PART OF THE SOLUTIONS TO EXERCISES IN ∅KSENDAL'S BOOK

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Exercise. 2.1

Proof. **(a)** $(\Rightarrow)$: Assume X is a random variable, i.e., a measurable function from $(\Omega, \mathcal{F})$ to $(\mathbb{R}, \mathcal{B})$. Since $\{a_k\} \in \mathcal{B}$ for all $k = 1, 2, ...,$ $X^{-1}(a_k) \in \mathcal{F}$ by definition of measurable function. $(\Leftarrow)$: Now assume $X^{-1}(a_k) \in \mathcal{F}$ for all $k = 1, 2, ...$ Then$\forall A \in \mathcal{B}$, we can show that

$$X^{-1}(A) = X^{-1}(A \cap \bigcup_{k=1}^{\infty} \{a_k\}) = \bigcup_{k=1}^{\infty} X^{-1}(A \cap \{a_k\}) \in \mathcal{F}$$

for $A \cap \{a_k\}$ equals either $\{a_k\}$ or $\emptyset$. By definition, X is a random variable.

(b) Since $\Omega = \bigcup_{k=1}^{\infty} \{X = a_k\}$ is a partition, then define nonnegative piecewise simple function as follows

$$\varphi_n(\omega) = \begin{cases} \sum_{k=1}^{n} |a_k| \, 1_{\{X=a_k\}} & \omega \in \bigcup_{k=1}^{n} \{X = a_k\} \\ 0 & \text{others} \end{cases}$$

See that $\varphi_n = |X|$ on $\bigcup_{k=1}^{n} \{X = a_k\}$ and obviously $\varphi_n \nearrow |X|$ as $n \to \infty$. Therefore, by the property of integral of nonnegative measurable function, as $E[|X|] = \int_\Omega |X| \, dP$,

$$E[|X|] = \lim_{n\to\infty} \int_\Omega \varphi_n dP = \lim_{n\to\infty} \sum_{k=1}^{n} |a_k| \int_{\{X=a_k\}} dP = \sum_{k=1}^{\infty} |a_k| \, P(X = a_k)$$

(c) When $E[|X|] < \infty$, $|X|$ is integrable and then $E[X] = \int_\Omega X dP \leq E[|X|] < \infty$. Similarly with (b) let's define

$$\varphi_n(\omega) = \begin{cases} \sum_{k=1}^{n} a_k 1_{\{X=a_k\}} & \omega \in \bigcup_{k=1}^{n} \{X = a_k\} \\ 0 & \text{others} \end{cases}$$

Clearly $\varphi_n \to X$ as $n \to \infty$. For $\varphi_n \leq |X|$and $|X|$ is integrable, by *Lebesgue Dominated Convergence Theorem*,

$$E[X] = \int_\Omega X dP = \lim_{n\to\infty} \int_\Omega \varphi_n dP = \sum_{k=1}^{\infty} a_k P(X = a_k)$$

(d) Since f is bounded and measurable, $\exists M > 0$, *s.t.* $\int_\Omega f(X)dP \leq M < \infty$. Similarly with (c), let's define

$$\varphi_n(\omega) = \begin{cases} \sum_{k=1}^n f(a_k)1_{\{X=a_k\}} & \omega \in \bigcup_{k=1}^n \{X = a_k\} \\ 0 & \text{others} \end{cases}$$

Then $\varphi_n \to f(X)$ as $n \to \infty$. Therefore, since $\varphi_n \leq M$ which is integrable, also by *Lebesgue Dominated Convergence Theorem* we show that

$$E[f(X)] = \int_\Omega f(X)dP = \lim_{n\to\infty} \int_\Omega \varphi_n dP = \sum_{k=1}^\infty f(a_k)P(X = a_k)$$

and ends the proof. □

Exercise. 2.2

Proof. **(a)** (i) By definition of probability measure, $0 \leq F(x) = P(X \leq x) \leq P(\Omega) = 1$. Secondly, since we know that $P(\emptyset) = 0$ then let $x_n \searrow -\infty$, then $\{X \leq x_n\} \searrow \emptyset$. By upper continuity of probability measure,

$$\lim_{n\to\infty} P(X \leq x_n) = P(\bigcap_{n=1}^\infty \{X \leq x_n\}) = 0$$

Hence $\forall \varepsilon > 0$, $\exists M > 0$, $N > 0$, *s.t.*$\forall x < -M < x_N$,

$$F(x) = P(X \leq x) \leq P(X \leq x_N) < \varepsilon$$

By definition, $\lim\limits_{x\to-\infty} F(x) = 0$. At last, almost completely the same, see that $\{X \geq x_n\} \searrow \emptyset$ as $x_n \nearrow \infty$, then in the same way $\lim_{x\to\infty} F(x) = 1$.

(ii) Clearly $F(x_1) = P(X \leq x_1) \leq P(X \leq x_2) = F(x_2)$, as $x_1 \leq x_2$, $x_1, x_2 \in \mathbb{R}$ and $\{X \leq x_1\} \subset \{X \leq x_2\}$.

(iii) For $x \in \mathbb{R}$, $F(x + h) - F(x) = P(x < X \leq x + h)$, where $h > 0$. Let $h_n \searrow 0$, then $P(x < X \leq x + h_n) \searrow 0$ for the same reason as (i). Then $\forall \varepsilon > 0$, $\exists \delta > 0$ and $N > 0$, *s.t.*$\forall 0 < h < \delta < h_N$,

$$P(x < X \leq x + h) \leq P(x < X \leq x + h_N) < \varepsilon$$

which means $F(x + h) - F(x) = P(x < X \leq x + h) \to 0$ as $h \to 0$ and ends the proof.

(b) Let $\{X^{-1}(A_n)\}_{n\in\mathbb{N}}$ be a measurable partition of Ω, where $A_n = (a_n, b_n] \in \mathcal{B}$, $a_{n+1} = b_n$, $a_0 = -\infty$, $b_n \nearrow \infty$. Via $E[|g(X)|] < \infty$ firstly we can show that

$$E[g(X)] = \int_\Omega g(X)dP = \sum_{n=0}^\infty \int_{\{X\in A_n\}} g(X)dP < \infty$$

As the property of expectation as a probability integral, we directly state that

$$\sum_{n=0}^{\infty}\int_{\{X\in A_n\}} g(X)dP \quad = E[g(X)] = \int_{\mathbb{R}} gP\circ X^{-1} = \quad \sum_{n=0}^{\infty}\int_{A_n} gdP\circ X^{-1}$$

Since for $(a_n, b_n]$, as we proved in (a), denote L-S measure induced by distribution function F by m_F, then

$$m_F(A_n) = F(b_n) - F(a_n) = P\circ X^{-1}(A_n)$$

By the uniqueness of extension of measure, $P\circ X^{-1} = m_F$. Then we are able to transform the expectation integral as follows:

$$\sum_{n=0}^{\infty}\int_{A_n} gdP\circ X^{-1} = \sum_{n=0}^{\infty}\int_{A_n} gdF = \int_{\mathbb{R}}\sum_{n=0}^{\infty} g1_{A_k}dF < \infty$$

According to the two equations above,

$$E[g(X)] = \int_{\mathbb{R}}\sum_{n=0}^{\infty} g1_{A_k}dF = \int_{-\infty}^{\infty} gdF < \infty$$

(c) Denote the density of B_t^2 by $p_{B_t^2}$, then for $y \geq 0$, show that

$$\int_{-\infty}^{y} p_{B_t^2}(x)dx = P(B_t^2 \leq y) = P(|B_t| \leq \sqrt{y}) = \int_{-\sqrt{y}}^{\sqrt{y}} p(x)d(\sqrt{y})$$

Then by simple calculation we have

$$p_{B_t^2}(y) = \frac{1}{2\sqrt{2y\pi t}}e^{-\frac{y}{2t}} + \frac{1}{2\sqrt{2y\pi t}}e^{-\frac{y}{2t}} = \frac{1}{\sqrt{2y\pi t}}e^{-\frac{y}{2t}}$$

□

Exercise. 2.3

Proof. Firstly, since $\emptyset \in \mathcal{H}_i$ for all $i \in I$, $\emptyset \in \bigcap_{i\in I}\mathcal{H}_i$. Secondly, $A^c \in \bigcap_{i\in I}\mathcal{H}_i$ given $A \in \bigcap_{i\in I}\mathcal{H}_i$, for $A^c \in \mathcal{H}_i$ for all $i \in I$. At last, let $\{A_n\}_{n\in\mathbb{N}}$ be a set sequence in $\bigcap_{i\in I}\mathcal{H}_i$, since in each $\mathcal{H}_i$, $\bigcup_{n=0}^{\infty} A_n \in \mathcal{H}_i$, hence $\bigcup_{n=0}^{\infty} A_n \in \bigcap_{i\in I}\mathcal{H}_i$. Based on all above we conclude that $\bigcap_{i\in I}\mathcal{H}_i$ is again a sigma algebra. □

Exercise. 2.8

Proof. **(a)** Directly by ***(2.2.3)***, let $k = n = 1$, we can conclude $E^0[\exp(iuB_t)] = \exp(-\frac{1}{2}u^2t)$, here $\forall u \in \mathbb{R}$.

(b) Denote $E[B_t^n] = m^{(n)}(B_t)$. For fixed t, in (a),

$$E[e^{iuB_t}] = \sum_{n=0}^{\infty} m^{(n)}(B_t)\frac{(iu)^n}{n!} = e^{-\frac{u^2}{2}t}$$

Then let $f(t) = e^{-\frac{u^2}{2}t}$,

$$\sum_{n=0}^{\infty} m^{(n)}(B_t)\frac{u^n i^n}{n!} = \sum_{n=0}^{\infty} f^{(n)}(0)\frac{t^n}{n!}$$

See that $f^{(n)}(0) = (-\frac{u^2}{2})^n$, therefore,

$$\sum_{n=0}^{\infty} m^{(n)}(B_t)\frac{u^n i^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n}\frac{u^{2n}t^n}{n!}$$

Compare the term of u of same power, we could see

$$m^{(2n)}(B_t) = \frac{(-1)^n 2n!}{2^n i^{2n} n!}t^n = \frac{2n!}{2^n n!}t^n$$

Finally let $n = 2$, we get $E[B_t^4] = 3t^2$ and proof ends.

(c) From *(2.2.2)* we know $P^0(B_t \in A) = \int_A p(t,0,y)dy$, here $A \in \mathcal{B}$ and $p(t,x,y) = \frac{1}{\sqrt{2\pi t}}\exp(-\frac{y^2}{2t})$. Then for measurable function f, by the conclusion of **2.2, (b)**,

$$E[f(B_t)] = \frac{1}{\sqrt{2\pi t}}\int_{\mathbb{R}} f(x)\exp(-\frac{x^2}{2t})dx$$

(d) Firstly see that $E^x[|B_t - B_s|^4] = E[\left|B_{|t-s|}\right|^4] = E[(\sum_{i=1}^n (B_{|t-s|}^{(i)})^2)^2]$. Then expand the summation, we get

$$E^x[|B_t - B_s|^4] = E[\sum_{i=1}^{n}(B_{|t-s|}^{(i)})^4 + \sum_{1\le j\ne k\le n}(B_{|t-s|}^{(j)})^2(B_{|t-s|}^{(k)})^2]$$

Since we know $E[(B_{|t-s|}^{(i)})^4] = 3\,|t-s|^2$ from (b), and $B_{|t-s|}^{(j)}$ and $B_{|t-s|}^{(k)}$ are independent where $j < k$, thus

$$\begin{aligned} E^x[|B_t - B_s|^4] &= 3n\,|t-s|^2 + n(n-1)E[(B_{|t-s|}^{(j)})^2]E[(B_{|t-s|}^{(k)})^2] \\ &= 3n\,|t-s|^2 + n(n-1)\,|t-s|^2 \\ &= n(n+2)\,|t-s|^2 \end{aligned}$$

□

Exercise. 2.16

Proof. Without loss of generality, assume $\{B_t\}_{t\ge0}$ starts at 0. In fact we can rewrite $\widetilde{B}_t = B_t - B_0$ by B_t. Since $\{B_t\}_{t\ge0}$ is a *Gaussian* process, thus $Z = (B_{t_1}, ..., B_{t_k})$ obeys multi normal distribution for any fixed $0 \le t_1 \le ... \le t_k$ and any $k = 1, 2, ...$ According to the property of *Gaussian* random variable, $\hat{B}_{t_i} = \frac{1}{c}B_{c^2 t_i}$ is also *Gaussian* random variable for $i = 1, ..., k$. Consequently, $\hat{Z} = (\hat{B}_{t_1}, ..., \hat{B}_{t_k})$ is also $k-$dimensional *Gaussian* vector.

Since k and $\{t_i\}_{i=1,...,k}$ is arbitrary, $\{\hat{B}_t\}$is a *Gaussian* process. Secondly by property of standard *Brownian motion* $\{B_t\}_{t\geq 0}$ show that

$$\begin{aligned} Cov(\hat{B}_s, \hat{B}_t) &= \frac{1}{c^2} Cov(B_{c^2 s}, B_{c^2 t}) = \frac{1}{c^2}\min\{c^2 s, c^2 t\} = \min\{s,t\} \\ E(\hat{B}_t) &= \frac{1}{c} E(B_{c^2 t}) = 0 \end{aligned}$$

for arbitrary fixed $t \geq 0$ and $s \geq 0$. Consequently, $\{\hat{B}_t\}_{t\geq 0}$ is a *Brownian motion*. Moreover , to directly prove by definition, see that for fixed $t \geq 0$, $B_t \sim N(0,t)$, hence $B_{c^2 t} \sim N(0, c^2 t)$ and $\hat{B}_t = \frac{1}{c} B_{c^2 t} \sim N(0,t)$. Then the $k-$dimension distribution of $\hat{B}_t$ generates

$$\hat{v}_{t_1 ... t_k}(A_1 \times ... \times A_k) = \int_{A_1 \times ... \times A_k} p(t_1, 0, x_1) ... p(t_k - t_{k-1}, x_{k-1}, x_k) dx_1 ... dx_k$$

as the same measure generated by the $k-$dimensional distribution of $\{B_t\}_{t\geq 0}$, where $p(t,x,y)$ is the density of normal distribution. So in the canonical defining way of *Brownian motion* by *Kolmogorov Extension Theorem*, $\{\hat{B}_t\}_{t\geq 0}$ is a *Browinian motion*. □

Exercise. 3.2

Proof. To begin with, let $0 = t_0 < t_1 < ... < t_n < t_{n+1} = t$ be a uniform partition of $[0,t]$, $t_{k+1} - t_k = t_k - t_{k-1}$ for all $1 \leq k \leq n$, and denote variation $\triangle_j^m(B_t^k) = (B_{t_{j+1}}^k - B_{t_j}^k)^m$, $\triangle_j(t) = t_{j+1} - t_j$, $j = 0, 1, ..., n$.

The proof is based on the key variation equation as below

$$\triangle_j(B_t^3) = \triangle_j^3(B_t) + 3B_{t_j}^2 \triangle_j (B_t) + 3B_{t_j} \triangle_j^2 (B_t)$$

and rewrite the equation by defining two variational summations

$$I_n^{(1)} := \sum_{j=0}^n B_{t_j}^2 \triangle_j (B_t) + \frac{1}{3}\sum_{j=0}^n \triangle_j^3(B_t) = \frac{1}{3}B_t^3 - \sum_{j=0}^n B_{t_j} \triangle_j^2 (B_t) := \frac{1}{3}B_t^3 - I_n^{(2)}$$

In order to prove the proposition directly by the definition of $It\hat{o}$ integral, we are to prove an equivalent statement that $I_n^{(1)}$ converges to $\int_0^t B_s^2 dB_s$ and $I_n^{(2)}$ in right side converges to $\int_0^t B_s ds$ both in sense of $L^2(P)$, as $n \to \infty$ (i.e., $t_{k+1} - t_k \to 0$).

Firstly, to deal with $I_n^{(1)}$, define elementary function sequence in the form of $\phi_n(s,\omega) := \sum_{j=0}^n B_{t_j}^2 \cdot 1_{[t_j, t_{j+1})}(s)$ and claim that

$$E[\int_0^t (\phi_n - B_s^2)^2 ds] = \sum_{j=0}^n \int_{t_j}^{t_{j+1}} E[(\phi_n - B_s^2)^2] ds \to 0$$

See $B_s \in L^4$ and $\phi_n \geq 0$, so that $(\phi_n - B_s^2)^2$ is dominated by B_s^4 . And we can also show

$$\phi_n - B_s^2 = B_{t_j}^2 - B_s^2 \xrightarrow{L_1} 0$$

as $n \to \infty$ (i.e., $t_{k+1} - t_k \to 0$). Since $(\cdot)^2$ is a continuous function, by *Lebesgue Dominated Convergence Theorem*, then we can take the limit in probability (implied by L_1 convergence) inside the expectation and obtain $E[(\phi_n - B_s^2)^2] \to 0$. Meanwhile, since $\frac{1}{3}\sum_{j=0}^{n}(B_{t_{j+1}} - B_{t_j})^3 \xrightarrow{L_1} 0$, by *Itô Isometry* we obtain $I_n^{(1)} \xrightarrow{L^2(P)} \int_0^t B_s^2 dB_s$.

Secondly, to deal with $I_n^{(2)}$, see that

$$\begin{aligned} E\left[\left|I_n^{(2)} - \int_0^t B_s ds\right|^2\right] &= E \mid \sum_{j=0}^{n} B_{t_j}(\triangle_j^2(B_t) - \triangle_j(t)) \\ &\quad + \sum_{j=0}^{n} B_{t_{j-1}} \triangle_j (t) - \int_0^t B_s ds \mid^2 \end{aligned}$$

Since $(x+y)^2 \le 2x^2+2y^2$ for all $x, y \in \mathbb{R}$, we can control $E\left[\left|I_n^{(2)} - \int_0^t B_s ds\right|^2\right]$ by inequality

$$\begin{aligned} 2E_1 + 2E_2 &\ge E\left[\left|I_n^{(2)} - \int_0^t B_s ds\right|^2\right] \\ E_1 &:= E\left[\sum_{j=0}^{n} B_{t_j}(\triangle_j^2(B_t) - \triangle_j(t))\right]^2 \\ E_2 &:= E\left[\sum_{j=0}^{n} B_{t_j} \triangle_j (t) - \int_0^t B_s ds\right]^2 \end{aligned}$$

Now we claim that both E_1 and E_2 converge to zero. For E_1, expand the sqaure into two parts as follows

$$E_1 := E_3 + E_4$$

$$E_3 := E[\sum_{j=0}^{n}(B_{t_j} - B_0)^2(\triangle_j^2(B_t) - \triangle_j(t))^2]$$

$$E_4 = \sum_{i<j} E_{i,j} := \sum_{i<j} E[B_{t_j} B_{t_i}(\triangle_j^2(B_t) - \triangle_j(t))(\triangle_i^2(B_t) - \triangle_i(t))]$$

About E_3, as $\sum_{j=0}^{n} E[\triangle_j^2(B_t) - \triangle_j(t)]^2 \to 0$, by *the independent increment* of $\{B_t\}_{t\ge 0}$, obviously

$$E_3 = \sum_{j=0}^{n} E[(B_{t_j} - B_0)^2] E[(\triangle_j^2(B_t) - \triangle_j(t))^2] \to 0$$

About E_4, via a partition of Ω see that

$$\begin{aligned} |E_{i,j}| &\leq \int_{\{|B_{t_j}|,|B_{t_i}|\leq M\}} \Big|B_{t_i}B_{t_j}\Big| \Big|\triangle_j^2(B_t) - \triangle_j(t)\Big| \Big|\triangle_i^2(B_t) - \triangle_i(t)\Big| dP \\ &\quad + \int_{\{|B_{t_j}|,|B_{t_i}|\leq M\}^c} \Big|B_{t_i}B_{t_j}\Big| \Big|\triangle_j^2(B_t) - \triangle_j(t)\Big| \Big|\triangle_i^2(B_t) - \triangle_i(t)\Big| dP \\ &\leq M^2 E[\Big|\triangle_j^2(B_t) - \triangle_j(t)\Big| \Big|\triangle_i^2(B_t) - \triangle_i(t)\Big|] + \varepsilon \end{aligned}$$

for the function inside the expectations are all integrable. Then as $M \nearrow 0$, the second integral above is bounded by given $\varepsilon > 0$. As $n \to 0$ (i.e., $t_{k+1} - t_k \to 0$) and $\varepsilon \to 0$, by *the independent increment* of $\{B_t\}_{t\geq 0}$ again, $E_4 \to 0$ and then $E_1 \to 0$.

In the second expectation, as we already know $\int_0^t s dB_s = tB_t - \int_0^t B_s ds$ and $E[\int_0^t f dB_s] = 0$ for all $f \in \mathcal{V}(0,t)$, by *Itô Isometry* we have

$$\begin{aligned} E[(\int_0^t B_s ds)^2] &= E[(\int_0^t s dB_s - tB_t)^2] \\ &= \int_0^t s^2 ds + E[t^2 B_t^2] - 2tE[\int_0^t B_t s dB_s] < \infty \end{aligned}$$

and therefore, the function inside E_2 is integrable. Fix $\omega \in \Omega$, as the trajectory $B_s(\omega)$ is continuous, then take limit $n \to \infty$ in the *Riemann sum*, we obtain $\sum_{j=0}^n B_{t_j}(\omega)\triangle_j(t) - \int_0^t B_s(\omega)ds \to 0$, i.e., $\sum_{j=0}^n B_{t_j}\triangle_j(t) \xrightarrow{a.s.} \int_0^t B_s ds$ (actually pointwise?). So $E_2 \to 0$ so that $I_n^{(2)} \xrightarrow{L_2(P)} \int_0^t B_s ds$. Consequently, we conclude that

$$\int_0^t B_s^2 dB_s \overset{L^2(P)}{=} \lim_{n\to\infty} I_n^{(1)} = \lim_{n\to\infty} (\frac{1}{3}B_t^3 - I_n^{(2)}) \overset{L^2(P)}{=} \frac{1}{3}B_t^3 - \int_0^t B_s ds$$

and ends the proof. □

Exercise. 3.10

Proof. By definition of *Itô* integral, in fact obviously we know

$$I = \int_0^T f(t,\omega)dB_t \overset{L_2(P)}{=} \lim_{n\to\infty} \sum_{j=0}^n f(t_j,\omega) \triangle_j (B_t)$$

For all $t'_j \in [t_j, t_{j+1}]$, in order to show the equality in sense of $L_1(P)$, we just prove

$$E\left[\left|I - \sum_{j=0}^n f(t'_j) \triangle_j (B_t)\right|\right] \to 0$$

Firstly by *Holder* Inequality,

$$\begin{aligned}
E\left[\left|\sum_{j=0}^{n}(f(t_j)-f(t'_j))\triangle_j(B_t)\right|\right] &\le \sum_{j=0}^{n}\left(E\left[\left|f(t_j)-f(t'_j)\right|^2\right]\right)^{\frac{1}{2}}\left(E\left[|\triangle_j(B_t)|^2\right]\right)^{\frac{1}{2}} \\
&\le K^{\frac{1+\varepsilon}{2}}\sum_{j=0}^{n}|\triangle_j(t)|^{\frac{1+\varepsilon}{2}}\cdot|\triangle_j(t)|^{\frac{1}{2}} \\
&\le \max_j|\triangle_j(t)|^{\frac{\varepsilon}{2}}\cdot\left(K^{\frac{1+\varepsilon}{2}}T\right)
\end{aligned}$$

simply by the smooth property of f we assumed. Let $\max_{0\le j\le n}\{\triangle_j(t)\}\to 0$,

$$E\left[\left|\sum_{j=0}^{n}(f(t_j)-f(t'_j))\triangle_j(B_t)\right|\right]\to 0$$

Secondly by definition of *Itô* integral and *Holder Inequality*, we can show

$$E\left[\left|I-\sum_{j=0}^{n}f(t_j)\triangle_j(B_t)\right|\right]\le\left(E\left[\left|I-\sum_{j=0}^{n}f(t_j)\triangle_j(B_t)\right|^2\right]\right)^{\frac{1}{2}}\to 0$$

By the two limits above, clearly by the inequality as below

$$\begin{aligned}
E\left[\left|I-\sum_{j=0}^{n}f(t'_j)\triangle_j(B_t)\right|\right] &\le E\left[\left|\sum_{j=0}^{n}(f(t_j)-f(t'_j))\triangle_j(B_t)\right|\right] \\
&\quad +E\left[\left|I-\sum_{j=0}^{n}f(t_j)\triangle_j(B_t)\right|\right]
\end{aligned}$$

the left side converges to zero as $n\to 0$, i.e., $\max_{0\le j\le n}\{\triangle_j(t)\}\to 0$. As a simple corollary,

$$\int_0^T f(t,\omega)dB(t,\omega)=\int_0^T f(t,\omega)\circ dB(t,\omega)$$

□

Exercise. 3.13

Proof. **a)** This is a quite obvious statement. $E[B^2(t,\omega)]=t<\infty$ for all $t\ge 0$. Then see that $E[(B_t-B_s)^2]=|s-t|\to 0$ as $s\to t$ for all $t\ge 0$, thus *Brownian motion* $\{B_t\}_{t\ge 0}$ is continuous in mean square.

b) Firstly we prove $E[f^2(B_t)]<\infty$. See that

$$E[(f(B_t)-f(B_0))^2]=E[f^2(B_t)]-2E[f(B_t)f(B_0)]+f^2(B_0)$$

therefore by property of f as a *Lipschitz* function, since $B_0=0$, thus $E[f(B_t)-f(0)]\le CE[|B_t|]$. Then we have

$$E[f^2(B_t)]\le C^2E[|B_t|^2]+2Cf(0)E[|B_t|]+f^2(0)$$

As for certain $t \geq 0$, $E[|B_t|^2] < \infty$, consequently, we have shown $E[f^2(B_t)] < \infty$. Secondly show that

$$E[(f(B_t) - f(B_s))^2] \leq CE[(B_t - B_s)^2] = C\,|t-s| \to 0$$

as $s \to t$ for all $t \geq 0$ and certain constant $0 < C < \infty$. To conclude, $Y_t = f(B_t)$ is continuous in mean square.

c) In order to prove the integral equality, see that

$$\begin{aligned} E\left[\int_S^T (X_t - \phi_n(t))^2 dt\right] &= \sum_{j=0}^n E\left[\int_{t_j}^{t_{j+1}} (X_t - X_{t_j})^2 dt\right] \\ &= \sum_{j=0}^n \int_{t_j}^{t_{j+1}} E[(X_t - X_{t_j})^2] dt \end{aligned}$$

Simply as X_t is continuous in mean square, then

$$0 \leq \sum_{j=0}^n \int_{t_j}^{t_{j+1}} E[(X_t - X_{t_j})^2] dt < \varepsilon(T - S) \to 0$$

Consequently, by definition, $\int_S^T \phi_n(t,\omega) dB(t,\omega) \stackrel{L_2(P)}{\longrightarrow} \int_S^T X_t dB_t$. □

Exercise. 3.17

Proof. **a)** As $E[X \mid \mathcal{G}]$ is $\mathcal{G}-$measurable, then $\forall c \in E[X \mid \mathcal{G}](\Omega)$,

$$E[X \mid \mathcal{G}]^{-1}(c) \in \mathcal{G}$$

Consequently we can conlude the finity of $E[X \mid \mathcal{G}](\Omega)$ simply for

$$\{E[X \mid \mathcal{G}]^{-1}(c) : c \in E[X \mid \mathcal{G}](\Omega)\} \subset \mathcal{G}$$

And by the property of such finite $\mathcal{G}$, there exists finite number of $G_{n_1}, ..., G_{n_k}$, $k = 1, ..., n$, such that

$$E[X \mid \mathcal{G}]^{-1}(c) = \bigcup_{j=1}^k G_{n_j}$$

which implies for any such G_i, $E[X \mid \mathcal{G}] \mid_{G_i} = c$. Since c is arbitrary, then we conclude that $E[X \mid \mathcal{G}]$ is a constant on each G_i.

b) By definition of $E[X \mid \mathcal{G}]$ and the conclusion of **a)**, let $E[X \mid \mathcal{G}] \mid_{G_i} = c_i$, $i = 1, ..., n$,

$$\int_{G_i} E[X \mid \mathcal{G}] dP = \int_{G_i} X dP = c_i P(G_i)$$

and directly we get $E[X \mid \mathcal{G}] = c_i = \frac{\int_{G_i} X dP}{P(G_i)}$ when $P(G_i) > 0$.

c) In the same way as **a)** we can prove $X \mid_{G_i} = a_{k_i}$ for some $k_i = 1, ..., m$. Thus when $P(G_i) > 0$, $P(X = a_k \mid G_i) = 0$ except $k = k_i$ where $P(X =$

$a_{k_i} \mid G_i) = 1$, so simply by showing two equalities

$$\begin{aligned} E[X \mid G_i] &= \sum_{k=1}^{m} a_k P(X = a_k \mid G_i) = a_{k_i} \\ \int_{G_i} (E[X \mid \mathcal{G}] - X) dP &= (c_i - a_{k_i}) P(G_i) = 0 \end{aligned}$$

we have $E[X \mid \mathcal{G}] \mid_{G_i} = c_i = a_{k_i} = E[X \mid G_i]$. However, when $P(G_i) = 0$, even if $E[X \mid \mathcal{G}] \mid_{G_i} = c_i \neq a_{k_i} = X \mid_{G_i}$, $\int_{G_i} E[X \mid \mathcal{G}] dP = \int_{G_i} X dP$ still holds, and without any contradiction, they two are inequal. □

Exercise. 3.18

Proof. Note that $M(x) = \exp(\sigma x - \frac{1}{2}\sigma^2 t)$ is a continuous function on $\mathbb{R}$, thus as B_t is $\mathcal{F}_t-$measurable, M_t is also $\mathcal{F}_t-$measurable. Secondly, clearly $M(x)$ is a *Lipschitz* function, so by conclusion of **3.13 b)**, $E[|M_t|] < \infty$ implied by $E[M_t^2] < \infty$. At last, for $s > t$, clearly see that

$$\begin{aligned} E[\exp(\sigma B_s - \frac{1}{2}\sigma^2 s) \mid \mathcal{F}_t] &= E[\exp(\sigma(B_s - B_t)) \cdot \exp(\sigma B_t - \frac{1}{2}\sigma^2 s) \mid \mathcal{F}_t] \\ &= E[\exp(\sigma(B_s - B_t))] \cdot E[\exp(\sigma B_t - \frac{1}{2}\sigma^2 s) \mid \mathcal{F}_t] \\ &= e^{\frac{1}{2}\sigma^2(s-t)} \cdot e^{-\frac{1}{2}\sigma^2 s} \cdot E[\exp(\sigma B_t) \mid \mathcal{F}_t] \\ &= \exp(\sigma B_t - \frac{1}{2}\sigma^2 t) = M_t \end{aligned}$$

Consequently, by definition, $\{M_t\}_{t \geq 0}$ is an $\mathcal{F}_t-$martingale. □

Exercise. 4.6

Proof. **(a)** Let $X_t = g(B_t, t) = \exp\{ct + \alpha B_t\}$, then by *Itô Formula*,

$$dX_t = cX_t dt + \alpha X_t dB_t + \frac{1}{2}\alpha^2 X_t (dB_t)^2 = (c + \frac{1}{2}\alpha^2) X_t dt + \alpha X_t dB_t$$

(b) Still let $X_t = g(B_t, t) = \exp\{ct + \sum_{j=1}^{n} \alpha_j B_j(t)\}$, then by the *Multi-dimensional Itô Formula*,

$$\begin{aligned} dX_t &= cX_t dt + X_t \sum_{j=1}^{n} \alpha_j dB_j(t) + \frac{1}{2} X_t \sum_{j=1}^{n} \alpha_j^2 (dB_j(t))^2 \\ &= \left(c + \frac{1}{2}\sum_{j=1}^{n} \alpha_j^2\right) X_t dt + X_t \left(\sum_{j=1}^{n} \alpha_j dB_j(t)\right) \end{aligned}$$

□

Exercise. 4.7

Proof. **(a)** Let $v = (1, 0, ..., 0) \in \mathcal{V}^n(0, T)$, then $X_t = B_t \in \mathbb{R}$. Obviously $X_t^2 = B_t^2$ is not a martingale.

(b) Let's define the filtration as $\mathcal{F}_t := \sigma(\{B_s\}_{0 \le s \le t})$. Obviously X_t and $\int_0^t v_s^2 ds$ are $\mathcal{F}_t-$measurable, thus M_t is $\mathcal{F}_t-$measurable. Secondly, since $|v_s| < \infty$, by *Itô Isometry*,

$$E[|M_t|] \le E[X_t^2] + E\left[\int_0^t v_s^2 ds\right] = 2E\left[\int_0^t v_s^2 ds\right] < \infty$$

At last, we need to show $E[M_t \mid \mathcal{F}_s] = M_s$ for all $t \ge s$. Show that

$$\begin{aligned} E[X_t^2 \mid \mathcal{F}_s] &= E\left[\left(\int_0^s v_u dB_u + \int_s^t v_u dB_u\right)^2 \mid \mathcal{F}_s\right] \\ &= \left(\int_0^s v_u dB_u\right)^2 + E\left[\left(\int_s^t v_u dB_u\right)^2 \mid \mathcal{F}_s\right] \\ &= \left(\int_0^s v_u dB_u\right)^2 + E\left[\int_s^t v_u^2 du \mid \mathcal{F}_s\right] \end{aligned}$$

Therefore we obtain

$$\begin{aligned} E[M_t \mid \mathcal{F}_s] &= \left(\int_0^s v_u dB_u\right)^2 + E\left[\int_s^t v_u^2 du \mid \mathcal{F}_s\right] - E\left[\int_0^t v_u^2 du \mid \mathcal{F}_s\right] \\ &= \left(\int_0^s v_u dB_u\right)^2 - \int_0^s v_u^2 du \\ &= M_s \end{aligned}$$

So far, we have justified that $\{M_t\}_{t \ge 0}$ is a $\mathcal{F}_t-$martingale *w.r.t.* the filtration we defined. □

Exercise. 4.8

Proof. **(a)** Apply the *Multidimensional Itô Formula* to $\{f(B_t)\}_{t \ge 0}$, then

$$df(B_t) = \sum_{j=1}^n \frac{\partial f}{\partial x_j}(B_t) dB_j(t) + \frac{1}{2} \sum_{j=1}^n \frac{\partial^2 f}{\partial x_j^2}(B_t)(dB_j(t))^2$$

By taking the integral of both sides, we obtain that

$$f(B_t) - f(B_0) = \int_0^t \nabla f(B_s) dB_s + \frac{1}{2}\int_0^t \triangle f(B_s) ds$$

(b) Since g' is *a.e.* differentiable, then it is absolutely continuous, and $g' \in C(\mathbb{R})$. By *Weiestrass Theorem*, there exists a polynomial sequence $\{f_n\}_{n \in \mathbb{N}}$ such that $f_n \xrightarrow{u} g$, $f_n' \xrightarrow{u} g'$. More importantly, as f_n' is differentiable,

$f_n'' \xrightarrow{a.e.} g''$, where $f_n'' \to g''$ outside $\{z_1, ..., z_N\}$. For each f_n, we can apply the result of **(a)** and get

$$f_n(B_t) - f(B_0) = \int_0^t f_n'(B_s)dB_s + \frac{1}{2}\int_0^t f_n''(B_s)ds$$

As $|g''(x)| \le M$, *a.e.*, then g' is also *a.e.* bounded on $[0,t]$. By *Lebesgue Bounded Convergence Theorem*, take *a.e.* limit of both sides and conclude that

$$g(B_t) = g(B_0) + \int_0^t g'(B_s)dB_s + \frac{1}{2}\int_0^t g''(B_s)ds$$

□

Exercise. 4.13

Proof. Apply *Itô Formula* to $M_t = \exp\{-\int_0^t u_r dB_r - \frac{1}{2}\int_0^t u_r^2 dr\}$,

$$dM_t = -\frac{1}{2}u^2 M_t dt - uM_t dB_t + \frac{1}{2}u^2 M_t dt = -uM_t dB_t$$

Then by the general *Integration by Parts Formula*,

$$\begin{aligned} dY_t &= X_t dM_t + M_t dX_t + dX_t dM_t \\ &= uX_t M_t dB_t + uM_t dt + M_t dB_t - uM_t dB_t(udt + dB_t) \\ &= (uX_t M_t + M_t)dB_t \end{aligned}$$

Hence $Y_t = \int_0^t (uX_r M_r + M_r)dB_r$ is a $\mathcal{F}_t-$martingale, where as u is bounded, $uX_r M_r + M_r \in \mathcal{V}(0,t)$ for all $t \ge 0$. □

Exercise. 4.16

Proof. **(a)** By the *Jensen Inequality*,

$$E[M_t^2] = E\left[|E[Y \mid \mathcal{F}_t]|^2\right] \le E[E[|Y|^2 \mid \mathcal{F}_t]] = E[|Y|^2] < \infty$$

for all $t \in [0,T]$.

(b) **(i)** Since $B_t^2 - t = 2\int_0^t B_s dB_s$ is a $\mathcal{F}_t-$martingale, then

$$E[M_0] - T = E[B_T^2 \mid \mathcal{F}_0] - T = E[B_T^2 - T \mid \mathcal{F}_0] = 0$$

As a result, show that

$$M_t = E[B_T^2 - T \mid \mathcal{F}_t] + T = E[M_0] + \int_0^t g dB_t$$

where we set $g := 2B_t$.

(ii) Since $B_t^3 - 3tB_t = \int_0^t B_s^2 dB_s - \int_0^t 3s dB_s$ is a $\mathcal{F}_t-$martingale, then

$$E[M_0] = E[B_T^3 - 3TB_T \mid \mathcal{F}_0] + 3TE[B_T \mid \mathcal{F}_0] = 0$$

As a result, show that

$$M_t = E[B_T^3 - 3TB_T \mid \mathcal{F}_t] + 3TE[B_T \mid \mathcal{F}_t] = E[M_0] + \int_0^t g dB_s$$

where we set $g(s,\omega) := 3T - 3s + B_s^2$.

(iii) Since $\exp\{\sigma B_t - \frac{1}{2}\sigma^2 t\}$ is a $\mathcal{F}_t-$martingale, then

$$E[M_0] = e^{\frac{1}{2}\sigma^2 T} E[\exp\{\sigma B_T - \frac{1}{2}\sigma^2 T\} \mid \mathcal{F}_0] = e^{\frac{1}{2}\sigma^2 T}$$

Then apply *Itô Formula* to $Y_t = \exp\{\sigma B_t - \frac{1}{2}\sigma^2 t\}$, show that

$$dY_t = -\frac{1}{2}\sigma^2 Y_t dt + \sigma Y_t dB_t + \frac{1}{2}\sigma^2 Y_t dt = \sigma Y_t dB_t$$

hence $Y_T = Y_0 + \int_0^T \sigma Y_t dB_t$. As $Y_0 = 1$, finally we obtain that

$$M_t = E[Y_T e^{\frac{1}{2}\sigma^2 T} \mid \mathcal{F}_t] = e^{\frac{1}{2}\sigma^2 T} + \int_0^t \sigma e^{\frac{1}{2}\sigma^2 T} Y_s dB_s = E[M_0] + \int_0^t g dB_s$$

so we set $g(s,\omega) = \sigma e^{\frac{1}{2}\sigma^2 T} Y_s$. □

Exercise. 5.1

Proof. (i) By *Ito Formla*,

$$\begin{aligned} dX_t &= 0dt + e^{B_t} dB_t + \frac{1}{2} e^{B_t} (dB_t)^2 \\ &= \frac{1}{2} X_t dt + X_t dB_t \end{aligned}$$

(ii) By *Ito Formula*,

$$\begin{aligned} dX_t &= -\frac{B_t}{(1+t)^2} dt + \frac{1}{1+t} dB_t - \frac{1}{2} \cdot 0 \cdot (dB_t)^2 \\ &= -\frac{1}{1+t} B_t dt + \frac{1}{1+t} dB_t \end{aligned}$$

(iii) By *Ito Formula*, for $t < \inf\{s > 0 : B_s \notin [-\frac{\pi}{2}, \frac{\pi}{2}]\}$, $\cos B_s \leq 1$, hence

$$\begin{aligned} dX_t &= \cos B_t dB_t - \frac{1}{2} \sin B_t (dB_t)^2 \\ &= -\frac{1}{2} X_t dt + \sqrt{1 - X_t^2} dB_t \end{aligned}$$

(iv) By *Ito Formula*,

$$\begin{aligned} dX_1(t) &= 1 \\ dX_2(t) &= e^t B_t dt + e^t dB_t \\ &= X_2(t) dt + e^{X_1} dB_t \end{aligned}$$

So we can verify that

$$\begin{bmatrix} dX_1 \\ dX_2 \end{bmatrix} = \begin{bmatrix} 1 \\ X_2 \end{bmatrix} dt + \begin{bmatrix} 0 \\ e^{X_1} \end{bmatrix} dB_t$$

(v) By *Ito Formula*,

$$\begin{aligned} dX_1(t) &= d\left(\frac{e^{B_t}+e^{-B_t}}{2}\right) \\ &= \frac{e^{B_t}-e^{-B_t}}{2}dB_t + \frac{1}{2}(\frac{e^{B_t}+e^{-B_t}}{2})d(B_t)^2 \\ &= \frac{1}{2}X_1(t)dt + X_2(t)dB_t \\ dX_2(t) &= d\left(\frac{e^{B_t}-e^{-B_t}}{2}\right) \\ &= \frac{e^{B_t}+e^{-B_t}}{2}dB_t + \frac{1}{2}(\frac{e^{B_t}-e^{-B_t}}{2})d(B_t)^2 \\ &= \frac{1}{2}X_2(t)dt + X_1(t)dB_t \end{aligned}$$

So we can verify that

$$\begin{bmatrix} dX_1 \\ dX_2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} dt + \begin{bmatrix} X_2 \\ X_1 \end{bmatrix} dB_t$$

□

Exercise. 5.5

Proof. (a) Multiplying the integrating factor $e^{-\mu t}$ to both sides of the equation we can see that

$$e^{-\mu t}d(X_t) = e^{-\mu t}\mu X_t dt + \sigma e^{-\mu t}dB_t$$

At the same time

$$d(e^{-\mu t}X_t) = -\mu e^{-\mu t}X_t dt + e^{-\mu t}dX_t$$

So that

$$d(e^{-\mu t}X_t) = \sigma e^{-\mu t}dB_t$$

Take integral of both sides, we obtain that

$$X_t = e^{\mu t}X_0 + \int_0^t \sigma e^{\mu(t-s)}dB_s$$

where $X_0 \in \mathbb{R}$ is the starting point.

(b) $EX_t = e^{\mu t}EX_0$, and by *Ito Isometry*,

$$\begin{aligned} VarX_t &= E\left[\sigma^2\left(\int_0^t e^{\mu(t-s)}dB_s\right)^2\right] \\ &= -\frac{\sigma^2}{2\mu}e^{2\mu t}E\left[\int_0^t(-2\mu)e^{-2\mu s}ds\right] \\ &= \frac{\sigma^2}{2\mu}e^{2\mu t}(1-e^{-2\mu t}) \\ &= \frac{\sigma^2}{2\mu}(e^{2\mu t}-1) \end{aligned}$$

□

Exercise. 5.10

Proof. Let's prove this by calculating straight forward, apply *Ito Isometry*, $(p+q+r)^2 \le 3p^2+3q^2+3r^2$ and *Holder Inequality* that $(\int_0^t bds)^2 \le \left((\int_0^t b^2ds)^{\frac{1}{2}}(\int_0^t ds)^{\frac{1}{2}}\right)^2 = t\int_0^t b^2ds$, so that for any $t \le T$ we have

$$\begin{aligned} E\left[|X_t|^2\right] &= E\left[\left(Z+\int_0^t bds+\int_0^t \sigma dB_s\right)^2\right] \\ &\le 3E\left[|Z|^2\right]+3TE\left[\int_0^t b^2ds\right]+3E\left[\int_0^t \sigma^2ds\right] \\ &\le 3E\left[|Z|^2\right]+3(T+1)E\left[\int_0^t(b^2+\sigma^2)ds\right] \\ &\le 3E\left[|Z|^2\right]+3(T+1)E\left[\int_0^t(|b|+|\sigma|)^2ds\right] \end{aligned}$$

Further more by *(5.2.1)* and apply the trick $(p+q)^2 \le 2p^2+2q^2$ again,

$$\begin{aligned} E\left[|X_t|^2\right] &\le 3E\left[|Z|^2\right]+3(T+1)\int_0^t E\left[(|b|+|\sigma|)^2\right]ds \\ &\le 3E\left[|Z|^2\right]+3(T+1)\int_0^t E\left[(C+C\,|X_s|)^2\right]ds \\ &\le 3E\left[|Z|^2\right]+6T(T+1)C^2+6(T+1)C^2\int_0^t |X_t|^2\,ds \\ &\le K_1+K_2\int_0^t |X_t|^2\,ds \end{aligned}$$

Here $K_1 := 3E\left[|Z|^2\right]+6T(T+1)C^2$ and $K_2 = 6(T+1)C^2$ as stated in the problem. Consequently by the *Gronwall Lemma*, easily we reached our aim as below

$$E\left[|X_t|^2\right] \leq K_1 \cdot \exp\{K_2 t\}$$

□

Exercise. 5.13

Proof. (i) Just check that

$$\begin{aligned} AX_t &= \begin{bmatrix} x_t' \\ -\omega^2 x_t - a_0 x_t' \end{bmatrix} \\ KX_t &= \begin{bmatrix} 0 \\ -\alpha_0 \eta x_t' \end{bmatrix} \end{aligned}$$

Hence the right side is

$$\begin{bmatrix} x_t' dt \\ -\omega^2 x_t dt - a_0 x_t' dt - a_0 \eta x_t' dB_t + T_0 \eta dB_t \end{bmatrix}$$

According to the original equation,

$$x_t'' = -\omega^2 x_t - a_0 x_t' + (T_0 - a_0 x_t')\eta dB_t$$

So it's verified that we can rewrite

$$dX_t = \begin{bmatrix} dx_t \\ dx_t' \end{bmatrix} = AX_t dt + KX_t dB_t + MdB_t$$

(ii) See that $X_t = e^{At}\int_0^t e^{-As}(KX_s + M)dB_t$, so obviously $\frac{\partial X_t}{\partial t} = AX_t$. Then by *Ito Formula*,

$$\begin{cases} dX_t = AX_t dt + \int_0^t e^{A(t-s)}(KX_s + M)dB_s & t \geq 0 \\ X_0 = 0 \end{cases}$$

(iii) At the right side of the equation,

$$(\cos\xi t + \frac{\lambda}{\xi}\sin\xi t)I + \frac{1}{\xi}A\sin\xi t = I\cos\xi t + \frac{1}{\xi}J\sin\xi t$$

Here we define $J := \lambda I + A = \begin{pmatrix} \lambda & 1 \\ -\omega^2 & -\lambda \end{pmatrix}$, and it's easy to check that $J^2 = -\xi^2 I$. Then we obtain

$$\begin{aligned} e^{Jt} &= I \cdot \sum_{n=0}^{\infty} \frac{(-1)^n (\xi t)^{2n}}{2n!} + \frac{1}{\xi} J \cdot \sum_{n=0}^{\infty} \frac{(-1)^n (\xi t)^{2n+1}}{(2n+1)!} \\ &= I\cos\xi t + \frac{1}{\xi}J\sin\xi t \end{aligned}$$

At the left side, similarly we can show

$$e^{At} = e^{Jt} \cdot \sum_{n=0}^{\infty} \frac{(-\lambda t)^n}{n!} I^n = e^{Jt} \cdot e^{-\lambda t} I$$

By the three equalities above, finally we justify that

$$e^{At} = e^{-\lambda t} \cdot e^{Jt} = \frac{e^{-\lambda t}}{\xi}\{(\xi \cos\xi t + \lambda \sin\xi t)I + A\sin\xi t\}$$

In the matrix form of the equation's solution,

$$\begin{aligned} \begin{bmatrix} x_t \\ x'_t \end{bmatrix} &= \int_0^t e^{A(t-s)} \begin{pmatrix} 0 \\ \eta(T_0 - \alpha x'_s) \end{pmatrix} dB_s \\ &= \int_0^t e^{A(t-s)} \begin{pmatrix} 0 \\ \eta(T_0 - \alpha x'_s) dBs \end{pmatrix} \end{aligned}$$

According to our result above, now denote $u := \cos\xi(t-s)$, $v := \sin\xi(t-s)$ and $y_t = x'_t$, then

$$e^{A(t-s)} = \frac{e^{-\lambda(t-s)}}{\xi} \begin{pmatrix} \xi u + \lambda v & v \\ -\omega^2 v & \xi u - \lambda v \end{pmatrix}$$

and we obtain that

$$\begin{aligned} \begin{bmatrix} x_t \\ x'_t \end{bmatrix} &= \int_0^t \frac{e^{-\lambda(t-s)}}{\xi} \cdot \begin{pmatrix} \eta v(T_0 - \alpha y_s) dB_s \\ (\xi u - \lambda v)\eta(T_0 - \alpha y_s) dB_s \end{pmatrix} \\ &= \begin{bmatrix} \eta \int_0^t \frac{e^{-\lambda(t-s)} v}{\xi} (T_0 - \alpha y_s) dB_s^{(2)} \\ \eta \int_0^t \frac{e^{-\lambda(t-s)}}{\xi} (\xi u - \lambda v)(T_0 - \alpha y_s) dB_s^{(2)} \end{bmatrix} \end{aligned}$$

As $\zeta := -\lambda + i\xi$, it's easy to check

$$\begin{aligned} e^{\zeta(t-s)} &= e^{-\lambda(t-s)} \cdot e^{i\xi(t-s)} = e^{-\lambda(t-s)}(u + iv) \\ \zeta e^{\zeta(t-s)} &= -e^{-\lambda(t-s)}(u\lambda + v\xi) + e^{-\lambda(t-s)}(u\xi - v\lambda)i \end{aligned}$$

Hence $g_t = \frac{1}{\xi}\mathrm{Im}(e^{\zeta t}) = \frac{1}{\xi}e^{-\lambda t} v$, and similarly $h_t = \frac{1}{\xi}\mathrm{Im}(\zeta e^{\zeta t}) = \frac{1}{\xi}e^{-\lambda t}(\xi u - \lambda v)$. Therefore,

$$\begin{bmatrix} x_t \\ x'_t \end{bmatrix} = \begin{bmatrix} \eta \int_0^t g_{t-s}(T_0 - \alpha_0 y_s) dB_s \\ \eta \int_0^t h_{t-s}(T_0 - \alpha_0 y_s) dB_s \end{bmatrix}$$

□

Exercise. 5.18

Proof. Consider second order differentiable function $x = e^y$ and let $Y_t = \ln X_t$, by *Ito Formula* we can rewrite the equation as below

$$dX_t = \left[\frac{\partial X_t}{\partial Y_t} \frac{\partial Y_t}{\partial t} + \frac{1}{2} \frac{\partial^2 X_t}{\partial Y_t \partial B_t} \frac{\partial Y_t}{\partial B_t} + \frac{1}{2} \frac{\partial X_t}{\partial Y_t} \frac{\partial^2 Y_t}{\partial B_t^2} \right] dt + \frac{\partial X_t}{\partial Y_t} \frac{\partial Y_t}{\partial B_t} dB_t = 0$$

As $\frac{\partial X_t}{\partial Y_t} = X_t$, then we have

$$dX_t = \left[\frac{\partial Y_t}{\partial t} + \frac{1}{2}(\frac{\partial Y_t}{\partial B_t})^2 + \frac{1}{2} \frac{\partial^2 Y_t}{\partial B_t^2} \right] X_t dt + \frac{\partial Y_t}{\partial B_t} X_t dB_t = 0$$

Now by the definition of certain X_t and Y_t, it's not difficult to check that $\frac{\partial Y_t}{\partial B_t} = \sigma$, $\frac{\partial^2 Y_t}{\partial B_t^2} = 0$, and

$$\begin{aligned} \frac{\partial Y_t}{\partial t} &= -k\ln x e^{-kt} + k(\alpha - \frac{\sigma^2}{2k})e^{-kt} - k\sigma e^{-kt}\int_0^t e^{ks}dB_s \\ &= -kY_t + k(\alpha - \frac{\sigma^2}{2k}) \\ &= k(\alpha - Y_t) - \frac{\sigma^2}{2} \end{aligned}$$

Therefore we have

$$\begin{aligned} dX_t &= [k(\alpha - Y_t) - \frac{1}{2}\sigma^2 + \frac{1}{2}\sigma^2]dt + \sigma X_t dB_t \\ &= k(\alpha - \ln X_t)X_t dt + \sigma X_t dB_t \end{aligned}$$

Obviously $X_0 = x$, thus such X_t is the solution to the SDE *(5.3.21)*.
(b) Firstly we know

$$E[X_t] = e^{e^{-kt}\ln x + \left(\alpha - \frac{\sigma^2}{2k}\right)(1-e^{-kt})} \cdot E\left[\exp\left\{\sigma \int_0^t e^{-k(t-s)}dB_s\right\}\right]$$

Let $Y_t = \exp\left\{\sigma \int_0^t e^{k(s-t)}dB_s\right\}$, then

$$dY_s = \sigma Y_s e^{k(s-t)}dB_s + \frac{1}{2}\sigma^2 Y_s e^{2k(s-t)}ds$$

So that

$$E[Y_t] = E[Y_0] + \frac{\sigma^2}{2}\int_0^t e^{2k(s-t)}E[Y_s]\,ds$$

Consider $E[Y_t]$ as a function of t, then

$$\frac{1}{E[Y_s]} \cdot \frac{dE[Y_s]}{ds} = -\frac{\sigma^2}{2}e^{2k(s-t)}$$

Solve this deterministic ODE we obtain that

$$\ln(E[Y_t]) = \frac{\sigma^2}{4k} - \frac{\sigma^2}{4k}e^{-2kt}$$

As a result,

$$E[X_t] = \exp\left\{e^{-kt}\ln x + \left(\alpha - \frac{\sigma^2}{2k}\right)(1 - e^{-kt}) + \frac{\sigma^2(1-e^{-2kt})}{4k}\right\}$$

□

Exercise. 7.5

Proof. Define $f(x) = |x|^2 = \sum_{i=1}^n x_i^2$ for $x \in \mathbb{R}^n$. Notice that $\frac{\partial^2 f}{\partial x_i \partial x_j} = 0$ for all $i \neq j$, thus

$$Af(x) = 2\sum_{i=1}^n b_i(x)x_i + \sum_{i=1}^n \sigma_i^2(x)$$

We know $\sum_{i=1}^n \sigma_i^2(x) \leq |\sigma|^2$ and

$$2\sum_{i=1}^n b_i(x)x_i \leq \sum_{i=1}^n b_i^2(x) + \sum_{i=1}^n x_i^2 = |b|^2 + |x|^2$$

where "$|\cdot|$" denotes the norm in $\mathbb{R}^n$. Then we apply the condition "$|b| + |\sigma| \leq C(1+|x|)$" and get

$$\begin{aligned} Af(x) &\leq |b|^2 + |x|^2 + |\sigma|^2 \\ &\leq C^2(1+|x|)^2 + |x|^2 \\ &\leq C^2 + (C^2+1)|x|^2 + 2C^2|x| \end{aligned}$$

Again see that $2C^2|x| \leq C^2 + C^2|x|^2$, so that for $K > \max\{2C^2, 2(C^2+1)\} \geq 0$,

$$Af(x) \leq 2C^2 + 2(C^2+1)|x|^2 \leq K(1+|x|^2)$$

Define $\tau := t \wedge \tau_R$, where $\tau_R = \inf\{s > 0 : |X_s| > R\}$. Certainly this is a stopping time *w.r.t.* $\{\mathcal{M}_t\}_{t\geq 0}$, and for all $R > x$,

$$E^x[\tau] = \frac{1}{n}(R^2 - x) < \infty$$

for certain $t \geq 0$. Therefore while applying Lemma 7.3.2, we know that C is independent with t, let $R \to \infty$ so that $\tau \to t$,

$$E^{X_0(\omega)}\left[|X_t|^2\right] \leq |X_0|^2 + K\int_0^t \left(1 + E^{X_0(\omega)}\left[|X_s|^2\right]\right) ds$$

As $E\left[|X_t|^2\right] = E\left[E^{X_0(\omega)}\left[|X_t|^2\right]\right]$, hence

$$1 + E\left[|X_t|^2\right] \leq 1 + E\left[|X_0|^2\right] + K\int_0^\tau \left(1 + E\left[|X_s|^2\right]\right) ds$$

According to *Gronwall Lemma*,

$$E\left[|X_t|^2\right] \leq \left(1 + E\left[|X_0|^2\right]\right)e^{Kt} - 1$$

□

Exercise. 7.9

Proof. (a) For any $f \in C_0^2(\mathbb{R})$,

$$Af(y) = ry \cdot \frac{\partial f}{\partial x}(y) + \frac{\alpha^2 y^2}{2} \cdot \frac{\partial^2 f}{\partial x^2}(y) \tag{0.1}$$

So the generator A of geometric *Brownian* motion X_t is given by operator $rx \cdot \frac{\partial f}{\partial x} + \frac{\alpha^2 x^2}{2} \cdot \frac{\partial^2 f}{\partial x^2}$ on $C_0^2(\mathbb{R})$. Since $f(x) = x^\gamma \in C_0^2(\mathbb{R})$, thus

$$Af(x) = \left(r + \frac{\alpha^2}{2}(\gamma - 1)\right)\gamma x^\gamma$$

(b) Choose a real number ρ such that $0 < \rho < x < R$, a function $f_\rho \in C_0^2(\mathbb{R})$ satisfying $f_\rho = f$ on (ρ, R), and define

$$\tau_{(\rho,R)} := \inf\{t > 0 : X_t \notin (\rho, R)\}$$

It's easy to confirm that $\tau_{(\rho,R)}$ is a stopping time *w.r.t.* $\{\mathcal{M}_t\}$. Via *Dynkin's Formula*, for all $k \in \mathbb{N}$ we have

$$E^x\left[f_\rho(X_{k\wedge\tau_{(\rho,R)}})\right] = f_\rho(x)$$

This is because

$$\begin{aligned} Af_\rho(x) &= \left(r + \frac{\alpha^2}{2}(\gamma_1 - 1)\right)\gamma_1 x^{\gamma_1} \\ &= \left(r + \frac{\alpha^2}{2}(1 - \frac{2r}{\alpha^2} - 1)\right)(1 - \frac{2r}{\alpha^2})x^{1-\frac{2r}{\alpha^2}} \\ &= 0 \end{aligned}$$

The condition that $r < \frac{1}{2}\alpha^2$ means $X_t \xrightarrow{a.s.} 0$ as $t \to \infty$, so that $P\left[\tau_{(\rho,R)} < \infty\right] = 1$. As a result, $f_\rho(X_{k\wedge\tau_{(\rho,R)}})$ is *a.s.* bounded by R^{γ_1}. For f_ρ is continuous, let $k \to \infty$,

$$f_\rho(x) = E^x\left[f_\rho(X_{\tau_{(\rho,R)}})\right]$$

See that either $X_{\tau(\rho,R)} = \rho$ or $X_{\tau(\rho,R)} = R$, so

$$\begin{cases} p := P^x\left[X_{\tau_{(\rho,R)}} = R\right] \\ 1 - p := P^x\left[X_{\tau_{(\rho,R)}} = \rho\right] \end{cases}$$

Thus we have

$$f_\rho(x) = f_\rho(\rho)(1-p) + f_\rho(R)p$$

Let $\rho \to 0$, by definition $\rho^{\gamma_1}(1-p) \to 0$, therefore we obtain that

$$p = \left(\frac{x}{R}\right)^{\gamma_1}$$

(c) Now we just change $f_\rho(x) = \ln x$ on (ρ, R),

$$Af_\rho(x) = rx \cdot \frac{1}{x} - \frac{\alpha^2 x^2}{2} \cdot \frac{1}{x^2} = r - \frac{1}{2}\alpha^2$$

Hence by *Dynkin's Formula*,

$$E^x\left[f_\rho(X_{k\wedge\tau_{(\rho,R)}})\right]=f_\rho(x)+(r-\frac{1}{2}\alpha^2)E^x\left[k\wedge\tau_{(\rho,R)}\right]$$

When $r>\frac{1}{2}\alpha^2$, $X_t\xrightarrow{a.s.}\infty$ as $t\to\infty$, which still implies $\tau_{(\rho,R)}<\infty$, $a.s.$. Similarly let $k\to\infty$,

$$E^x\left[\tau_{(\rho,R)}\right]=\frac{f_\rho(\rho)(1-p)+f_\rho(R)p-f_\rho(x)}{r-\frac{1}{2}\alpha^2}$$

Let $\rho\to 0$, still via $\ln(\rho)(1-p)\to 0$,

$$E^x\left[\tau_{(\rho,R)}\right]=\frac{\ln\frac{R}{x}}{r-\frac{1}{2}\alpha^2}$$

□

Exercise. 7.10

Proof. (a) According to the *Markov* property of *Itô* diffusion X_t, denote $h=T-t$

$$E^x\left[X_T\mid\mathcal{F}_t\right]=E^{X_t^x(\omega)}\left[X_h\right]=E\left[X_h^{X_t^x(\omega)}\right]$$

So that we have $E^x\left[X_T\mid\mathcal{F}_t\right]=X_t^x(\omega)e^{r(T-t)}$ for

$$E\left[X_h^{X_t(\omega)}\right]=X_t^x(\omega)\exp\left\{(T-t)\left[(r-\frac{1}{2}\alpha^2)+\frac{1}{2}\alpha^2\right]\right\}=X_t^xe^{r(T-t)}$$

(b) As $M_t=\exp\left(\alpha B_t-\frac{1}{2}\alpha^2t\right)$ is a martingale $w.r.t.$ $\{\mathcal{F}_t\}$,

$$E^x\left[X_T\mid\mathcal{F}_t\right]=xe^{rT}E\left[M_T\mid\mathcal{F}_t\right]=xe^{rT}M_t$$

Then as $X_t=xe^{rt}M_t$,

$$E^x\left[X_T\mid\mathcal{F}_t\right]=xe^{rT}e^{\alpha B_t-\frac{1}{2}\alpha^2t}=X_te^{r(T-t)}$$

□

Exercise. 8.13

Proof. (a) As $b:\mathbb{R}\to\mathbb{R}$ is a *Lipschitz* function, X_t is a well-defined *Itô* diffusion. Write $Y_t=X_t-x$, then we still have $dY_t=b(Y_t+x)dt+dB_t$. Since b is *Lipschitz* continuous, the *Novikov* condition that $E\left[\exp\left(\frac{1}{2}\int_0^T b^2(Y_s+x)ds\right)\right]<\infty$ certainly holds for $t\le T<\infty$, therefore

$$M_t=\exp\left\{-\int_0^t b(Y_s+x)dB_s-\frac{1}{2}\int_0^t b^2(Y_s+x)ds\right\}$$

is a martingale *w.r.t.* $\mathcal{F}_t$ and P. Thus according to *Girsanov Theorem I*, Y_t is a *Brownian* motion *w.r.t.* the *Girsanov* transformed probability measure Q, so that

$$P\left[X_t^x \geq M\right] = P\left[Y_t \geq M - x\right] = \int_{\{Y_t \geq M-x\}} M_T dQ$$

We know that $M_T = \exp\left\{-\int_0^T b dB_s - \frac{1}{2}\int_0^T b^2 ds\right\} > 0$, a.s., and *w.r.t.* Q, Y_t is a Brownian motion, so that

$$Q[Y_t \geq M - x] = \int_{M-x}^{\infty} \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} dt$$

Hence $\int_{\{Y_t \geq M-x\}} M_T dQ > 0$, i.e., $P\left[X_t^x \geq M\right] > 0$ for sure.

(b) Let $b = -r$, then $dX_t^x = -rdt + dB_t$. Obviously

$$X_t^x = x - rt + B_t$$

for all $t \geq 0$. Therefore as $t \to \infty$, $X_t^x \to -\infty$. Notice that the *Novikov* condition only holds for finite time interval $[0, T]$ if only with b is Lipschitz function. So when $t \to \infty$, we can no longer use *Novikov* condition to ensure M_t defined above to be a martingale and *Girsanov* Theorem is valid to apply here. In this case, it's obviously reasonable that X_t^x might not satisfy $[X_t^x \geq M] > 0$. □

Exercise. 12.1

Proof. (a) (⇒) Let $\{\theta_t\}_{t\leq T}$ be an arbitrage in the market $\{X_t\}_{t\leq T}$, then for the normalized market $\{\overline{X}_t\}_{t\leq T}$:

(i) θ is self-financing, i.e., $d\overline{V}_t^\theta = \theta_t d\overline{X}_t$, which is shown as follows,

$$\begin{aligned} d\overline{V}_t^\theta &= X_0^{-1}(t)dV_t^\theta + V_t^\theta dX_0^{-1}(t) \\ &= X_0^{-1}(t)\theta_t dX_t - \rho_t X_0^{-1}(t)V_t^\theta dt \\ &= X_0^{-1}(t)\theta_t\left[dX_t - \rho_t X_t dt\right] \\ &= \theta_t d\overline{X}_t \end{aligned}$$

(ii) θ is admissible. We know that $\overline{V}_t^\theta = \exp\left(-\int_0^t \rho_s ds\right) V_t^\theta$, and V_t^θ is (t, ω) *a.s.* lower bounded, so is $\overline{V}_t^\theta$.

(iii) θ is an arbitrage, just because $V_t^\theta > 0$ is equivalent to $\overline{V}_t^\theta > 0$.

Consequently, $\{\theta_t\}_{t\leq T}$ is an arbitrage in $\{\overline{X}_t\}_{t\leq T}$ if it is an arbitrage in $\{X_t\}_{t\leq T}$.

(⇐) Conversely, just replace ρ by $-\rho$, then the fact that $\exp\left(-\int_0^t (-\rho_s) ds\right) \overline{V}_t^\theta = V_t^\theta$ enables us to confirm $\{\theta_t\}_{t\leq T}$ is an arbitrage in $\{X_t\}_{t\leq T}$ if it is an arbitrage in the normalized market $\{\overline{X}_t\}_{t\leq T}$.

(b) Firstly in a normalized market $\{X_t\}_{t\leq T}$, construct the arbitrage portfolio. Define $\tilde{\theta} = \{\tilde{\theta}(t)\}_{t\leq T} \in \mathbb{R}^{n+1} \times [0,T]$ as follows: $\tilde{\theta}_i(t) = \theta_i$ for $i = 1, ..., n$, $\theta_0(t)$ satisfies two conditions as below:

(i) $V_0^{\tilde{\theta}} = 0$: just let $\tilde{\theta}_0(0) = -\sum_{i=1}^n \theta_i(0)X_i(0)$;

(ii) As $V_t^{\tilde{\theta}} = \tilde{\theta}_0(t) + \sum_{i=1}^n \theta_i(t)X_i(t)$ and $V_t^{\theta} = \theta_0(t) + \sum_{i=1}^n \theta_i(t)X_i(t)$, then let $\tilde{\theta}_0(t) := \theta_0(t) - V_0^{\theta}$,

$$V_t^{\tilde{\theta}} = V_t^{\theta} - V_0^{\theta} = \int_0^t \tilde{\theta}(s)dX_s = \int_0^t \theta(s)dX_s - V_0^{\theta}$$

Secondly we prove the equivalence of the existences of an arbitrage and such admissible portfolio $\hat{\theta}$ satisfying **(12.3.82)**.

($\Rightarrow$) Let $\hat{\theta}$ be an arbitrage, then $V_0^{\hat{\theta}} = 0$, $V_T^{\hat{\theta}} \geq 0$ and $P\left[V_T^{\hat{\theta}} > 0\right] > 0$, so that it obviously satisfies **(12.3.82)**: $V_T^{\hat{\theta}} \geq V_0^{\hat{\theta}}$ and $P\left[V_T^{\hat{\theta}} > V_0^{\hat{\theta}}\right] > 0$.

($\Leftarrow$) Let θ be an admissible portfolio satisfying **(12.3.82)**, then $\tilde{\theta}$ constructed above certainly satisfies that:

(i) $V_0^{\tilde{\theta}} = 0$;

(ii) $V_T^{\tilde{\theta}} = V_t^{\theta} - V_0^{\theta} \geq 0$;

(iii) $P\left[V_t^{\tilde{\theta}} > 0\right] = P\left[V_t^{\theta} - V_0^{\theta} > 0\right] > 0$;

Therefore, $\tilde{\theta}$ is a well-defined arbitrage. □

Exercise. 12.10

Proof. We know $X_t = X_0 \exp\left((\alpha - \frac{1}{2}\beta^2)t + \beta B_t\right)$. As X_t is defined by original *Brownian* motion B_t,

$$E^{X_0}\left[h(X_{T-t})\right] = X_0 e^{\alpha(T-t)} \tag{0.2}$$

Firstly $\frac{\partial}{\partial X_0}E^{X_0}\left[X_{T-t}\right] = e^{\alpha(T-t)}$ exists. Secondly define

$$\phi(t) = e^{\alpha(T-t)}\beta X_t = \beta X_0 e^{\alpha T - \frac{1}{2}\beta^2 t + \beta B_t} \in \mathcal{V}(0,T) \tag{0.3}$$

and it's easy to see that

$$E^{X_0}\left[\int_0^t \phi^2 ds\right] = \beta^2 e^{2\alpha(T-t)} E^{X_0}\left[\int_0^t X_s^2 ds\right] < \infty \tag{0.4}$$

which is confirmed by the property of *Itô* process. Then via *Theorem 12.3.3*,

$$X_T = X_0 \exp\left((\alpha - \frac{1}{2}\beta^2)t + \frac{1}{2}\beta^2 t\right) + e^{\alpha(T-t)}\beta \int_0^T X_t dB_t \tag{0.5}$$

and $z = X_0 e^{\alpha t} \in \mathbb{R}$ is what we need. □

Exercise. M.1

Proof. **(a)** The smallest sets in $\mathcal{F}$ are $A_1 \cap A_2 = \{5,6\}$, $A_1 \setminus A_2 = \{1,3\}$ and $A_2 \setminus A_1 = \{2,4\}$ which is just a partition of Ω. Therefore all sets in $\mathcal{F}$ are:

$\emptyset$
Two Elements:$A_1 \setminus A_2 = \{1,3\}, A_2 \setminus A_1 = \{2,4\}, A_1 \cap A_2 = \{5,6\}$
Four Elements:$A_1 \triangle A_2 = \{1,2,3,4\}, A_1 = \{1,3,5,6\}, A_2 = \{2,4,5,6\}$
Ω

which contains totally eight sets.

(b) As $X(\Omega)$ is $\{-1,2\}$, then the (actually the simplest) $\sigma-$algebra on the range of X is just $\{\emptyset, \{-1\}, \{2\}, \{-1,2\}\}$. After checking one by one:

$$\begin{aligned} X^{-1}(\{-1\}) &= A_2 \in \mathcal{F} \\ X^{-1}(\{2\}) &= A_1 \setminus A_2 \in \mathcal{F} \end{aligned}$$

it's confirmed the preimage of every measurable sets on the range of f is in $\mathcal{F}$, and by definition f is $\mathcal{F}-$measurable.

(c) Just let $X = 1_{\{1\}}$, obviously $X^{-1}(\{1\}) = \{1\}$ and $X^{-1}(\{0\}) = \{2,3,4,5,6\}$ are both not $\mathcal{F}-$measurable. Thus such X is not a $\mathcal{F}-$measurable mapping. □

Exercise. M.2

Proof. **(Approach I)** As $\{B_t\}_{t\geq 0}$ is a *Gaussian* process, the $k-$dimension random vector $Z := (B_{t_1}, ..., B_{t_k})$ obeys $k-$dimension *Gaussian* distribution, $k \geq 1$. Thus with $X_0 := B_0 = 0$, $X_t := tB_{1/t}$, $X := (t_1 B_{t_1^{-1}}, ..., t_k B_{t_k^{-1}})$ also $k-$dimension *Gaussian* random vector, where $t_j > 0$, $1 \leq j \leq k$. So as $\{t_1, ..., t_k\}$ and $k \geq 1$ are both arbitrary, $\{X_t\}_{t\geq 0}$ is also *Gaussian* process. Secondly by the property of *Brownian motion*, show that for any $s, t > 0$,

$$Cov(X_s, X_t) = st \cdot Cov(B_{s^{-1}}, B_{t^{-1}}) = st \cdot \min\{\frac{1}{s}, \frac{1}{t}\} = \min\{s,t\}$$

and when $\min\{s,t\} = 0$, $Cov(X_s, X_t) \equiv 0 = \min\{s,t\}$. Meanwhile for arbitrary $t \geq 0$, $E[X_t] = tE[B_{t^{-1}}] = 0$. Therefore, $\{X_t\}_{t\geq 0}$ is a *Brownian motion.*

(Approach II) Given arbitrary finitely many time intevals $\{(s_i, t_i]\}_{1\leq i\leq n}$ pairwise disjoint, where $s_i, t_i \neq 0$, $\{[\frac{1}{t_i}, \frac{1}{s_i})\}_{i\leq n}$ is also pairewise disjoint, so that $\{X_{t_i} - X_{s_i} = B_{t_i^{-1}} - B_{s_i^{-1}}\}_{i\leq n}$ are independent. Once some s_i or $t_i = 0$, the independence still holds obviously as $X_0 := 0$. Secondly, when $s \neq 0$, $r > 0$,

$$X_{s+r} - X_s = (s+r)B_{(s+r)^{-1}} - sB_{s^{-1}} \sim N(0, (s+r)\lambda - s\lambda) = N(0, r\lambda)$$

and when $s = 0$, $X_r - X_0 = rB_{r^{-1}} \sim N(0, r\lambda)$. At last $\forall \omega \in \Omega$, $tB_{t^{-1}}(\omega)$ is obviously continuous respect to $t \geq 0$. To conclude, $\{X_t\}_{t\geq 0}$ defined above is verified to have independent, stationary and normal distributed increments and continuous trajectory everywhere on Ω, so is a *Brownian motion.* □

Exercise. M.3

Proof. **(a)** Let $0 = t_0 < t_1 < ... < t_n < t_{n+1} = t$, $\triangle_j^p(B_t^q) = (B_{t_{j+1}}^q - B_{t_j}^q)^p$ and $\triangle_j(t) = t_{j+1} - t_j$. In order to make the approximation terms match up, assume γ_j, $j = 0, 1, ..., n$ satisfy that

$$\gamma_j + 4B_{t_j}^3 \triangle_j (B_t) = \triangle_j(B_t^4) - 6B_{t_j}^2 \triangle_j^2 (B_t)$$

Then after simplification we obtain that

$$\gamma_j = [2B_{t_{j+1}}B_{t_j} + B_{t_{j+1}}^2 - 3B_{t_j}^2] \triangle_j^2 (B_t)$$

Then take $j-$summation of both sides of (3), we have

$$\sum_{j=0}^n B_{t_j}^3 \triangle_j (B_t) + \frac{1}{4}\sum_{j=0}^n \gamma_j = \frac{1}{4}B_t^4 - \frac{3}{2}\sum_{j=0}^n B_{t_j}^2 \triangle_j^2 (B_t)$$

In the left side of (5), just define the approximator as $\varphi_n := \sum_{j=0}^n B_{t_j}^3 1_{[t_j,t_{j+1})}$ which is $\mathcal{F}_{t_j}-$measurable, then

$$E\left[\int_0^t (\varphi_n - B_s^3)^2 ds\right] = \sum_{j=0}^n \int_{t_j}^{t_{j+1}} E[(\varphi_n - B_s^3)^2]ds$$

Now as as $\max_{0\le j\le n}\{\triangle_j(t)\} \to 0$, we know firstly $\varphi_n - B_s^3 \xrightarrow{L_1} 0$ and secondly $(\varphi_n - B_s^3)^2$ is dominated by integrable (finite expectation) function $(|\varphi_n| + |B_s|^3)^2$, we can apply *Lebesgue Dominated Convergence Theorem* together with *Itô Isometry* then see that as $n \to 0$,

$$E\left[(\int_0^t \varphi_n dB_s - \int_0^t B_s^3 dB_s)^2\right] = E\left[\int_0^t (\varphi_n - B_s^3)^2 ds\right] \longrightarrow 0 \quad L^2(P)$$

At the same time, since it has term "$\triangle_j^2 (B_t)$", the other term in the left side of (5) satisfies that

$$\sum_{j=0}^n \gamma_j = \sum_{j=0}^n [2B_{t_{j+1}}B_{t_j} + B_{t_{j+1}}^2 - 3B_{t_j}^2] \triangle_j^2 (B_t) \xrightarrow{L_1} 0$$

Hence in the left sides of (5) holds that $\sum_{j=0}^n B_{t_j}^3 \triangle_j (B_t) + \frac{1}{4}\sum_{j=0}^n \gamma_j \longrightarrow \int_0^t B_s^3 dB_s$ in $L^2(P)$ sense.

In the right side of (5), we are to prove $\sum_j B_{t_j}^2 \triangle_j^2 (B_t) \to \int_0^t B_s^2 ds$ in $L^2(P)$ sense. Obviously it holds that

$$\begin{aligned} 2E_1 + 2E_2 &\geq E\left[\left|\sum_{j=0}^n B_{t_j}^2 \triangle_j^2 (B_t) - \int_0^t B_s^2 ds\right|^2\right] \\ E_1 &:= E\left[\sum_{j=0}^n B_{t_j}^2 (\triangle_j^2(B_t) - \triangle_j(t))\right]^2 \\ E_2 &:= E\left[\sum_{j=0}^n B_{t_j}^2 \triangle_j (t) - \int_0^t B_s^2 ds\right]^2 \end{aligned}$$

Regarding to E_1, expand the sqaure into two parts as follows

$$E_1 := E_3 + E_4$$

$$E_3 := E[\sum_{j=0}^n B_{t_j}^4 (\triangle_j^2(B_t) - \triangle_j(t))^2]$$

$$E_4 = 2\sum_{i<j} E_{i,j} := 2\sum_{i<j} E[B_{t_j}^2 B_{t_i}^2 (\triangle_j^2(B_t) - \triangle_j(t))(\triangle_i^2(B_t) - \triangle_i(t))]$$

About E_3, it is easy to see that $\sum_{j=0}^n E[\triangle_j^2(B_t) - \triangle_j(t)]^2 \to 0$,

$$E_3 = \sum_{j=0}^n E[B_{t_j}^4] E[(\triangle_j^2(B_t) - \triangle_j(t))^2]$$

About E_4, based on their integrability we can apply *Cauchy-Schwarz Inequality*,

$$|E_{i,j}| \leq \left(E\left[B_{t_i}^4 \left|\triangle_i^2(B_t) - \triangle_i(t)\right|^2\right]\right)^{\frac{1}{2}} \left(E\left[B_{t_j}^4 \left|\triangle_j^2(B_t) - \triangle_j(t)\right|^2\right]\right)^{\frac{1}{2}}$$

Then by the independent increment of $\{B_t\}_{t\geq 0}$, (9) and (10) above certainly implies that as $n \to \infty$, i.e., $\max_{0\leq j\leq n}\{\triangle_j(t)\} \to 0$, $E_1 = E_3 + E_4 \to 0$.

For E_2 just fix $\omega \in \Omega$, the trajectory $B_s(\omega)$ is $a.s.$ continuous, so take the limit $n \to \infty$, i.e., $\max_{0\leq j\leq n}\{\triangle_j(t)\} \to 0$ in the *Riemann Sum* as below, we can have

$$\sum_{j=0}^n B_{t_j}^2(\omega) \triangle_j (t) \longrightarrow \int_0^t B_s^2(\omega) ds$$

Hence $\sum_{j=0}^n B_{t_j}^2 \triangle_j (t) \longrightarrow \int_0^t B_s^2 ds$ pointwisely on Ω results in that $E_2 \to 0$. So far, we have justified that

$$\sum_{j=0}^n B_{t_j}^2 \triangle_j^2 (B_t) \to \int_0^t B_s^2 ds \quad (L^2(P))$$

Consequently based on all above, take $L^2(P)$ limit of both sides of (5), we finally conclude that

$$\int_0^t B_s^3 dB_s = \frac{1}{4}B_t^4 - \frac{3}{2}\int_0^t B_s^2 ds$$

and the proof ends.

(b) Let $g(t, B_t) = \frac{1}{4}B_t^4$. By the 1-dimensional *Itô Formula*,

$$d(\frac{1}{4}B_t^4) = B_t^3 dB_t + \frac{3}{2}B_t^2(dB_t)^2 = B_t^3 dB_t + \frac{3}{2}B_t^2 dt$$

Therefore $\frac{1}{4}B_t^4 = \int_0^t B_s^3 dB_s + \frac{3}{2}\int_0^t B_s^2 ds$. □

Exercise. M.4

Proof. Firstly we find the expectation of $I_1(t)$ and $I_2(t)$. About $I_2(t)$, we have two ways to show that

$$E\left[I_2(t)\right] = \begin{cases} \int_0^t (EB_s^2)ds = \frac{1}{2}t^2 & B_s \in L^2(P) \\ E\left[(\int_0^t B_s dB_s)^2\right] = E\left[(\frac{1}{2}B_t^2 - \frac{1}{2}t)^2\right] = \frac{1}{2}t^2 & Itô\, Isometry \end{cases}$$

For $I_1(t)$, we also has two ways to calculate its expectation. Let $g(t, B_t) = \frac{1}{3}(B_t + t)^3$, by *Itô Formula,*

$$\frac{1}{3}(B_t + t)^3 = \int_0^t [(B_t + t)^2 + (B_t + t)]dt + I_1(t)$$

and then after simplification, we can have $E[I_1(t)] = 0$. Also we can obtain this for: $(B_s + s)^2 \in \mathcal{V}(0, t)$ implies the *Itô* integral of it has null expectation or, $B_s + s \in L^4(P)$ enables us to switch the integral.

Here I just usually try to get rid of switching the integrals which causes problems frequently.

Meanwhile here to switch the integral is necessary. We definitely know $B_s + s \in L^4(P)$, thus we are allowed to switch the itegrals. Then by $E[I_1(t)] = 0$,

$$Var[I_1(t)] = E\left[\int_0^t (B_s + s)^4 ds\right] = \int_0^t E\left[(B_s + s)^4\right] ds = \frac{1}{5}t^5 + \frac{3}{2}t^4 + t^3$$

To find the variance of $I_2(t)$ is more difficult. Show that

$$E\left[\left(\triangle(t)\sum_i B_{t_i}^2\right)\left(\triangle(t)\sum_j B_{t_j}^2\right)\right] = \triangle^2(t)\sum_{i,j} E\left[B_{t_i}^2 B_{t_j}^2\right] \to E[I_2^2]$$

and simultaneously we also have the double *Rieman sum*'s limit as

$$\triangle^2(t)\sum_{i,j} E\left[B_{t_i}^2 B_{t_j}^2\right] \to \int_0^t \int_0^t E[B_s^2 B_u^2]dsdu$$

By (14) and (15), we know that $E[(I_2(t))^2] = \int_0^t \int_0^t E[B_s^2 B_u^2] dsdu$. According to the independent increment of *Brownian motion* we are able to show that

$$E\left[B_s^2 B_u^2\right] = E\left[B_s^2 (B_t - B_s)^2 + 2B_s^3 (B_s + (B_t - B_s)) - B_s^4\right]$$

Hence we can calculate $E[(I_2(t))^2]$ as below

$$E[(I_2(t))^2] \quad = \quad \int_0^t \int_0^t (s\,|u-s| + 3u^2) dsdu = \frac{7}{12} t^4$$

□

Exercise. M.5

Proof. **(a)** Directly apply *Itô Formula*, we have

$$M_t = \int_0^t (\beta - \frac{1}{2}\alpha^2) e^{\beta t} \cos \alpha B_t dt - \int_0^t \alpha e^{\beta t} \sin \alpha B_t dB_t \tag{0.6}$$

We know that if $M_t = \int_0^t \alpha e^{\beta t} \sin \alpha B_t dB_t$ then M_t is a $\mathcal{F}_t-$martingale. As a result $\beta = \frac{1}{2}\alpha^2$ can be a sufficient condition to make M_t a $\mathcal{F}_t-$martingale.

(b) By the result of **(a)**, $N_t = e^{8t} E[\cos 4B_t]$ is a $\mathcal{F}_t-$martingale which means

$$E\left[\cos 4B_1\right] = e^{-8} E\left[N_1\right] = e^{-8} E\left[N_0\right] = e^{-8}$$

Hence such r.v. $Z := B_1 \sim N(0,1)$ finishes the proof. □