum_{n=1}^{\infty} nI_{n-1}[f_n(\cdot,t) \cdot \mathcal{X}_{[0,t]}^{\otimes(n-1)}(\cdot)] dW(t) = \int_0^T \sum_{n=1}^{\infty} n(n-1)! J_{n-1}[f_n(\cdot,t)\mathcal{X}_{[0,t]}^{\otimes(n-1)}] dW(t)$$

$$= \sum_{n=1}^{\infty} n! J_n[f_n(\cdot)] = \sum_{n=1}^{\infty} I_n[f_n] = \sum_{n=0}^{\infty} I_n[f_n] - I_0[f_0] = F - E[F].$$

□

The generalized Clark-Ocone formula

We proceed to prove the *generalized* Clark-Ocone formula. This formula expresses an $\mathcal{F}_T$-measurable random variable $F(\omega)$ as a stochastic integral with respect to a process of the form

$$\widetilde{W}(t,\omega) = \int_0^t \theta(s,\omega) ds + W(t,\omega); \qquad 0 \le t \le T \tag{5.8}$$

where $\theta(s,\omega)$ is a given $\mathcal{F}_s$-adapted stochastic process satisfying some additional conditions. By the Girsanov theorem (see Exercise 5.1) the process $\widetilde{W}(t) = \widetilde{W}(t,\omega)$ is a Wiener process under the new probability measure Q defined on $\mathcal{F}_T$ by

$$dQ(\omega) = Z(T,\omega) dP(\omega) \tag{5.9}$$

where

$$Z(t) = Z(t,\omega) = \exp\{-\int_0^t \theta(s,\omega) dW(s) - \frac{1}{2}\int_0^t \theta^2(s,\omega) ds\}; \qquad 0 \le t \le T \tag{5.10}$$

We let E_Q denote expectation w.r.t. Q, while $E_P = E$ denotes expectation w.r.t. P.

THEOREM 5.9. (The generalized Clark-Ocone formula [KO])
Suppose $F \in \mathbb{D}_{1,2}$ is $\mathcal{F}_t$-measurable and that

$$E_Q[|F|] < \infty \tag{5.11}$$

$$E_Q[\int_0^T |D_t F|^2 dt] < \infty \tag{5.12}$$

$$E_Q[|F| \cdot \int_0^T (\int_0^T D_t\theta(s,\omega) dW(s) + \int_0^T D_t\theta(s,\omega)\theta(s,\omega) ds)^2 dt] < \infty \tag{5.13}$$

Then

$$F(\omega) = E_Q[F] + \int_0^T E_Q[(D_tF - F\int_t^T D_t\theta(s,\omega)d\widetilde{W}(s))|\mathcal{F}_t]d\widetilde{W}(t). \tag{5.14}$$

Remark. Note that we cannot obtain a representation as an integral w.r.t. $\widetilde{W}$ simply by applying the Clark-Ocone formula to our new Wiener process $(\widetilde{W}(t), Q)$, because F is only assumed to be $\mathcal{F}_T$-measurable, not $\widetilde{\mathcal{F}}_T$-measurable, where $\widetilde{\mathcal{F}}_T$ is the σ-algebra generated by $\widetilde{W}(t,\cdot)$; $t \le T$. In general we have $\widetilde{\mathcal{F}}_T \subseteq \mathcal{F}_T$ and usually $\widetilde{\mathcal{F}}_T \ne \mathcal{F}_T$. Nevertheless, the Ito integral w.r.t. $\widetilde{W}$ in (5.14) does make sense, because $\widetilde{W}(t)$ is a martingale w.r.t. $\mathcal{F}_t$ and Q (see Exercise 5.1)).

The proof of Theorem 5.9 is split up into several useful results of independent interest:

LEMMA 5.10. Let μ and ν be two probability measures on a measurable space $(\Omega, \mathcal{G})$ such that $d\nu(\omega) = f(\omega)d\mu(\omega)$ for some $f \in L^1(\mu)$. Let X be a random variable on $(\Omega, \mathcal{G})$ such that $X \in L^1(\nu)$. Let $\mathcal{H} \subset \mathcal{G}$ be a σ-algebra. Then

$$E_\nu[X|\mathcal{H}] \cdot E_\mu[f|\mathcal{H}] = E_\mu[fX|\mathcal{H}] \tag{5.15}$$

Proof. See e.g. [Ø, Lemma 8.24].

COROLLARY 5.11. Suppose $G \in L^1(Q)$. Then

$$E_Q[G|\mathcal{F}_t] = \frac{E[Z(T)G|\mathcal{F}_t]}{Z(t)} \tag{5.16}$$

The next result gives a useful connection between differentiation and Skorohod integration:

THEOREM 5.12. Let $u(s,\omega)$ be a stochastic process such that

$$E[\int_0^T u^2(s,\omega)ds] < \infty \tag{5.17}$$

and assume that $u(s,\cdot) \in \mathbb{D}_{1,2}$ for all $s \in [0,T]$, that $D_tu \in \text{Dom}(\delta)$ for all $t \in [0,T]$, and that

$$E[\int_0^T (\delta(D_tu))^2 dt] < \infty \tag{5.18}$$

Then $\int_0^T u(s,\omega)\delta W(s) \in \mathbb{D}_{1,2}$ and

$$D_t(\int_0^T u(s,\omega)\delta W(s)) = \int_0^T D_tu(s,\omega)\delta W(s) + u(t,\omega). \tag{5.19}$$

Proof. First assume that

$$u(s,\omega) = I_n(f_n(\cdot,s)),$$

where $f_n(t_1, \dots, t_n, s)$ is symmetric with respect to $t_1, \dots, t_n$. Then

$$\int_0^T u(s,\omega)\delta W(s) = I_{n+1}[\widetilde{f}_n],$$

where

$$\widetilde{f}_n(x_1, \dots, x_{n+1}) = \frac{1}{n+1}[f_n(\cdot, x_1) + \cdots + f_n(\cdot, x_{n+1})]$$

is the symmetrization of f_n as a function of all its $n+1$ variables. Hence

$$D_t(\int_0^T u(s,\omega)\delta W(s)) = (n+1)I_n[\widetilde{f}_n(\cdot, t)], \tag{5.20}$$

where

$$\widetilde{f}_n(\cdot, t) = \frac{1}{n+1}[f_n(t,\cdot, x_1) + \cdots + f_n(t, \cdot, x_n) + f_n(\cdot, t)] \tag{5.21}$$

(since f_n is symmetric w.r.t. its first n variables, we may choose t to be the first of them, in the first n terms on the right hand side). Combining (5.20) with (5.21) we get

$$D_t(\int_0^T u(s,\omega)\delta W(s)) = I_n[f_n(t, \cdot, x_1) + \cdots + f_n(t, \cdot, x_n) + f_n(\cdot, t)] \tag{5.22}$$

(integration in I_n is w.r.t. $x_1, \dots, x_n$).

To compare this with the right hand side of (5.19) we consider

$$\delta(D_t u) = \int_0^T D_t u(s,\omega)\delta W(S) = \int_0^T nI_{n-1}[f_n(\cdot, t, s)]\delta W(s) = nI_n[\widehat{f}_n(\cdot, t, \cdot)], \tag{5.23}$$

where

$$\widehat{f}_n(x_1, \dots, x_{n-1}, t, x_n) = \frac{1}{n}[f_n(t, \cdot, x_1) + \cdots + f_n(t, \cdot, x_n)] \tag{5.24}$$

is the symmetrization of $f_n(x_1, \dots, x_{n-1}, t, x_n)$ w.r.t. $x_1, \dots, x_n$.

From (5.23) and (5.24) we get

$$\int_0^T D_t u(s,\omega)\delta W(s) = I_n[f_n(t, \cdot, x_1) + \cdots + f_n(t, \cdot, x_n)]. \tag{5.25}$$

Comparing (5.22) and (5.25) we obtain (5.19).

Next, consider the general case when

$$u(s,\omega) = \sum_{n=0}^{\infty} I_n[f_n(\cdot, s)].$$

Define

$$u_m(s,\omega) = \sum_{n=0}^{m} I_n[f_n(\cdot,s)]; \qquad m = 1,2,\dots \tag{5.26}$$

Then by the above we have

$$D_t(\delta(u_m)) = \delta(D_t u_m) + u_m(t) \quad \text{for all } m. \tag{5.27}$$

By (5.23) we see that (5.18) is equivalent to saying that

$$E[\int_0^T (\delta(D_t u))^2 dt] = \sum_{n=0}^{\infty} n^2 n! \int_0^T \|\widehat{f}_n(\cdot,t,\cdot)\|^2_{L^2([0,T]^n)} dt$$

$$= \sum_{n=0}^{\infty} n^2 n! \|\widehat{f}_n\|^2_{L^2([0,T]^{n+1})} < \infty \quad \text{since } D_t u \in \text{Dom}(\delta). \tag{5.28}$$

Hence

$$\|\delta(D_t u_m) - \delta(D_t u)\|^2_{L^2([0,T]\times\Omega)} = \sum_{n=m+1}^{\infty} n^2 n! \|\widehat{f}_n\|^2_{L^2([0,T]^{n+1})} \to 0 \qquad \text{as } m \to \infty. \tag{5.29}$$

Therefore, by (5.27)

$$D_t(\delta(u_m)) \to \delta(D_t u) + u(t) \quad \text{in} \quad L^2([0,T]\times\Omega) \tag{5.30}$$

as $m \to \infty$. Note that from (5.21) and (5.24) we have

$$(n+1)\widetilde{f}_n(\cdot,t) = n\widehat{f}_n(\cdot,t,\cdot) + f_n(\cdot,t)$$

and hence

$$(n+1)!\|\widetilde{f}_n\|^2_{L^2([0,T]^{n+1})} \le \frac{2n^2 n!}{n+1}\|\widehat{f}_n\|^2_{L^2([0,T]^{n+1})} + \frac{2n!}{n+1}\|f_n\|^2_{L^2([0,T]^{n+1})}$$

Therefore,

$$E[(\delta(u_m) - \delta(u))^2] = \sum_{n=m+1}^{\infty} (n+1)!\|\widetilde{f}_n\|^2_{L^2([0,T]^{n+1})} \to 0 \tag{5.31}$$

as $m \to \infty$. From (5.30) and (5.31) we conclude that $\delta(u) \in \mathbb{D}_{1,2}$ and that

$$D_t(\delta(u)) = \delta(D_t u) + u(t), \qquad \text{which is (5.19).}$$

□

COROLLARY 5.13. Let $u(s,\omega)$ be as in Theorem 5.12 and assume in addition that

$$u(s,\omega) \qquad \text{is } \mathcal{F}_s\text{-adapted.}$$

Then

$$D_t(\int_0^T u(s,\omega)dW(s)) = \int_t^T D_t u(s,\omega)dW(s) + u(t,\omega). \tag{5.32}$$

Proof. This is an immediate consequence of Theorem 5.12 and Corollary 5.7. □

LEMMA 5.14. Let F, θ be as in Theorem 5.9 and let Q and $Z(t)$ be as in (5.9), (5.10). Then

$$D_t(Z(T)F) = Z(T)[D_tF - F\{\theta(t,\omega) + \int_t^T D_t\theta(s,\omega)d\widetilde{W}(s)\}]. \tag{5.33}$$

Proof. By the chain rule we have, using Corollary 5.13, $D_t(Z(T)F) = Z(T)D_tF + F\,D_tZ(T)$ and

$$\begin{aligned} D_tZ(T) &= Z(T)\{-D_t(\int_0^T \theta(s,\omega)dW(s)) - \frac{1}{2}D_t(\int_0^T \theta^2(s,\omega)ds)\} \\ &= Z(T)\{-\int_t^T D_t\theta(s,\omega)dW(s) - \theta(t,\omega) - \int_0^T \theta(s,\omega)D_t\theta(s,\omega)ds\} \\ &= Z(T)\{-\int_t^T D_t\theta(s,\omega)d\widetilde{W}(s) - \theta(t,\omega)\}. \end{aligned}$$

□

Proof of Theorem 5.9: Suppose that (5.11)–(5.13) hold and put

$$Y(t) = E_Q[F|\mathcal{F}_t] \tag{5.34}$$

and

$$\Lambda(t) = Z^{-1}(t) = \exp\{\int_0^t \theta(s,\omega)dW(s) + \frac{1}{2}\int_0^t \theta^2(s,\omega)ds\}. \tag{5.35}$$

Note that

$$\Lambda(t) = \exp\{\int_0^t \theta(s,\omega)d\widetilde{W}(s) - \frac{1}{2}\int_0^t \theta^2(s,\omega)ds\}. \tag{5.36}$$

By Corollary 5.11, Theorem 5.8 and Corollary 5.7 we can write

$$\begin{aligned} Y_t &= \Lambda(t)E[Z(T)F|\mathcal{F}_t] \\ &= \Lambda(t)\{E[E[Z(T)F|\mathcal{F}_t]] + \int_0^T E[D_sE[Z(T)F|\mathcal{F}_t]|\mathcal{F}_s]dW(s)\} \\ &= \Lambda(t)\{E[Z(T)F] + \int_0^t E[D_s(Z(T)F)|\mathcal{F}_s]dW(s)\} \\ &=: \Lambda(t)U(t). \end{aligned} \tag{5.37}$$

By (5.36) and the Ito formula we have

$$d\Lambda(t) = \Lambda(t)\theta(t)d\widetilde{W}(t) \tag{5.38}$$

Combining (5.37), (5.38) and (5.33) we get

$$\begin{aligned} dY(t) &= \Lambda(t) \cdot E[D_t(Z(T)F)|\mathcal{F}_t]dW(t) \\ &\quad +\Lambda(t)\theta(t)U(t)d\widetilde{W}(t) \\ &\quad +\Lambda(t)\theta(t)E[D_t(Z(T)F)|\mathcal{F}_t]dW(t)d\widetilde{W}(t) \\ &= \Lambda(t)E[D_t(Z(T)F)|\mathcal{F}_t]d\widetilde{W}(t) + \theta(t)Y(t)d\widetilde{W}(t) \\ &= \Lambda(t)\{E[Z(T)D_tF|\mathcal{F}_t] - E[Z(T)F\theta(t)|\mathcal{F}_t] \\ &\quad -E[Z(T)F\int_t^T D_t\theta(s)d\widetilde{W}(s)|\mathcal{F}_t]\}d\widetilde{W}(t) + \theta(t)Y(t)d\widetilde{W}(t) \end{aligned}$$

Hence

$$\begin{aligned} dY(t) &= \{E_Q[D_tF|\mathcal{F}_t] - E_Q[F\theta(t)|\mathcal{F}_t] - E_Q[F\int_t^T D_t\theta(s)d\widetilde{W}(s)|\mathcal{F}_t]\}d\widetilde{W}(t) \\ &\quad +\theta(t)E_Q[F|\mathcal{F}_t]d\widetilde{W}(t) \\ &= E_Q[(D_tF - F\int_t^T D_t\theta(s)d\widetilde{W}(s))|\mathcal{F}_t]d\widetilde{W}(t). \end{aligned} \tag{5.39}$$

Since

$$Y(T) = E_Q[F|\mathcal{F}_T] = F$$

and

$$Y(0) = E_Q[F|\mathcal{F}_0] = E_Q[F],$$

we see that Theorem 5.9 follows from (5.39). The conditions (5.11)–(5.13) are needed to make all the above operations valid. We omit the details. □

Application to finance

We end this section by explaining how the generalized Clark-Ocone theorem can be applied in portfolio analysis:

Suppose we have two possible investments:

a) A safe investment (e.g. a bond), with price dynamics

$$dA(t) = \rho(t) \cdot A(t)dt \tag{5.40}$$

b) A risky investment (e.g. a stock), with price dynamics

$$dS(t) = \mu(t)S(t)dt + \sigma(t)S(t)dW(t) \tag{5.41}$$

Here $\rho(t) = \rho(t,\omega)$, $\mu(t) = \mu(t,\omega)$ and $\sigma(t) = \sigma(t,\omega)$ are $\mathcal{F}_t$-adapted processes. In the following we will not specify further conditions, but simply assume that these processes are sufficiently nice to make the operations convergent and well-defined.

Let $\xi(t) = \xi(t,\omega)$, $\eta(t) = \eta(t,\omega)$ denote the number of units invested at time t in investments a), b), respectively. Then the *value* at time t, $V(t) = V(t,\omega)$, of this *portfolio* $(\xi(t), \eta(t))$ is given by

$$V(t) = \xi(t)A(t) + \eta(t)S(t) \tag{5.42}$$

The portfolio $(\xi(t), \eta(t)$ is called *self-financing* if

$$dV(t) = \xi(t)dA(t) + \eta(t)dS(t). \tag{5.43}$$

Assume from now on that $(\xi(t), \eta(t))$ is self-financing. Then by substituting

$$\xi(t) = \frac{V(t) - \eta(t)S(t)}{A(t)} \tag{5.44}$$

from (5.42) in (5.43) and using (5.40) we get

$$dV(t) = \rho(t)(V(t) - \eta(t)S(t))dt + \eta(t)dS(t). \tag{5.45}$$

Then by (5.41) this can be written

$$dV(t) = [\rho(t)V(t) + (\mu(t) - \rho(t))\eta(t)S(t)]dt + \sigma(t)\eta(t)S(t)dW(t) \tag{5.46}$$

Suppose now that we are required to find a portfolio $(\xi(t), \eta(t))$ which leads to a given value

$$V(T,\omega) = F(\omega) \quad \text{a.s.} \tag{5.47}$$

at a given (deterministic) future time T, where the given $F(\omega)$ is $\mathcal{F}_T$-measurable. Then the problem is:

What initial fortune $V(0)$ is needed to achieve this, and what portfolio $(\xi(t), \eta(t))$ should we use? Is $V(0)$ and $(\xi(t), \eta(t))$ unique?

This type of question appears in option pricing.

For example, in the classical Black-Scholes model we have

$$F(\omega) = (S(T,\omega) - K)^+$$

where K is the exercise price and then $V(0)$ is the *price of the option.*

Because of the relation (5.44) we see that we might as well consider $(V(t), \eta(t))$ to be the unknown $\mathcal{F}_t$-adapted processes. Then (5.46)–(5.47) constitutes what is known as a *stochastic backward* differential equation (SBDE): The *final* value $V(T,\omega)$ is given and one seeks the value of $V(t), \eta(t)$ for $0 \le t \le T$. Note that since $V(t)$ is $\mathcal{F}_t$-adapted, we have that $V(0)$ is $\mathcal{F}_0$-measurable and therefore a *constant*. The general theory of SBDE gives that (under reasonable conditions on F, ρ, μ and σ) equation (5.46)–(5.47) has a *unique* solution of $\mathcal{F}_t$-adapted processe $V(t), \eta(t)$. See e.g. [PP]. However, this general theory says little about how to find this solution explicitly. This is where the generalized Clark-Ocone theorem enters the scene:

Define

$$\theta(t) = \theta(t, \omega) = \frac{\mu(t) - \rho(t)}{\sigma(t)} \tag{5.48}$$

and put

$$\widetilde{W}(t) = \int_0^t \theta(s) ds + W(t). \tag{5.49}$$

Then $\widetilde{W}(t)$ is a Wiener process w.r.t. the measure Q defined by (5.9), (5.10). In terms of $\widetilde{W}(t)$ equation (5.46) gets the form

$$\begin{aligned} dV(t) &= [\rho(t)V(t) + (\mu(t) - \rho(t))\eta(t)S(t)]dt + \sigma(t)\eta(t)S(t)d\widetilde{W}(t) \\ &\quad -\sigma(t)\eta(t)S(t)\sigma^{-1}(t)(\mu(t) - \rho(t))dt \end{aligned}$$

i.e.

$$dV(t) = \rho(t)V(t)dt + \sigma(t)\eta(t)S(t)d\widetilde{W}(t). \tag{5.50}$$

Define

$$U(t) = e^{-\int_0^t \rho(s,\omega)ds} V(t). \tag{5.51}$$

Then, substituting in (5.50), we get

$$dU(t) = e^{-\int_0^t \rho ds} \sigma(t)\eta(t)S(t)d\widetilde{W}(t) \tag{5.52}$$

or

$$e^{-\int_0^T \rho ds} V(T) = V(0) + \int_0^T e^{-\int_0^t \rho ds} \sigma(t)\eta(t)S(t)d\widetilde{W}(t). \tag{5.53}$$

By the generalized Clark-Ocone theorem applied to

$$G(\omega) := e^{-\int_0^T \rho(s,\omega)ds} F(\omega) \tag{5.54}$$

we get

$$G(\omega) = E_Q[G] + \int_0^T E_Q[(D_t G - G \int_t^T D_t \theta(s,\omega) d\widetilde{W}(s)) | \mathcal{F}_t] d\widetilde{W}, \tag{5.55}$$

By uniqueness we conclude from (5.53) and (5.55) that

$$V(0) = E_Q[G] \tag{5.56}$$

and the required risky investment at time t is

$$\eta(t) = e^{\int_0^t \rho ds}\sigma^{-1}(t)S^{-1}(t)E_Q[(D_tG - G\int_t^T D_t\theta(s)d\widetilde{W}(s))|\mathcal{F}_t]. \tag{5.57}$$

EXAMPLE 5.14. Suppose $\rho(t,\omega) = \rho$, $\mu(t,\omega) = \mu$ and $\sigma(t,\omega) = \sigma \neq 0$ are constants. Then

$$\theta(t,\omega) = \theta = \frac{\mu - \rho}{\sigma}$$

is constant also and hence $D_t\theta = 0$. Therefore by (5.57)

$$\eta(t) = e^{\rho(t-T)}\sigma^{-1}S^{-1}(t)E_Q[D_tF|\mathcal{F}_t].$$

For example, if the payoff function is

$$F(\omega) = \exp(\alpha W(T)) \qquad (\alpha \neq 0 \text{ constant})$$

then by the chain rule we get

$$\begin{aligned} \eta(t) &= e^{\rho(t-T)}\sigma^{-1}S^{-1}(t)E_Q[\alpha\exp(\alpha W(T))|\mathcal{F}_t] \\ &= e^{\rho(t-T)}\alpha\sigma^{-1}S^{-1}(t)Z^{-1}(t)E[Z(T)\exp(\alpha W(T))|\mathcal{F}_t]. \end{aligned} \tag{5.58}$$

Note that

$$Z(T)\exp(\alpha W(T)) = M(T)\exp(\frac{1}{2}(\alpha-\theta)^2T)$$

where $M(t) := \exp\{(\alpha-\theta)W(t) - \frac{1}{2}(\alpha-\theta)^2t)\}$ is a martingale. This gives

$$\begin{aligned} \eta(t) &= e^{\rho(t-T)}\alpha\sigma^{-1}S^{-1}(t)Z^{-1}(t)M(t)\exp(\frac{1}{2}(\alpha-\theta)^2T) \\ &= e^{\rho(t-T)}\alpha\sigma^{-1}\exp\{(\alpha-\sigma)W(t) + (\frac{1}{2}\sigma^2 + \frac{1}{2}\theta^2 - \mu)t + \frac{1}{2}(\alpha-\theta)^2(T-t)\}. \end{aligned}$$

EXAMPLE 5.15 (The Black and Scholes formula)

Finally, let us illustrate the method above by using it to prove the celebrated Black and Scholes formula (see e.g. [Du]). As in Example 5.14 let us assume that $\rho(t,\omega) = \rho$, $\mu(t,\omega) = \mu$ and $\sigma(t,\omega) = \sigma \neq 0$ are constants. Then

$$\theta = \frac{\mu - \rho}{\sigma}$$

is constant and hence $D_t\theta = 0$. Hence

$$\eta(t) = e^{\rho(t-T)}\sigma^{-1}S^{-1}(t)E_Q[D_tF \mid \mathcal{F}_t] \tag{5.59}$$

as in Example 5.14. However, in this case $F(\omega)$ represents the payoff at time T (fixed) of a (European call) option which gives the owner the right to buy the stock with value

$S(T,\omega)$ at a fixed exercise price K, say. Thus if $S(T,\omega) > K$ the owner of the option gets the profit $S(T,\omega) - K$ and if $S(T,\omega) \le K$ the owner does not exercise the option and the profit is 0. Hence in this case

$$F(\omega) = (S(T,\omega) - K)^+. \tag{5.60}$$

Thus we may write

$$F(\omega) = f(S(T,\omega)) \tag{5.61}$$

where

$$f(x) = (x - K)^+. \tag{5.62}$$

The function f is not differentiable at $x = K$, so we cannot use the chain rule directly to evaluate $D_t G$ from (5.61). However, we can approximate f by C^1 functions f_n with the property that

$$f_n(x) = f(x) \qquad \text{for} \quad |x - K| \ge \frac{1}{n} \tag{5.63}$$

and

$$0 \le f_n'(x) \le 1 \qquad \text{for all } x. \tag{5.64}$$

Putting

$$F_n(\omega) = f_n(S(T,\omega))$$

we then see that

$$\begin{aligned} D_t F(\omega) &= \lim_{n\to\infty} D_t F_n(\omega) = \mathcal{X}_{[K,\infty]}(S(T,\omega)) D_t S(T,\omega) \\ &= \mathcal{X}_{[K,\infty]}(S(T,\omega)) \cdot S(T,\omega) \cdot \sigma \end{aligned} \tag{5.65}$$

Hence by (5.59)

$$\eta(t) = e^{\rho(t-T)} S^{-1}(t) E_Q[S(T) \cdot \mathcal{X}_{[K,\infty]}(S(T)) | \mathcal{F}_t] \tag{5.66}$$

By the Markov property of $S(t)$ this is the same as

$$\eta(t) = e^{\rho(t-T)} S^{-1}(t) E_Q^y[S(T-t) \cdot \mathcal{X}_{[K,\infty]}(S(T-t))]_{y=S(t)} \tag{5.66}$$

where E_Q^y is the expectation when $S(0) = y$. Since

$$\begin{aligned} dS(t) &= \mu S(t)dt + \sigma S(t) dW(t) \\ &= (\mu - \sigma\theta) S(t) dt + \sigma S(t) d\widetilde{W}(t) \\ &= \rho S(t) dt + \sigma S(t) d\widetilde{W}(t), \end{aligned}$$

we have

$$S(t) = S(0) \exp((\rho - \frac{1}{2}\sigma^2)t + \sigma \widetilde{W}(t)) \tag{5.67}$$

and hence

$$\underline{\underline{\eta(t) = e^{\rho(t-T)} S^{-1}(t) E^y[Y(T-t)\mathcal{X}_{[K,\infty]}(Y(T-t))]_{y=S(t)}}}\,, \tag{5.68}$$

where

$$\underline{\underline{Y(t) = S(0)\exp((\rho - \tfrac{1}{2}\sigma^2)t + \sigma W(t))}}\,. \tag{5.69}$$

Since the distribution of $W(t)$ is well-known, we can express the solution (5.68) explicitly in terms of quantities involving $S(t)$ and the normal distribution function.

In this model $\eta(t)$ represents the number of units we must invest in the risky investment at times $t \leq T$ in order to be guaranteed to get the payoff $F(\omega) = (S(T,\omega) - K)^+$ (a.s.) at time T. The constant $V(0)$ represents the corresponding initial fortune needed to achieve this. Thus $V(0)$ is the (unique) initial fortune which makes it possible to establish a (self-financing) portfolio with the same payoff at time T as the option gives. Hence $V(0)$ deserves to be called *the right price* for such an option. By (5.56) this is given by

$$\begin{aligned} V(0) &= E_Q[e^{-\rho T} F(\omega)] = e^{-\rho T} E_Q[(S(T) - K)^+] \\ &= \underline{\underline{e^{-\rho T} E[(Y(T) - K)^+]}}\,, \end{aligned} \tag{5.70}$$

which again can be expressed explicitly by the normal distribution function.

Final remarks In the *Markovian* case, i.e. when the price $S(t)$ is given by a stochastic differential equation of the form

$$dS(t) = \mu(S(t))S(t)dt + \sigma(S(t))S(t)dW(t)$$

where $\mu\colon \mathbf{R} \to \mathbf{R}$ and $\sigma\colon \mathbf{R} \to \mathbf{R}$ are given functions, then there is a well-known alternative method for finding the option price $V(0)$ and the corresponding replicating portfolio $\eta(t)$: One assumes that the value process has the form

$$V(t,\omega) = f(t, S(t,\omega))$$

for some function $f\colon \mathbf{R}^2 \to \mathbf{R}$ and deduces a (deterministic) partial differential equation which determines f. Then η is given by

$$\eta(t,\omega) = \left[\frac{\partial f(t,x)}{\partial x}\right]_{x=S(t,\omega)}\,.$$

However, the method does not work in the non-markovian case. The method based on the Clark-Ocone formula has the advantage that it does not depend on a Markovian setup.

Exercises

5.1 Recall the *Girsanov theorem* (see e.g. [Ø1], Th. 8.26): Let $Y(t) \in \mathbf{R}^n$ be an Ito process of the form

$$dY(t) = \beta(t,\omega)dt + \gamma(t,\omega)dW(t); \qquad t \leq T \tag{5.71}$$

where $\beta(t,\omega) \in \mathbf{R}^n$, $\gamma(t,\omega) \in \mathbf{R}^{n\times m}$ are $\mathcal{F}_t$-adapted and $W(t)$ is m-dimensional. Suppose there exist $\mathcal{F}_t$-adapted processes $\theta(t,\omega) \in \mathbf{R}^m$ and $\alpha(t,\omega) \in \mathbf{R}^n$ such that

$$\gamma(t,\omega)\theta(t,\omega) = \beta(t,\omega) - \alpha(t,\omega) \tag{5.72}$$

and such that Novikov's condition

$$E[\exp(\tfrac{1}{2}\int_0^T \theta^2(s,\omega)ds)] < \infty \tag{5.73}$$

holds. Put

$$Z(t,\omega) = \exp(-\int_0^t \theta(s,\omega)dW(s) - \tfrac{1}{2}\int_0^t \theta^2(s,\omega)ds); \qquad t \le T \tag{5.74}$$

and define a measure Q on $\mathcal{F}_T$ by

$$dQ(\omega) = Z(T,\omega)dP(\omega) \qquad \text{on } \mathcal{F}_T \tag{5.75}$$

Then

$$\widetilde{W}(t,\omega) := \int_0^t \theta(s,\omega) + W(t,\omega); \qquad 0 \le t \le T \tag{5.76}$$

is a Wiener process w.r.t. Q, and in terms of $\widetilde{W}(t,\omega)$ the process $Y(t,\omega)$ has the stochastic integral representation

$$dY(t) = \alpha(t,\omega)dt + \gamma(t,\omega)d\widetilde{W}(t). \tag{5.77}$$

a) Show that $\widetilde{W}(t)$ is an $\mathcal{F}_t$-martingale w.r.t. Q.
(Hint: Apply Ito's formula to $Y(t) := Z(t)\widetilde{W}(t)$.)

b) Suppose $X(t) = at + W(t) \in \mathbf{R}$, $t \le T$, where $a \in \mathbf{R}$ is a constant. Find a probability measure Q on $\mathcal{F}_T$ such that $X(t)$ is a Wiener process w.r.t. Q.

c) Let $a, b, c \neq 0$ be real constants and define

$$dY(t) = bY(t)dt + cY(t)dW(t).$$

Find a probability measure Q and a Wiener process $\widetilde{W}(t)$ w.r.t. Q such that

$$dY(t) = aY(t)dt + cY(t)d\widetilde{W}(t).$$

5.2 Verify the Clark-Ocone formula

$$F(\omega) = E[F] + \int_0^T E[D_tF \mid \mathcal{F}_t]dW(t)$$

for the following $\mathcal{F}_T$-measurable random variables F

a) $F(\omega) = W(T)$

b) $F(\omega) = \int_0^T W(s)ds$

c) $F(\omega) = W^2(T)$

d) $F(\omega) = W^3(T)$

e) $F(\omega) = \exp W(T)$

f) $F(\omega) = (W(T) + T)\exp(-W(T) - \frac{1}{2}T)$

5.3 Let $\widetilde{W}(t) = \int_0^t \theta(s,\omega)ds + W(t)$ and Q be as in Exercise 5.1. Use the generalized Clark-Ocone formula to find the $\mathcal{F}_t$-adapted process $\widetilde{\varphi}(t,\omega)$ such that

$$F(\omega) = E_Q[F] + \int_0^T \widetilde{\varphi}(t,\omega)d\widetilde{W}(t)$$

in the following cases:

a) $F(\omega) = W^2(T)$, $\theta(s,\omega) = \theta(s)$ is deterministic

b) $F(\omega) = \exp(\int_0^T \lambda(s)dW(s))$, $\lambda(s)$ and $\theta(s)$ are deterministic.

c) $F(\omega)$ like in b), $\theta(s,\omega) = W(s)$.

5.4 Suppose we have the choice between the investments (5.40), (5.41). Find the initial fortune $V(0)$ and the number of units $\eta(t,\omega)$ which must be invested at time t in the risky investment in order to produce the terminal value $V(T,\omega) = F(\omega) = W(T,\omega)$ when $\rho(t,\omega) = \rho > 0$ (constant) and the price $S(t)$ of the risky investment is given by

a) $dS(t) = \mu S(t)dt + \sigma S(t)dW(t)$; μ, σ constants ($\sigma \neq 0$)

b) $dS(t) = c dW(t)$; $c \neq 0$ constant

c) $dS(t) = \mu S(t)dt + c dW(t)$; μ, c constants (the Ornstein-Uhlenbeck process).
Hint:

$$S(t) = e^{\mu t}S(s) + c\int_0^T e^{\mu(t-s)}dW(s).$$

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6 Solutions to the exercises

1.1. a)

$$\begin{aligned}\exp(tx-\frac{t^2}{2}) &= \exp(\frac{1}{2}x^2)\cdot\exp(-\frac{1}{2}(x-t)^2)\\ &= \exp(\frac{1}{2}x^2)\cdot\sum_{n=0}^{\infty}\{\frac{1}{n!}\frac{d^n}{dt^n}(\exp(-\frac{1}{2}(x-t)^2))_{t=0}t^n\}\\ (u=x-t) &= \exp(\frac{1}{2}x^2)\cdot\sum_{n=0}^{\infty}\{\frac{1}{n!}\frac{d^n}{du^n}(\exp(-\frac{1}{2}u^2))_{u=x}(-1)^n t^n\}\\ &= \exp(\frac{1}{2}x^2)\sum_{n=0}^{\infty}\{\frac{1}{n!}(-1)^n\frac{d^n}{dx^n}(\exp(-\frac{1}{2}x^2))t^n\}\\ &= \sum_{n=0}^{\infty}\frac{t^n}{n!}h_n(x).\end{aligned}$$

1.1. b) $u = t\sqrt{\lambda}$ gives

$$\exp(tx-\frac{t^2\lambda}{2}) = \exp(u\frac{x}{\sqrt{\lambda}}-\frac{u^2}{2})$$

$$\text{(by a))} = \sum_{n=0}^{\infty}\frac{u^n}{n!}h_n(\frac{x}{\sqrt{\lambda}}) = \sum_{n=0}^{\infty}\frac{t^n\lambda^{n/2}}{n!}h_n(\frac{x}{\sqrt{\lambda}}).$$

1.1. c) If we choose $x=\theta$, $\lambda=\|g\|^2$ and $t=1$ in b), we get

$$\exp(\int_0^T g\,dW-\frac{1}{2}\|g\|^2) = \sum_{n=0}^{\infty}\frac{\|g\|^n}{n!}h_n(\frac{\theta}{\|g\|}).$$

1.1. d) In particular, if we choose $g(s)=\mathcal{X}_{[0,t]}(s)$, we get

$$\exp(W(t)-\frac{1}{2}t) = \sum_{n=0}^{\infty}\frac{t^{n/2}}{n!}h_n(\frac{W(t)}{\sqrt{t}}).$$

1.2. a) $\varphi(\omega) = W(t,\omega) = \int_0^T \mathcal{X}_{[0,t]}(s)dW(s)$, so $\underline{\underline{f_0=0}}$, $\underline{\underline{f_1=\mathcal{X}_{[0,t]}}}$ and $\underline{\underline{f_n=0 \text{ for } n\geq 2}}$.

1.2. b) $\varphi(\omega) = \int_0^T g(s)dW(s) \Rightarrow \underline{\underline{f_0=0,\ f_1=g,\ f_n=0 \text{ for } n\geq 2}}$

1.2. c) Since

$$\int_0^t(\int_0^{t_2}1\,dW(t_1))dW(t_2) = \int_0^t W(t_2)dW(t_2) = \frac{1}{2}W^2(t)-\frac{1}{2}t\,,$$

we get that

$$W^2(t) = t + 2\int_0^t\int_0^{t_2} dW(t_1)dW(t_2)$$

$$= t + 2\int_0^T\int_0^{t_2} \mathcal{X}_{[0,t]}(t_1)\mathcal{X}_{[0,t]}(t_2)dW(t_1)dW(t_2) = t + I_2[f_2],$$

where

$$f_2(t_1,t_2) = \mathcal{X}_{[0,t]}(t_1)\mathcal{X}_{[0,t]}(t_2) = \underline{\underline{\mathcal{X}_{[0,t]}^{\otimes 2}}}.$$

So $\underline{\underline{f_0 = t}}$ and $\underline{\underline{f_n = 0 \text{ for } n \neq 2}}$.

1.2. d) By Exercise 1.1c) and (1.14) we have

$$\varphi(\omega) = \exp(\int_0^T g(s)dW(s))$$

$$= \exp(\frac{1}{2}\|g\|^2)\sum_{n=0}^{\infty}\frac{\|g\|^n}{n!}h_n(\frac{\theta}{\|g\|})$$

$$= \exp(\frac{1}{2}\|g\|^2)\sum_{n=0}^{\infty}J_n[g^{\otimes n}] = \sum_{n=0}^{\infty}\frac{1}{n!}\exp(\frac{1}{2}\|g\|^2)I_n[g^{\otimes n}].$$

Hence

$$f_n = \underline{\underline{\frac{1}{n!}\exp(\frac{1}{2}\|g\|^2)g^{\otimes n}}}\ ; \qquad n = 0,1,2,\ldots$$

where

$$g^{\otimes n}(x_1,\ldots,x_n) = g(x_1)g(x_2)\cdots g(x_n).$$

1.3. a) Since $\int_0^T W(t)dW(t) = \frac{1}{2}W^2(T) - \frac{1}{2}T$, we have

$$F(\omega) = W^2(T) = T + 2\int_0^T W(t)dW(t).$$

Hence

$$E[F] = \underline{\underline{T}} \quad \text{and} \quad \underline{\underline{\varphi(t,\omega) = 2W(t)}}.$$

1.3. b) Define $M(t) = \exp(W(t) - \frac{1}{2}t)$. Then by the Ito formula

$$dM(t) = M(t)dW(t)$$

and therefore

$$M(T) = 1 + \int_0^T M(t)dW(t)$$

or

$$F(\omega) = \exp W(T) = \exp \frac{1}{2}T + \exp(\frac{1}{2}T) \cdot \int_0^T \exp(W(t) - \frac{1}{2}t)dW(t).$$

Hence

$$E[F] = \underline{\underline{\exp(\frac{1}{2}T)}} \quad \text{and} \quad \underline{\underline{\varphi(t,\omega) = \exp(W(t) + \frac{1}{2}(T-t))}}.$$

1.3. c) Integration by parts gives

$$F(\omega) = \int_0^T W(t)dt = TW(T) - \int_0^T t dW(t) = \int_0^T (T-t)dW(t).$$

Hence $E[F] = \underline{\underline{0}}$ and $\varphi(t,\omega) = \underline{\underline{T-t}}$.

1.3. d) By the Ito formula

$$d(W^3(t)) = 3W^2(t)dW(t) + 3W(t)dt$$

Hence

$$F(\omega) = W^3(T) = 3\int_0^T W^2(t)dW(t) + 3\int_0^T W(t)dt$$

Therefore, by 1.3.c) we get

$$E[F] = \underline{\underline{0}} \quad \text{and} \quad \varphi(t,\omega) = \underline{\underline{3W^2(t) + 3T(1-t)}}.$$

1.3. e) Put $X(t) = e^{\frac{1}{2}t}$, $Y(t) = \cos W(t)$, $N(t) = X(t)Y(t)$. Then

$$\begin{aligned} dN(t) &= X(t)dY(t) + Y(t)dX(t) + dX(t)dY(t) \\ &= e^{\frac{1}{2}t}[-\sin W(t)dW(t) - \frac{1}{2}\cos W(t)dt] + \cos W(t) \cdot e^{\frac{1}{2}t} \cdot \frac{1}{2}dt \\ &= -e^{\frac{1}{2}t}\sin W(t)dW(t). \end{aligned}$$

Hence

$$e^{\frac{1}{2}T}\cos W(T) = 1 - \int_0^T e^{\frac{1}{2}t}\sin W(t)dW(t)$$

or

$$F(\omega) = \cos W(T) = e^{-\frac{1}{2}T} - e^{-\frac{1}{2}T}\int_0^T e^{\frac{1}{2}t}\sin W(t)dW(t).$$

Hence $E[F] = \underline{\underline{e^{-\frac{1}{2}T}}}$ and $\varphi(t,\omega) = \underline{\underline{-e^{\frac{1}{2}(t-T)}\sin W(t)}}$.

1.4. a) By Ito's formula and Kolmogorov's backward equation we have

$$\begin{aligned} dY(t) &= \frac{\partial g}{\partial t}(t,X(t))dt + \frac{\partial g}{\partial x}(t,X(t))dX(t) + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t,X(t))(dX(t))^2 \\ &= \frac{\partial}{\partial t}[P_{T-t}f(\xi)]_{\xi=X(t)}dt + \sigma(X(t))\frac{\partial}{\partial\xi}[P_{T-t}f(\xi)]_{\xi=X(t)}dW(t) \\ &\quad +\{b(X(t))\frac{\partial}{\partial\xi}[P_{T-t}f(\xi)]_{\xi=X(t)} + \frac{1}{2}\sigma^2(X(t))\frac{\partial^2}{\partial\xi^2}[P_{T-t}f(\xi)]_{\xi=X(t)}\}dt \\ &= \frac{\partial}{\partial t}[P_{T-t}f(\xi)]_{\xi=X(t)}dt + \sigma(X(t))\frac{\partial}{\partial\xi}[P_{T-t}f(\xi)]_{\xi=X(t)}dW(t) \\ &\quad +\frac{\partial}{\partial u}[P_u f(\xi)]_{\substack{\xi=X(t)\\ u=T-t}}dt \\ &= \sigma(X(t))\frac{\partial}{\partial\xi}[P_{T-t}f(\xi)]_{\xi=X(t)}dW(t). \end{aligned}$$

Hence

$$Y(T) = Y(0) + \int_0^T [\sigma(x)\frac{\partial}{\partial\xi}P_{T-t}f(\xi)]_{\xi=X(t)}dW(t).$$

Since $Y(T) = g(T,X(T)) = [P_0 f(\xi)]_{\xi=X(T)} = f(X(T))$ and $Y(0) = g(0,X(0)) = P_T f(X)$, (1.48) follows.

1.4. b) If $F(\omega) = W^2(T)$ we apply a) to the case when $f(\xi) = \xi^2$ and $X(t) = x + W(t)$ (assuming $W(0) = 0$ as before). This gives

$$P_s f(\xi) = E^{\xi}[f(X(x))] = E^{\xi}[X^2(s)] = \xi^2 + s$$

and hence

$$E[F] = P_T f(x) = \underline{\underline{x^2 + T}}$$

and

$$\varphi(t,\omega) = [\frac{\partial}{\partial\xi}(\xi^2 + s)]_{\xi=x+W(t)} = \underline{\underline{2W(t) + 2x}}.$$

1.4. c) If $F(\omega) = W^3(T)$ we choose $f(\xi) = \xi^3$ and $X(t) = x + W(t)$ and get

$$P_s f(\xi) = E^{\xi}[X^3(s)] = \xi^3 + 3s\xi\,.$$

Hence

$$E[F] = P_T f(x) = \underline{\underline{x^3 + 3Tx}}$$

and

$$\varphi(t,\omega) = [\frac{\partial}{\partial\xi}(\xi^3 + 3(T-t)\xi)]_{\xi=x+W(t)} = \underline{\underline{3(x+W(t))^2 + 3(T-t)}}$$

1.4. d) In this case $f(\xi) = \xi$ so

$$P_s f(\xi) = E^\xi[X(s)] = \xi \exp(\rho s)$$

so

$$E[F] = P_T f(x) = \underline{\underline{x \exp(\rho T)}}$$

and

$$\begin{aligned}\varphi(t,\omega) &= [\alpha\xi \frac{\partial}{\partial \xi}\{\xi \exp(\rho(T-t))\}]_{\xi=X(t)} \\ &= \alpha X(t)\exp(\rho(T-t)) = \underline{\underline{\alpha x \exp(\rho T - \frac{1}{2}\alpha^2 t + \alpha W(t))}}.\end{aligned}$$

1.4. e) We proceed as in a) and put

$$Y(t) = g(t, X(t)) \qquad \text{with} \quad g(t,x) = P_{T-t} f(x)$$

and

$$dX(t) = b(X(t))dt + \sigma(X(t))dW(t)\,; \qquad X_0 = x \in \mathbf{R}^n$$

where

$$b\colon \mathbf{R}^n \to \mathbf{R}^n, \quad \sigma\colon \mathbf{R}^n \to \mathbf{R}^{n\times m} \qquad \text{and} \quad W(t) = (W_1(t), \ldots, W_m(t))$$

is the m-dimensional Wiener process.

Then by Ito's formula and (1.50) we have

$$\begin{aligned}dY(t) &= \frac{\partial g}{\partial t}(t, X(t))dt + \sum_{i=1}^n \frac{\partial g}{\partial x_i}(t, X(t))dX_i(t) \\ &\quad + \frac{1}{2}\sum_{i,j} \frac{\partial^2 g}{\partial x_i \partial x_j}(t, X(t))dX_i(t)dX_j(t) \\ &= \frac{\partial}{\partial t}[P_{T-t}f(\xi)]_{\xi=X(t)}dt + [\sigma^T(\xi)\nabla_\xi(P_{T-t}f(\xi))]_{\xi=X(t)}dW(t) \\ &\quad + [L_\xi(P_{T-t}f(\xi))]_{\xi=X(t)}dt\end{aligned}$$

where

$$L_\xi = \sum_{i=1}^n b_i(\xi)\frac{\partial}{\partial \xi_i} + \frac{1}{2}\sum_{i,j=1}^n (\sigma\sigma^T)_{ij}(\xi)\frac{\partial^2}{\partial \xi_i \partial \xi_j}$$

is the generator of the Ito diffusion $X(t)$. So by the Kolmogorov backward equation we get

$$dY(t) = [\sigma^T(\xi)\nabla_\xi(P_{T-t}f(\xi)]_{\xi=X(t)}dW(t)$$

and hence, as in a),

$$Y(T) = f(X(T)) = P_T f(x) + \int_0^T [\sigma^T(\xi)\nabla_\xi(P_{T-t}f(\xi))]_{\xi=X(t)}dW(t),$$

which gives, with $F = f(X(T))$,

$$\underline{\underline{E[F] = P_T f(x)}} \qquad \text{and} \qquad \underline{\underline{\varphi(t,\omega) = [\sigma^T(\xi)\nabla_\xi(P_{T-t}f(\xi))]_{\xi=X(t)}}}.$$

2.1. a) Since $W(t)$ is $\mathcal{F}_t$-adapted, we have

$$\int_0^T W(t)\delta W(t) = \int_0^T W(t)dW(t) = \underline{\underline{\frac{1}{2}W^2(T) - \frac{1}{2}T}}.$$

2.1. b) $\int_0^T (\int_0^T gdW)\delta(W) = \int_0^T I_1[f_1(t_1,t)]\delta W(t_1) = I_2[\widetilde{f}_1]$ where

$$f_1(t_1,t) = g(t_1).$$

This gives

$$\widetilde{f}_1(t_1,t) = \frac{1}{2}(g(t_1) + g(t))$$

and hence

$$\begin{aligned} I_2[\widetilde{f}_1] &= 2\int_0^T \int_0^{t_2} \widetilde{f}_1(t_1,t_2)dW(t_1)dW(t_2) \\ &= \int_0^T \int_0^{t_2} g(t_1)dW(t_1)dW(t_2) + \int_0^T \int_0^{t_2} g(t_2)dW(t_1)dW(t_2). \end{aligned} \tag{6.1}$$

Using integration by parts (i.e. the Ito formula) we see that

$$\begin{aligned} &\int_0^T \int_0^{t_2} g(t_1)dW(t_1)dW(t_2) = \\ &= (\int_0^T gdW)W(T) - \int_0^T g(t)W(t)dW(t) - \int_0^T g(t)dt. \end{aligned} \tag{6.2}$$

Combining (6.1) and (6.2) we get

$$\int_0^T (\int_0^T gdW)\delta W = \underline{\underline{(\int_0^T gdW)\cdot W(T) - \int_0^T g(s)ds.}}$$

2.1. c) By Exercise 1.2. c) we have

$$\int_0^T W^2(t_0)\sigma W(t) = \int_0^T (t_0 + I_2[f_2])\delta W(t), \tag{6.3}$$

where

$$f_2(t_1,t_2,t) = \mathcal{X}_{[0,t_0]}(t_1)\mathcal{X}_{[0,t_0]}(t_2).$$

Now

$$\widetilde{f}_2(t_1,t_2,t) = \frac{1}{3}[f_2(t_1,t_2,t)+f_2(t,t_2,t_1)+f_2(t_1,t,t_2)]$$
$$= \frac{1}{3}[\mathcal{X}_{[0,t_0]}(t_1)\mathcal{X}_{[0,t_0]}(t_2)+\mathcal{X}_{[0,t_0]}(t)\mathcal{X}_{[0,t_0]}(t_2)+\mathcal{X}_{[0,t_0]}(t_1)\mathcal{X}_{[0,t_0]}(t)]$$
$$= \frac{1}{3}[\mathcal{X}_{\{t_1,t_2<t_0\}}+\mathcal{X}_{\{t,t_2<t_0\}}+\mathcal{X}_{\{t_1,t<t_0\}}]$$
$$= \mathcal{X}_{\{t,t_1,t_2<t_0\}}+\frac{1}{3}\mathcal{X}_{\{t_1,t_2<t_0<t\}}+\frac{1}{3}\mathcal{X}_{\{t,t_2<t_0<t_1\}}+\frac{1}{3}\mathcal{X}_{\{t,t_1<t_0<t_2\}} \tag{6.4}$$

and hence, using (1.14),

$$\int_0^T W^2(t_0)\delta W(t) = t_0W(T)+\int_0^T I_2[f_2]\delta W(t)$$
$$= t_0W(T)+I_3[\widetilde{f}_2]$$
$$= t_0W(T)+6J_3[\widetilde{f}_2]$$
$$= t_0W(T)+6\int_0^T\int_0^{t_3}\int_0^{t_2}\mathcal{X}^{\otimes 3}_{[0,t_0]}(t_1,t_2,t_3)dW(t_1)dW(t_2)dW(t_3)$$
$$+6\int_0^T\int_0^{t_3}\int_0^{t_2}\frac{1}{3}\mathcal{X}_{\{t_1,t_2<t_0<t_3\}}dW(t_1)dW(t_2)dW(t_3)$$
$$= t_0W(T)+t_0^{3/2}h_3\left(\frac{W(t_0)}{\sqrt{t_0}}\right)+2\int_{t_0}^T\int_0^{t_0}\int_0^{t_2}dW(t_1)dW(t_2)dW(t_3)$$
$$= t_0W(T)+t_0^{3/2}\left(\frac{W^3(t_0)}{t_0^{3/2}}-3\frac{W(t_0)}{\sqrt{t_0}}\right)+2\int_{t_0}^T\left(\frac{1}{2}W^2(t_0)-\frac{1}{2}t_0\right)dW(t_3)$$
$$= t_0W(T)+W^3(t_0)-3t_0W(t_0)+(W^2(t_0)-t_0)(W(T)-W(t_0))$$
$$= \underline{\underline{W^2(t_0)W(T)-2t_0W(t_0)}}.$$

2.1. d) By Exercise 1.2. d) and (1.14) we get

$$\int_0^T \exp(W(T))\delta W(t) = \int_0^T\left(\sum_{n=0}^\infty \frac{1}{n!}\exp(\frac{1}{2}T)I_n[1]\right)\delta W(t)$$
$$= \sum_{n=0}^\infty\frac{1}{n!}\exp(\frac{1}{2}T)I_{n+1}[1] = \underline{\underline{\exp(\frac{1}{2}T)\sum_{n=0}^\infty\frac{1}{n!}T^{\frac{n+1}{2}}h_{n+1}\left(\frac{W(T)}{\sqrt{T}}\right)}}$$

3.1. a) $\int_0^T W(T)\delta W(t) = \int_0^T W(T)\diamond \overset{\bullet}{W}(t)dt = W(T)\diamond\int_0^T \overset{\bullet}{W}(t)dt = W(T)\diamond W(T) =$ $\underline{\underline{W^2(T)-T}}$, by (3.58).

3.1. b) $\int_0^T (\int_0^T g dW) \diamond \overset{\bullet}{W}(t)dt = (\int_0^T g dW) \diamond \int_0^T \overset{\bullet}{W}(t)dt = (\int_0^T g dW) \diamond W(T)$

$= \underline{\underline{(\int_0^T g dW) W(T) - \int_0^T g(s)ds}},$ by (3.45).

3.1. c) $\int_0^T W^2(t_0)\delta W(t) = \int_0^T (W^{\diamond 2}(t_0) + t_0)\delta W(t)$

$$\begin{aligned}
&= W^{\diamond 2}(t_0) \diamond W(T) + t_0 W(T) \\
&= W^{\diamond 2}(t_0 \diamond (W(T) - W(t_0)) + W^{\diamond 2}(t_0) \diamond W(t_0) + t_0 W(T) \\
&= W^{\diamond 2}(t_0) \cdot (W(T) - W(t_0)) + W^{\diamond 3}(t_0) + t_0 W(T) \\
&= (W^2(t_0) - t_0) \cdot (W(T) - W(t_0)) + W^3(t_0) - 3t_0 W(t_0) + t_0 W(T) \\
&= \underline{\underline{W^2(t_0)W(T) - 2t_0 W(t_0)}},
\end{aligned}$$

where we have used (3.55) and (3.58).

3.1. d) $\int_0^T \exp(W(T))\delta W(t) = \exp(W(T)) \diamond \int_0^T \overset{\bullet}{W}(t)dt = \exp(W(T)) \diamond W(T) =$

$\exp^{\diamond}(W(T) + \frac{1}{2}T) \diamond W(T) = \exp(\frac{1}{2}T) \sum_{n=0}^{\infty} \frac{1}{n!} W(T)^{\diamond(n+1)} = \exp(\frac{1}{2}T) \sum_{n=0}^{\infty} \frac{T^{\frac{n+1}{2}}}{n!} h_{n+1}\left(\frac{W(T)}{\sqrt{T}}\right).$

4.1. a) $D_t W(T) = \mathcal{X}_{[0,T]}(t) = \underline{\underline{1}}$ (for $t \in [0,T]$), by (4.16).

4.1. b) By the chain rule (Lemma 4.9) we get

$$D_t(\exp W(t_0)) = \underline{\underline{\exp W(t_0) \cdot \mathcal{X}_{[0,t_0]}(t)}}.$$

4.1. c) By (4.15) we get

$$D_t(\int_0^T s^2 dW(s)) = \underline{\underline{t^2}}.$$

4.1. d) By Theorem 4.16 we have

$$D_t(\int_0^T (\int_0^{t_2} \cos(t_1 + t_2)dW(t_1))dW(t_2)) = D_t(\frac{1}{2} I_2[\cos(t_1 + t_2)])$$

$$= \frac{1}{2} \cdot 2 \cdot I_1[\cos(\cdot + t)] = \underline{\underline{\int_0^T \cos(t_1 + t)dW(t_1)}}.$$

4.1. e) $D_t(3W(s_0)W^2(t_0)+\ln(1+W^2(s_0)))$

$$= \underline{\underline{[3W^2(t_0)+\frac{2W(s_0)}{1+W^2(s_0)}]\cdot\mathcal{X}_{[0,s_0]}(t)}}$$

$$\underline{\underline{+6W(s_0)W(t_0)\mathcal{X}_{[0,t_0]}(t)}}$$

4.1. f) By Exercise 2.1. b) we have

$$D_t(\int_0^T W(t_0)\delta W(t)) = D_t(W(t_0)W(T)-t_0)$$
$$= W(t_0)\cdot\mathcal{X}_{[0,T]}(t)+W(T)\mathcal{X}_{[0,t_0]}(t) = \underline{\underline{W(t_0)+W(T)\mathcal{X}_{[0,t_0]}(t)}}.$$

4.2. a)

$$D_t(\exp(\int_0^T g(s)dW(s))) = D_t(\sum_{n=0}^{\infty} I_n[f_n])$$
$$= \sum_{n=1}^{\infty} nI_{n-1}[f_n(\cdot,t)]$$
$$= \sum_{n=1}^{\infty} n\cdot\frac{1}{n!}\exp(\frac{1}{2}\|g\|^2)I_{n-1}[g(t_1)\dots g(t_{n-1})g(t)]$$
$$= g(t)\sum_{n=1}^{\infty}\frac{1}{(n-1)!}\exp(\frac{1}{2}\|g\|^2)I_{n-1}[g^{\otimes(n-1)}]$$
$$= \underline{\underline{g(t)\exp(\int_0^T g(s)dW(s))}},$$

where we have used Theorem 4.16.

4.2. b) The chain rule gives

$$D_t(\exp(\int_0^T g(s)dW(s))) = \exp(\int_0^T g(s)dW(s))D_t(\int_0^T g(s)dW(t))$$
$$= \underline{\underline{g(t)\exp(\int_0^T g(s)dW(s))}} \qquad \text{by (4.15).}$$

4.3 With the given F and φ the left hand side of (4.28) becomes

$$E[D_\gamma F\cdot\varphi] = E[D_\gamma F] = E[\int_0^T D_tF\cdot g(t)dt] = \int_0^T \psi(t)g(t)dt,$$

while the right hand side becomes

$$E[F\cdot\varphi\cdot\int_0^T gdW]-E[F\cdot D_\gamma\varphi]$$
$$=E[(\int_0^T \psi dW)\cdot(\int_0^T gdW)],$$

which is the same as the left hand side by the Ito isometry.

5.1. a) If $s>t$ we have

$$\begin{aligned}E_Q[\widetilde{W}(s)|\mathcal{F}_t]&=\frac{E[Z(T)\widetilde{W}(s)|\mathcal{F}_t]}{E[Z(T)|\mathcal{F}_t]}\\&=\frac{E[Z(T)\widetilde{W}(s)|\mathcal{F}_t]}{Z(t)}=Z^{-1}(t)E[E[Z(T)\widetilde{W}(s)|\mathcal{F}_s]|\mathcal{F}_t]\\&=Z^{-1}(t)E[\widetilde{W}(s)E[Z(T)|\mathcal{F}_s]|\mathcal{F}_t]\\&=Z^{-1}(t)E[\widetilde{W}(s)Z(s)|\mathcal{F}_t].\end{aligned}\tag{6.5}$$

Applying Ito's formula to $Y(t):=Z(t)\widetilde{W}(t)$ we get

$$\begin{aligned}dY(t)&=Z(t)d\widetilde{W}(t)+\widetilde{W}(t)dZ(t)+d\widetilde{W}(t)dZ(t)\\&=Z(t)[\theta(t)dt+dW(t)]+\widetilde{W}(t)[-\theta(t)Z(t)dW(t)]-\theta(t)Z(t)dt\\&=Z(t)[1-\theta(t)\widetilde{W}(t)]dW(t),\end{aligned}$$

and hence $Y(t)$ is an $\mathcal{F}_t$-martingale (w.r.t. P). Therefore, by (6.5),

$$E_Q[\widetilde{W}(s)|\mathcal{F}_t]=Z^{-1}(t)E[Y(s)|\mathcal{F}_t]=Z^{-1}(t)Y(t)=\widetilde{W}(t).$$

□

5.1. b) We apply the Girsanov theorem to the case with $\theta(t,\omega)=a$. Then $X(t)$ is a Wiener process w.r.t. the measure Q defined by

$$dQ(\omega)=Z(T,\omega)dP(\omega)\qquad\text{on } \mathcal{F}_T,$$

where

$$Z(t)=\exp(-aW(t)-\frac{1}{2}a^2t)\,;\qquad 0\le t\le T.$$

5.1. c) In this case we have

$$\beta(t,\omega)=bY(t,\omega),\quad \alpha(t,\omega)=aY(t,\omega),\quad \gamma(t,\omega)=cY(t,\omega)$$

and hence we put

$$\theta=\frac{\beta(t,\omega)-\alpha(t,\omega)}{\gamma(t,\omega)}=\frac{b-a}{c}$$

and

$$Z(t) = \exp(-\theta W(t) - \frac{1}{2}\theta^2 t); \qquad 0 \le t \le T.$$

Then

$$\widetilde{W}(t) := \theta t + W(t)$$

is a Wiener process w.r.t. the measure Q defined by $dQ(\omega) = Z(T,\omega)dP(\omega)$ on $\mathcal{F}_T$ and

$$dY(t) = bY(t)dt + cY(t)[d\widetilde{W}(t) - \theta dt] = aY(t)dt + cY(t)d\widetilde{W}(t).$$

□

5.2. a) $F(\omega) = W(T) \Rightarrow D_t F(\omega) = \mathcal{X}_{[0,T]}(t) = 1$ for $t \in [0,T]$ and hence

$$E[F] + \int_0^T E[D_t F|\mathcal{F}_t]dW(t) = \int_0^T 1 \cdot dW(t) = W(T) = F.$$

□

5.2. b) $F(\omega) = \int_0^T W(s)ds \Rightarrow D_t F(\omega) = \int_0^T D_t W(s)ds = \int_0^T \mathcal{X}_{[0,s]}(t)ds = \int_t^T ds = T - t,$

which gives

$$E[F] + \int_0^T E[D_t F|\mathcal{F}_t]dW(t) = \int_0^T (T-t)dW(t)$$
$$= \int_0^T W(s)dW(s) = F,$$

using integration by parts. □

5.2. c) $F(\omega) = W^2(T) \Rightarrow D_t F(\omega) = 2W(T) \cdot D_t W(T) = 2W(T)$. Hence

$$E[F] + \int_0^T E[D_t F|\mathcal{F}_t]dW(t) = T + \int_0^T E[2W(T)|\mathcal{F}_t]dW(t)$$
$$= T + 2\int_0^T W(t)dW(t) = T + W^2(T) - T = W^2(T) = F.$$

□

5.2. d) $F(\omega) = W^3(T) \Rightarrow D_t F(\omega) = 2W^2(T)$. Hence

$$E[F] + \int_0^T E[D_t F|\mathcal{F}_t]dW(t) = \int_0^T E[3W^2(T)|\mathcal{F}_t]dW(t)$$

$$= 3\int_9^T E[(W(T) - W(t))^2 + 2W(t)W(T) - W^2(t)|\mathcal{F}_t]dW(t)$$

$$= 3\int_0^T (T-t)dW(t) + 6\int_0^T W^2(t)dW(t) - 3\int_0^T W^2(t)dW(t)$$

$$= 3\int_0^T W^2(t)dW(t) - 3\int_0^T W(t)dt = W^3(T),$$

by Ito's formula. □

5.2. e) $F(\omega) = \exp W(T) \Rightarrow D_tF(\omega) = \exp W(T)$. Hence

$$\begin{aligned} RHS &= E[F] + \int_0^T E[D_tF|\mathcal{F}_t]dW(t) \\ &= \exp(\frac{1}{2}T) + \int_0^T E[\exp W(T)|\mathcal{F}_t]dW(t) \\ &= \exp(\frac{1}{2}T) + \int_0^T E[\exp(W(T) - \frac{1}{2}T) \cdot \exp(\frac{1}{2}T)|\mathcal{F}_t]dW(t) \\ &= \exp(\frac{1}{2}T) + \exp(\frac{1}{2}T)\int_0^T \exp(W(t) - \frac{1}{2}t)dW(t). \qquad (6.6) \end{aligned}$$

Here we have used that

$$M(t) := \exp(W(t) - \frac{1}{2}t)$$

is a martingale. In fact, by Ito's formula we have $dM(t) = M(t)dW(t)$. Combined with (6.6) this gives

$$RHS = \exp(\frac{1}{2}T) + \exp(\frac{1}{2}T)(M(T) - M(0)) = \exp W(T) = F\,.$$

□

5.2. f) $F(\omega) = (W(T) + T)\exp(-W(T) - \frac{1}{2}T)$
$\Rightarrow D_tF = \exp(-W(T) - \frac{1}{2}T)[1 - W(T) - T]$. Note that

$$Y(t) := (W(t) + t)N(t), \quad \text{with} \quad N(t) = \exp(-W(t) - \frac{1}{2}t)$$

is a martingale, since

$$dY(t) = (W(t)+t)N(t)(-dW(t)) + N(t)(dW(t)+dt) - N(t)dt = N(t)[1-t-W(t)]dW(t)$$

Therefore

$$
\begin{aligned}
&E[F] + \int_0^T E[D_t F|\mathcal{F}_t]dW(t) \\
&= \int_0^T E[N(T)(1-(W(T)+T))|\mathcal{F}_t]dW(t) \\
&= \int_0^T N(t)(1-(W(t)+t))dW(t) = \int_0^T dY(t) = Y(T) - Y(0) \\
&= (W(T)+T)\exp(-W(T) - \tfrac{1}{2}T) = F.
\end{aligned}
$$

□

5.3. a) $\widetilde{\varphi}(t,\omega) = E_Q[D_t F - F \int_t^T D_t\theta(s,\omega)d\widetilde{W}(s)|\mathcal{F}_t]$

If $\theta(s,\omega) = \theta(s)$ is deterministic, then $D_t\theta = 0$ and hence

$$
\begin{aligned}
\widetilde{\varphi}(t,\omega) &= E_Q[D_t F|\mathcal{F}_t] = E_Q[2W(T)|\mathcal{F}_t] \\
&= E_Q[2\widetilde{W}(T) - 2\int_0^T \theta(s)ds|\mathcal{F}_t] = 2\widetilde{W}(t) - 2\int_0^T \theta(s)ds = \underline{\underline{2W(t) - 2\int_t^T \theta(s)ds}}.
\end{aligned}
$$

5.3. b) $\widetilde{\varphi}(t,\omega) = E_Q[D_t F|\mathcal{F}_t]$

$$
\begin{aligned}
&= E_Q[\exp(\int_0^T \lambda(s)dW(s))\lambda(t)|\mathcal{F}_t] \\
&= \lambda(t)E_Q[\exp(\int_0^T \lambda(s)d\widetilde{W}(s) - \int_0^T \lambda(s)\theta(s)ds)|\mathcal{F}_t] \\
&= \lambda(t)\exp(\int_0^T (\tfrac{1}{2}\lambda^2(s) - \lambda(s)\theta(s))ds)E_Q[\exp(\int_0^T \lambda(s)d\widetilde{W}(s) - \tfrac{1}{2}\int_0^T \lambda^2(s)ds)|\mathcal{F}_t] \\
&= \lambda(t)\exp(\int_0^T \lambda(s)(\tfrac{1}{2}\lambda(s) - \theta(s))ds)\exp(\int_0^t \lambda(s)d\widetilde{W}(s) - \tfrac{1}{2}\int_0^t \lambda^2(s)ds) \\
&= \underline{\underline{\lambda(t)\exp(\int_0^t \lambda(s)dW(s) + \int_t^T \lambda(s)(\tfrac{1}{2}\lambda(s) - \theta(s))ds).}}
\end{aligned}
$$

5.3. c)

$$
\begin{aligned}
\widetilde{\varphi}(t,\omega) &= E_Q[D_t F - F\int_t^T D_t\theta(s,\omega)d\widetilde{W}(s)|\mathcal{F}_t] \\
&= E_Q[\lambda(t)F|\mathcal{F}_t] - E_Q[F\int_t^T d\widetilde{W}(s)|\mathcal{F}_t]
\end{aligned}
$$

$$(6.7) \qquad = A - B\,, \qquad \text{say.}$$

Now $\widetilde{W}(t) = W(t) + \int_0^t \theta(s,\omega)ds = W(t) + \int_0^t W(s)ds$ or

$$dW(t) + W(t)dt = d\widetilde{W}(t).$$

We solve this equation for $W(t)$ by multiplying by the "integrating factor" e^t and get

$$d(e^t W(t)) = e^t d\widetilde{W}(t)$$

Hence

$$W(u) = e^{-u}\int_0^u e^s d\widetilde{W}(s). \tag{6.8}$$

or

$$dW(u) = -e^{-u}\int_0^u e^s d\widetilde{W}(s)du + d\widetilde{W}(u) \tag{6.9}$$

Using (6.9) we may rewrite $F(\omega)$ as follows:

$$\begin{aligned}
F(\omega) &= \exp(\int_0^T \lambda(s)dW(s)) \\
&= \exp(\int_0^T \lambda(s)d\widetilde{W}(s) - \int_0^T \lambda(u)e^{-u}(\int_0^u e^s d\widetilde{W}(s))du) \\
&= \exp(\int_0^T \lambda(s)d\widetilde{W}(s) - \int_0^T (\int_0^T \lambda(u)e^{-u}du)e^s d\widetilde{W}(s)) \\
&= K(T)\cdot\exp(\frac{1}{2}\int_0^T \xi^2(s)ds),
\end{aligned}$$

where

$$\xi(s) = \lambda(s) - e^s\int_s^T \lambda(u)e^{-u}du \tag{6.10}$$

and

$$K(t) = \exp(\int_0^t \xi(s)d\widetilde{W}(s) - \frac{1}{2}\int_0^t \xi^2(s)ds); \qquad 0 \le t \le T. \tag{6.11}$$

Hence

$$\begin{aligned}
A &= E_Q[\lambda(t)F|\mathcal{F}_t] = \lambda(t)\exp(\frac{1}{2}\int_0^T \xi^2(s)ds)E[K(T)|\mathcal{F}_t] \\
&= \lambda(t)\exp(\frac{1}{2}\int_0^T \xi^2(s)ds)K(t). \qquad (6.12)
\end{aligned}$$

Moreover, if we put

$$H = \exp(\frac{1}{2}\int_0^T \xi^2(s)ds), \tag{6.13}$$

we get

$$\begin{aligned} B &= E_Q[F(\widetilde{W}(T) - \widetilde{W}(t))|\mathcal{F}_t] \\ &= H \cdot E_Q[K(T)(\widetilde{W}(T) - \widetilde{W}(t))|\mathcal{F}_t] \\ &= H \cdot E_Q[K(t) \cdot \exp(\int_t^T \xi(s)d\widetilde{W}(s) - \frac{1}{2}\int_t^T \xi^2(s)ds)(\widetilde{W}(T) - \widetilde{W}(t))|\mathcal{F}_t] \\ &= H \cdot K(t)E_Q[\exp(\int_t^T \xi(s)d\widetilde{W}(s) - \frac{1}{2}\int_t^T \xi^2(s)ds)(\widetilde{W}(T) - \widetilde{W}(t))] \\ &= H \cdot K(t)E[\exp(\int_t^T \xi(s)dW(s) - \frac{1}{2}\int_t^T \xi^2(s)ds)(W(T) - W(t))]. \end{aligned} \tag{6.14}$$

This last expectation can be evaluated by using Ito's formula: Put

$$X(t) = \exp(\int_{t_0}^t \xi(s)dW(s) - \frac{1}{2}\int_{t_0}^t \xi^2(s)ds)$$

and

$$Y(t) = X(t) \cdot (W(t) - W(t_0)).$$

Then

$$\begin{aligned} dY(t) &= X(t)dW(t) + (W(t) - W(t_0))dX(t) + dX(t)dW(t) \\ &= X(t)[1 + (W(t) - W(t_0))\xi(t)]dW(t) + \xi(t)X(t)dt \end{aligned}$$

and hence

$$\begin{aligned} E[Y(T)] &= E[Y(t_0)] + E[\int_{t_0}^T \xi(s)X(s)ds] \\ &= \int_{t_0}^T \xi(s)E[X(s)]ds = \int_{t_0}^T \xi(s)ds \end{aligned} \tag{6.15}$$

Combining (6.7), (6.10)–(6.15) we conclude that

$$\begin{aligned} \widetilde{\varphi}(t,\omega) &= \lambda(t)HK(t) - H\,K(t)\int_t^T \xi(s)ds \\ &= \underline{\underline{\exp(\frac{1}{2}\int_0^T \xi^2(s)ds) \cdot \exp(\int_0^t \xi(s)d\widetilde{W}(s) - \frac{1}{2}\int_0^t \xi^2(s)ds)[\lambda(t) - \int_t^T \xi(s)ds]}} \end{aligned}$$

5.4. a) Since $\theta = \frac{\mu-\rho}{\sigma}$ is constant we get by (5.57)

$$\eta(t) = e^{\rho t}\sigma^{-1}S^{-1}(t)E_Q[e^{-\rho T}D_t W(T)|\mathcal{F}_t] = \underline{\underline{e^{\rho(t-T)}\sigma^{-1}S^{-1}(t)}}.$$

5.4. b) Here $\mu = 0$, $\sigma(s,\omega) = c\,S^{-1}(s)$ and hence

$$\theta(s,\omega) = \frac{\mu-\rho}{\sigma} = -\frac{\rho}{c}S(s) = -\rho(W(s) + S(0)).$$

Hence

$$\int_t^T D_t\theta(s,\omega)d\widetilde{W}(s) = \rho[\widetilde{W}(t) - \widetilde{W}(T)].$$

Therefore

$$B := E_Q[F\int_t^T D_t\theta(s)d\widetilde{W}(s)|\mathcal{F}_t] = \rho E_Q[e^{-\rho T}W(T)(\widetilde{W}(t) - \widetilde{W}(T))|\mathcal{F}_t]. \tag{6.16}$$

To proceed further, we need to express $W(t)$ in terms of $\widetilde{W}(t)$: Since

$$\widetilde{W}(t) = W(t) + \int_0^t \theta(s,\omega)ds = W(t) - \rho S(0)t - \rho\int_0^t W(s)ds$$

we have

$$d\widetilde{W}(t) = dW(t) - \rho W(t)dt - \rho S(0)dt$$

or

$$e^{-\rho t}dW(t) - e^{-\rho t}\rho W(t)dt = e^{-\rho t}(d\widetilde{W}(t) + \rho S(0)dt)$$

or

$$d(e^{-\rho t}W(t)) = e^{-\rho t}d\widetilde{W}(t) + \rho e^{-\rho t}S(0)dt$$

Hence

$$W(t) = S(0)[e^{\rho t} - 1] + e^{\rho t}\int_0^t e^{-\rho s}d\widetilde{W}(s). \tag{6.17}$$

Substituting this into (6.16) we get

$$\begin{aligned}
B &= \rho E_Q[\int_0^T e^{-\rho s}d\widetilde{W}(s)\cdot(\widetilde{W}(t) - \widetilde{W}(T))|\mathcal{F}_t] \\
&= \rho E_Q[\int_0^t e^{-\rho s}d\widetilde{W}(s)\cdot(\widetilde{W}(t) - \widetilde{W}(T))|\mathcal{F}_t] \\
&\quad +\rho E_Q[\int_t^T e^{-\rho s}d\widetilde{W}(s)\cdot(\widetilde{W}(t) - \widetilde{W}(T))|\mathcal{F}_t] \\
&= \rho E_Q[\int_t^T e^{-\rho s}d\widetilde{W}(s)\cdot(\widetilde{W}(t) - \widetilde{W}(T))] \\
&= \rho\int_t^T e^{-\rho s}(-1)ds = e^{-\rho T} - e^{-\rho t}.
\end{aligned}$$

Hence

$$\begin{aligned}\eta(t) &= e^{\rho t}c^{-1}(E_Q[D_t(e^{-\rho T}W(T))|\mathcal{F}_t] - B)\\ &= e^{\rho t}c^{-1}(e^{-\rho T} - e^{-\rho T} + e^{-\rho t}) = \underline{\underline{c^{-1}}} \qquad \text{(as expected).}\end{aligned}$$

5.4. c) Here $\sigma = c\,S^{-1}(t)$ and hence

$$\theta(s,\omega) = \frac{\mu-\rho}{c}S(s) = \frac{\mu-\rho}{c}[e^{\mu s}S(0) + c\int_0^s e^{\mu(s-r)}dW(r)]$$

So

$$D_t\theta(s,\omega) = (\mu-\rho)e^{\mu(s-t)}\mathcal{X}_{[0,s]}(t)$$

Hence

$$\begin{aligned}\eta(t,\omega) &= e^{\rho t}c^{-1}E_Q[(D_t(e^{-\rho T}W(T)) - e^{-\rho T}W(T)\int_t^T D_t\theta(s,\omega)d\widetilde{W}(s)|\mathcal{F}_t]\\ &= e^{\rho(t-T)}c^{-1}(1 - (\mu-\rho)E_Q[W(T)\int_t^T e^{\mu(s-t)}d\widetilde{W}(s)|\mathcal{F}_t]). \qquad (6.18)\end{aligned}$$

Again we try to express $W(t)$ in terms of $\widetilde{W}(t)$: Since

$$\begin{aligned}d\widetilde{W}(t) &= dW(t) + \theta(t,\omega)dt\\ &= dW(t) + \frac{\mu-\rho}{c}[e^{\mu t}S(0) + c\int_0^t e^{\mu(t-r)}dW(r)]dt\end{aligned}$$

we have

$$e^{-\mu t}d\widetilde{W}(t) = e^{-\mu t}dW(t) + [\frac{\mu-\rho}{c}S(0) + (\mu-\rho)\int_0^t e^{-\mu r}dW(r)]dt \qquad (6.19)$$

If we put

$$X(t) = \int_0^t e^{-\mu r}dW(r), \qquad \widetilde{X}(t) = \int_0^t e^{-\mu r}d\widetilde{W}(r),$$

(6.19) can be written

$$d\widetilde{X}(t) = dX(t) + \frac{\mu-\rho}{c}S(0)dt + (\mu-\rho)X(t)dt$$

or

$$d(e^{(\mu-\rho)t}X(t)) = e^{(\mu-\rho)t}d\widetilde{X}(t) + \frac{\mu-\rho}{c}S(0)e^{[\mu-\rho)t}dt$$

or

$$\begin{aligned}X(t) &= e^{(\rho-\mu)t}\int_0^t e^{-\rho s}d\widetilde{W}(s) + \frac{\mu-\rho}{c}S(0)e^{(\rho-\mu)t}\int_0^t e^{(\mu-\rho)s}ds\\ &= e^{(\rho-\mu)t}\int_0^t e^{-\rho s}d\widetilde{W}(s) + \frac{S(0)}{c}[1 - e^{(\rho-\mu)t}]\end{aligned}$$

From this we get

$$
\begin{aligned}
e^{-\mu t}dW(t) &= e^{(\rho-\mu)t}e^{-\rho t}d\widetilde{W}(t) + (\rho-\mu)e^{(\rho-\mu)t}(\int_0^t e^{-\rho s}d\widetilde{W}(s))dt \\
&\quad -\frac{S(0)}{c}(\rho-\mu)e^{(\rho-\mu)t}dt
\end{aligned}
$$

or

$$
dW(t) = d\widetilde{W}(t) + (\rho-\mu)e^{\rho t}(\int_0^t e^{-\rho s}d\widetilde{W}(s))dt - \frac{S(0)}{c}(\rho-\mu)e^{\rho t}dt
$$

In particular,

$$
W(T) = \widetilde{W}(T) + (\rho-\mu)\int_0^T e^{\rho s}(\int_0^s e^{-\rho r}d\widetilde{W}(r))ds - \frac{S(0)}{\rho c}(\rho-\mu)(e^{\rho T}-1) \tag{6.20}
$$

Substituted in (6.18) this gives

$$
\begin{aligned}
\eta(t,\omega) &= e^{\rho(t-T)}c^{-1}\{1-(\mu-\rho)E_Q[\widetilde{W}(T)\cdot\int_t^T e^{\mu(s-t)}d\widetilde{W}(s)|\mathcal{F}_t] \\
&\quad +(\mu-\rho)^2E_Q[\int_0^T e^{\rho s}(\int_0^s e^{-\rho r}d\widetilde{W}(r))ds\cdot\int_t^T e^{\mu(s-t)}d\widetilde{W}(s)|\mathcal{F}_t]\} \\
&= e^{\rho(t-T)}c^{-1}\{1-(\mu-\rho)\int_t^T e^{\mu(s-t)}ds \\
&\quad +(\mu-\rho)^2\int_t^T e^{\rho s}E_Q[(\int_t^s e^{-\rho r}d\widetilde{W}(r))\cdot(\int_t^T e^{\mu(r-t)}d\widetilde{W}(r))|\mathcal{F}_t]ds\} \\
&= e^{\rho(t-T)}c^{-1}\{1-\frac{\mu-\rho}{\mu}(e^{\mu(T-t)}-1)+(\mu-\rho)^2\int_t^T e^{\rho r}(\int_t^s e^{-\rho r}\cdot e^{\mu(r-t)}dr)ds\} \\
&= \underline{\underline{e^{\rho(t-T)}c^{-1}\{1-\frac{\mu-\rho}{\rho}(e^{\rho(T-t)}-1)\}}}.
\end{aligned}
$$