On trajectories of analytic gradient vector fields

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Abstract

Let $f : \mathbf{R}^n, 0 \longrightarrow \mathbf{R}, 0$ be an analytic function defined in a neighbourhood of the origin, having a critical point at 0. We show that the set of non–trivial trajectories of the equation $\dot{x} = \nabla f(x)$ attracted by the origin has the same Čech–Alexander cohomology groups as the real Milnor fibre of f.

1 Introduction

Let $f : \mathbf{R}^n, 0 \longrightarrow \mathbf{R}, 0$ be a smooth function defined in a neighbourhood of the origin, having a critical point at 0. Let ϕ_t denote the flow associated with the differential equation $\dot{x} = \nabla f(x)$. Let W^+ denote the set of points attracted by the origin, i.e. $W^+ = \{x \mid \lim_{t\to\infty} \phi_t(x) = 0\}$.

If f has a non–degenerate critical point at 0 then, according to the Hadamard–Perron theorem, W^+ is locally diffeomorphic to $\mathbf{R}^\lambda$, where λ is the Morse index at the origin. In this paper we investigate the topology of W^+ in the case where f is an arbitrary analytic function. In particular, we do not exclude the case where 0 is not isolated in the set of critical points.

We call $F_- = \{x \mid ||x|| \leq r, f(x) = -a\}$, where $0 < a << r << 1$, the real Milnor fibre of f. It is well known (see [17]) that F_- is either void or an $(n-1)$ – dimensional compact manifold with boundary. Applying the Łojasiewicz inequalities we shall prove that

(i) there is a positive integer N such that if $r > 0$ is small enough then F_- is isotopic to $F_r = \{x \mid ||x|| \leq r, f(x) = -r^{2N}\}$;

(ii) if $x \in W^+ \setminus \{0\}$ then its trajectory $\phi_t(x)$ cuts F_r transversally at exactly one point. Hence there is one–to–one correspondence between non–trivial trajectories in W^+ and $\Gamma_r = W^+ \cap F_r$, and then $(W^+, 0)$ is locally homeomorphic to a cone over Γ_r;

(iii) the inclusion $\Gamma_r \longrightarrow F_r$ induces an isomorphism of the Čech– Alexander cohomology module $\bar{H}^*(\Gamma_r)$ to $H^*(F_r) \cong H^*(F_-)$. Thus the set of non–trivial trajectories of the equation $\dot{x} = \nabla f(x)$ attracted by the origin has the same Čech–Alexander cohomology groups as the real Milnor fibre of f.

The proof is given for an arbitrary analytic function, so it is affected by the fact that 0 might be a non-isolated critical point. This is why it requires techniques and concepts of singularity theory and analytic geometry such as the Łojasiewicz inequality, the Milnor fibre, a conical structure of $f^{-1}(0)$, and especially the fact that the function $f + ||x||^{2N}$ has an isolated critical point at 0 for every positive integer N large enough.

It has been pointed out by the referee that if f has an isolated critical point at 0 then one may get the same cohomological result for a class of singularities more general than analytic ones. This alternative proof follows from the existence of so-called "cylindrical" neighbourhoods as introduced by E. Rothe in [19], and used among others by O. Cornea [4], N. Dancer [5], and F. Takens [25]. In Section 4 we present in detail arguments suggested by the referee.

We should mention that there are several papers [2], [8], [11], [12], [13], [14], [16], [18], [20], [26] devoted to geometric properties of trajectories of analytic gradient vector fields. References [1], [3], [6], [7] [10], [21], [22], [23], [24] present effective methods for computing topological invariants associated with the Milnor fibre.

The authors wish to express their gratitude to the referee for several helpful comments and for suggesting an alternative approach in the case of an isolated singularity.

2 Real analytic functions

Let $f : \mathbf{R}^n, 0 \longrightarrow \mathbf{R}, 0$ be a real analytic function. We shall assume that 0 belongs to the closure of $\{f < 0\}$. (In the other case there are not trajectories attracted by the origin and the Milnor fibre F_- is void.) The Łojasiewicz inequality (see [15]) states that in some neighbourhood U_0 of the origin

$$||\nabla f|| \geq c|f|^\rho$$

for some $c > 0$, $0 < \rho < 1$. Hence if $x \in U_0$ and $\nabla f(x) = 0$ then $f(x) = 0$.

Let $S_r = \{x \mid ||x|| = r\}$. Denote by $Q_r(f)$ (resp. $Q_r(f + ||x||^{2N})$, where N is a positive integer) the set of critical values of $f_{|S_r}$ (resp. $(f + ||x||^{2N})_{|S_r}$). According to [21, p. 411], there are $C,\ \alpha > 0$ such that if $r > 0$ is small enough and $y \in Q_r(f)\setminus\{0\}$ then $|y| \geq Cr^{2\alpha}$. Thus $|y| > 2r^{2N}$ for any positive integer with $2N > 2\alpha$. Since $||x||^{2N}_{|S_r} = r^{2N}$, functions $f_{|S_r}$ and $(f + ||x||^{2N})_{|S_r} = f_{|S_r} + r^{2N}$ have the same set of critical points, and $y \in Q_r(f)$ if and only if $y + r^{2N} \in Q_r(f + ||x||^{2N})$. Thus $0 \notin Q_r(f + ||x||^{2N})$ for all $r > 0$ small enough. In particular, if $\nabla(f + ||x||^{2N}) = 0$ at $x \neq 0$ close to the origin, then $f(x) + ||x||^{2N} \neq 0$, which contradicts the Łojasiewicz inequality. Hence $f + ||x||^{2N}$ has an isolated critical point at the origin.

Let ϕ_t denote the flow associated with the differential equation $\dot{x} = \nabla f(x)$. Let $x \in U_0$ be such a point that $f(x) \leq 0$. The function f is increasing on the trajectory $\phi_t(x)$. Put $b(x) = \sup\{t \mid f(\phi_t(x)) \leq 0\}$.

The Łojasiewicz inequality implies that if $\phi_t(x)$ lies in U_0 for $t \in [u, w]$, then the lenght of the segment of the trajectory between $\phi_u(t)$ and $\phi_w(x)$ is bounded by

$$c_1(|f(\phi_u(x))|^{1-\rho} - |f(\phi_w(x))|^{1-\rho}),$$

where $c_1 = [c(1-\rho)]^{-1}$ (see [16], [13, p. 765]).

Consequently, if x is sufficiently close to the origin then the lenght of the curve $\phi_t(x)$, where $t \in [0, b(x))$, is bounded by $c_1|f(x)|^{1-\rho}$. In particular,

$$\omega(x) = \lim_{t \to b(x)} \phi_t(x)$$

does exist.

Then $\omega(x) \in f^{-1}(0)$ is either the point of intersection of the trajectory and the set $f^{-1}(0)$ (if the intersection is not empty) or the limit point of the trajectory. The length $l(x, \omega(x))$ of the trajectory from x to $\omega(x)$ is bounded by $c_1|f(x)|^{1-\rho}$, which implies that ω is continuous.

Lemma 1. *If N is a sufficiently large positive integer, $x \neq 0$ is sufficiently close to the origin and $-||x||^{2N} \leq f(x) < 0$, then $\omega(x) \neq 0$ and $l(x, \omega(x)) < \frac{1}{2}||x||$. In other words, if $x \neq 0$ and $\omega(x) = 0$ then $f(x) < -||x||^{2N}$.*

Proof. Since $l(x, \omega(x)) \leq c_1|f(x)|^{1-\rho} \leq c_1||x||^{2N(1-\rho)}$, we can choose N such that $2N(1-\rho) > 1$ and then $l(x, \omega(x)) < \frac{1}{2}||x||$ for $x \neq 0$ in some neighbourhood of the origin. In particular, $\omega(x) \neq 0$.

□

Lemma 2. *If N is sufficiently large, $f(x) + ||x||^{2N} = 0$ and $x \neq 0$ is sufficiently close to the origin then the scalar product*

$$\nabla f(x) \cdot \nabla(f(x) + ||x||^{2N}) = \nabla f(x) \cdot (\nabla f(x) + 2N||x||^{2N-2} \cdot x) > 0.$$

Proof. We can choose N such that $2N\rho < 2N - 1$ and then

$$||\nabla f(x)|| \geq c|f(x)|^{\rho} = c||x||^{2N\rho} > 2N||x||^{2N-1}$$

for x sufficiently close to the origin. Thus

$$\nabla f(x) \cdot \nabla(f(x) + ||x||^{2N}) = \nabla f(x) \cdot (\nabla f(x) + 2N||x||^{2N-2} \cdot x) >$$

$$> ||\nabla f(x)||^2 - 2N||\nabla f(x)||\ ||x||^{2N-1} > 0.$$

□

Let us denote $B_r = \{x \mid ||x|| < r\}$, $C_r = \{x \in B_r \mid -2r^{2N} < f(x) < 0\}$ and $D_r = \{x \in S_r \mid -2r^{2N} < f(x) < 0\}$. Then C_r (resp. D_r) is either void or an n-dimensional (resp. $(n-1)$-dimensional) manifold, and $D_r \subset \partial C_r$. Since $\{x \mid -2r^{2N} < f(x) < 0\}$ is open and the pair (B_r, S_r) satisfies Whitney's conditions (see [9], [15]), the pair (C_r, D_r) satisfies Whitney's conditions too.

For N sufficiently large, r_0 sufficiently small and any r such that $0 < r < r_0$, the interval $(-2r^{2N}, 0)$ consists of regular values of $f_{|C_r}$ as well as $f_{|D_r}$. Thom's First Isotopy Lemma (see [9, p. 41]) implies that $f : (C_r, D_r) \to (-2r^{2N}, 0)$ is a trivial fibre bundle. In particular, for any $0 < a << r < r_0$ we have $-2r^{2N} < -r^{2N} < -a < 0$, and then the real Milnor fibre $F_- = \{x \mid ||x|| \leq r,\ f(x) = -a\}$ is isotopic to $F_r = \{x \mid ||x|| \leq r, f(x) = -r^{2N}\}$.

3 Trajectories of gradient vector fields

Lemma 3. *If $g : [0, b) \longrightarrow \mathbf{R}$ is a differentiable mapping such that $g'(t) > 0$ if $g(t) = 0$, then $g^{-1}(0)$ consists at most of one point.* □

Lemma 4. *For N sufficiently large and $r > 0$ sufficiently small the trajectory $\phi_t(x)$ going through $x \in F_r$ can intersect the set $Z = \{x \mid f(x) + ||x||^{2N} = 0\}$ at most once, and then is transversal to it. If that is the case then the point of intersection is not 0 and $\omega(x) \neq 0$.*

Proof. Choose $r > 0$ such that if $||x|| \leq r$ then all the points on the trajectory $\phi_t(x)$ are so close to the origin that Lemma 2 holds. Then the trajectory $\phi_t(x)$ can intersect the set Z only transversally. Let us define $g(t) = f(\phi_t(x)) + ||\phi_t(x)||^{2N}$. By Lemma 2, we have

$$g'(t) = \nabla(f(\phi_t(x)) + 2N||\phi_t(x)||^{2N}\phi_t(x)) \cdot \nabla f(\phi_t(x)) > 0$$

if $g(t) = 0$. So Lemma 3 implies that the trajectory $\phi_t(x)$ can intersect the set Z at most once. If that is the case, let x' denote the intersection point. From Lemma 1, $\omega(x') \neq 0$, and then $\omega(x) = \omega(x') \neq 0$.

□

Lemma 5. *Assume that $x \neq 0$ lies close to the origin and $f(x) \leq -||x||^{2N}$. If the trajectory $\phi_t(x)$ does not intersect the set Z then $\omega(x) = 0$.*

Proof. If $\omega(x) \neq 0$ then we have $f(x) + ||x||^{2N} \leq 0$ and $f(\omega(x)) + ||\omega(x)||^{2N} = ||\omega(x)||^{2N} > 0$, so by the Darboux property there exists such a point x' on the trajectory that $f(x') + ||x'||^{2N} = 0$. So $x' \in Z$. Hence the trajectory intersects the set Z.

□

Assume that $f(x) \leq -||x||^{2N}$. Let $\gamma(x) \in Z$ be either the point of intersection of the trajectory $\phi_t(x)$ and the set Z (if the intersection is not empty) or the limit point of the trajectory. From Lemmas 4, 5, $\gamma(x) \neq 0$ and $\omega(x) \neq 0$ in the first case, $\gamma(x) = \omega(x) = 0$ in the second one. Thus we have $\gamma^{-1}(0) \cap F_r = \omega^{-1}(0) \cap F_r$.

Let N and r be such that all the above facts are true. We may also assume that $Z_r = \{x \mid f(x) + ||x||^{2N} = 0, ||x|| \leq r\} = Z \cap \bar{B}_r$ is homeomorphic to a cone over $\partial Z_r = Z \cap S_r$ (see [17]). We have $F_r \subset \{x \mid f(x) + ||x||^{2N} \leq 0, ||x|| \leq r\}$.

Let us define

$$V_r := \{x \mid ||x|| < r \text{ and } -r^{2N} < f(x) < -||x||^{2N}\}.$$

V_r is open and bounded. It is easy to see that $\partial F_r = \partial Z_r = \{x \mid f(x) = -r^{2N}, ||x|| = r\}$, so $F_r \cup Z_r = \partial V_r$. Hence $F_r \cup Z_r$ is the boundary of V_r and $\nabla f \neq 0$ for $x \in V_r$. Denote $\gamma_r = \gamma|_{F_r}$ and $\Gamma_r = \gamma_r^{-1}(0)$.

Theorem 6. *For N sufficiently large and $r > 0$ sufficiently small there is one-to-one correspondence between Γ_r and the set of non-trivial trajectories attracted by the origin.*

Proof. Let $\phi_t(x)$ be a non-trivial trajectory with $\omega(x) = 0$. From Lemma 1, the trajectory enters into V_r and it does not intersect Z_r, so it must intersect F_r. Let $x' \in F_r$ be the point of intersection. Of course, $\gamma_r(x') = \omega(x) = 0$.

On the other hand, if $\gamma_r(x) = 0$ then $\phi_t(x)$ is a non-trivial trajectory with $\omega(x) = 0$.

□

The above theorem allows us to equip the set of non-trivial trajectories attracted by the origin with the topology induced from Γ_r. In the remainder of the paper we shall show that this space has the same Čech-Alexander cohomology groups as F_-.

Lemma 7. *$\gamma_r : F_r \longrightarrow Z_r$ is a continuous function.*

Proof. First we will show that γ_r is well defined. If $x \in F_r$ then $\gamma_r(x) \in Z$. It is enough to see that $\gamma_r(x) \in Z_r$.

For $x \in F_r$ we have $f(x) = -r^{2N}$ and f is increasing on the trajectory $\phi_t(x)$, so the trajectory enters into the set V_r, and $\partial V_r = F_r \cup Z_r$. Since it cannot intersect F_r for the second time, and f has no critical points in V_r, $\gamma_r(x) \in Z_r$.

Let $x \in F_r$ be such that $\gamma_r(x) \neq 0$. The trajectory $\phi_t(x)$ is transversal to both F_r and Z_r, so γ_r is the Poincaré mapping in some neighbourhood of x. Hence γ_r is continuous at x.

Let $x \in F_r$ be such that $\gamma_r(x) = 0$. From Lemma 5, $\omega(x) = 0$. Let us assume that γ_r is not continuous at x, so there exists a sequence $(x_n) \subset F_r$ and $\delta > 0$ such that $x_n \to x$ and $||\gamma_r(x_n)|| > \delta$.

From Lemma 1 we have $l(\gamma_r(x_n), \omega(x_n)) \leq \frac{1}{2}||\gamma_r(x_n)||$. Hence

$$||\omega(x_n)|| \geq ||\gamma_r(x_n)|| - ||\gamma_r(x_n) - \omega(x_n)|| \geq \frac{1}{2}||\gamma_r(x_n)|| \geq \frac{1}{2}\delta.$$

It contradicts $\omega(x_n) \to \omega(x) = 0$. So γ_r is continuous at x.

□

Lemma 8. $\gamma(F_r) = Z_r$.

Proof. We will show that $Z_r \subset \gamma_r(F_r)$. Let $y \in Z_r \setminus \{0\}$. We consider the trajectory $\phi_{-t}(y)$. The function f is decreasing on the trajectory. The trajectory goes through the bounded set V_r whose boundary is $F_r \cup Z_r$. It cannot intersect Z_r twice, so it has to intersect F_r. Then there exists a point x of the trajectory such that $x \in F_r$. Hence $y = \gamma_r(x) \in \gamma_r(F_r)$.

Assume that $\gamma_r(x) \neq 0$ for every $x \in F_r$. Then $\gamma_r(F_r) = Z_r \setminus \{0\}$, but F_r is compact and γ_r is continuous, so it contradicts $\gamma_r(F_r)$ being compact. Hence $\gamma_r(F_r) = Z_r$.

□

We have $\nabla f(x) \cdot \nabla(f(x) + ||x||^{2N}) > 0$ for $x \in Z_r, x \neq 0$, so different trajectories with $\gamma_r(x) \neq 0$ cannot intersect Z_r at the same point. We will show that $\gamma_r : F_r \setminus \Gamma_r \longrightarrow Z_r \setminus \{0\}$ is a homeomorphism.

Let us define $\psi : Z_r \setminus \{0\} \longrightarrow F_r \setminus \Gamma_r$ such that $\psi(y)$ is the point of intersection of the trajectory $\phi_{-t}(y)$ and the set F_r. The mapping ψ is well defined and continuous because it is a Poincaré mapping in some neighbourhood of $y \in Z_r \setminus \{0\}$. Of course $\psi\gamma_r|_{F_r\setminus\Gamma_r} = \mathrm{id}_{F_r\setminus\Gamma_r}$ and $\gamma_r|_{F_r\setminus\Gamma_r}\psi = \mathrm{id}_{Z_r\setminus\{0\}}$.

Lemma 9. *For any open neighbourhood U of Γ_r in F_r the image $\gamma_r(U)$ is open in Z_r.*

Proof. U is open, so $F_r \setminus U$ is compact. Then $\gamma_r(F_r \setminus U)$ is compact in Z_r, so $Z_r \setminus \gamma_r(F_r \setminus U)$ is open in Z_r. It is enough to show that $Z_r \setminus \gamma_r(F_r \setminus U) = \gamma_r(U)$.

We have $Z_r \setminus \gamma_r(F_r \setminus U) \subset \gamma_r(U)$ because $\gamma_r(F_r) = Z_r$. Let $y \in \gamma_r(U)$. If $y = 0$ then $y \notin \gamma_r(F_r \setminus U)$ because $\Gamma_r \subset U$. If $y \neq 0$ then $y = \gamma_r(x)$ for some $x \in U \setminus \Gamma_r$. But $\gamma_r|_{F_r\setminus\Gamma_r}$ is a homeomorphism, so $y \notin \gamma_r(F_r \setminus U)$.

□

Lemma 10. *There is a descending family $F_r = U_1 \supset U_2 \supset \ldots$ of open neighbourhoods of Γ_r in F_r such that*

(i) every inclusion $U_{n+1} \subset U_n$ is a homotopy equivalence, so that the induced homomorphisms $H^*(U_n) \longrightarrow H^*(U_{n+1})$ and $H^*(F_r) \longrightarrow H^*(U_n)$ are isomorphisms;

(ii) for every open neighbourhood U of Γ_r in F_r there is n such that $U_n \subset U$.

Proof. Since Z_r is homeomorphic to a cone with the vertex at 0, there is a descending family $Z_r = W_1 \supset W_2 \supset \ldots$ of open neighbourhoods of 0 such that every inclusion $W_{n+1} \subset W_n$ is a homotopy equivalence, and for every open neighbourhood W of 0 in Z_r there is n such that $W_n \subset W$. Set $U_n = \gamma_r^{-1}(W_n)$. Each U_n is an open neighbourhood of Γ_r.

$\gamma_r : F_r \setminus \Gamma_r \to Z_r \setminus \{0\}$ is a homeomorphism, hence *(i)* holds. Let U be an open neighbourhood of Γ_r in F_r. Frome Lemma 9, $\gamma_r(U)$ is an open neighbourhood of 0 in Z_r, and then there is n with $W_n \subset \gamma_r(U)$. Hence $U_n \subset U$.

□

Theorem 11. *The Čech-Alexander cohomology modules $\bar{H}^*(F_r) = H^*(F_r)$ and $\bar{H}^*(\Gamma_r)$ are isomorphic.*

Proof. The family $U_1 \supset U_2 \supset \ldots$ described in Lemma 10 is cofinal in the family of all open neighbourhoods of Γ_r in F_r with the natural ordering induced by "$\supseteq$", i.e. $U \leq U'$ if $U \supseteq U'$. By Lemma 10 we have an isomorphism of direct limits

$$\bar{H}^*(\Gamma_r) = \varinjlim_{U} H^*(U) \cong \varinjlim_{U_n} H^*(U_n) \cong H^*(F_r)$$

□

We have already proved, that F_r is isotopic to the real Milnor fibre F_-. From Theorems 6, 11 we get

Theorem 12. *Let $f : \mathbf{R}^n, 0 \longrightarrow \mathbf{R}, 0$ be a real analytic function defined in some neighbourhood of the origin. The set of non-trivial trajectories of the equation $\dot{x} = \nabla f(x)$ attracted by the origin has the same Čech–Alexander cohomology groups as the real Milnor fibre F_-.*

□

Example. Let $f(x, y, z) = xyz^2 - x^2z - y^2z^2 - x^3 - 2y^4$. Since the z-axis consists of critical points of f, the origin is a non-isolated critical point. It is easy to verify that the point $(0, 0, r)$, for each $r > 0$ small enough, is isolated in $A_r = \{x \in S_r \mid f(x) \geq 0\}$, $A_r \setminus \{(0, 0, r)\} \neq \emptyset$, and so A_r has at least two components. The Alexander duality theorem implies that $H^1(S_r \setminus A_r)$ is non-trivial. It is well known (see [17, p. 53]) that F_- has the same homotopy type as $S_r \setminus A_r$, so that $H^1(F_-) \cong \bar{H}^1(\Gamma_r) \neq 0$. In particular, the set of trajectories attracted by the origin is infinite.

4 The case of an isolated critical point

Consider a smooth function $f : \mathbf{R}^n, 0 \longrightarrow \mathbf{R}, 0$ with an isolated critical point at the origin. For $\epsilon,\ r > 0$ consider the neighbourhoods

$$U(\epsilon, r) = \{x \in f^{-1}[-\epsilon, \epsilon] \mid \overline{\phi_R(x)} \cap f^{-1}(0) \in \bar{B}_r\},$$

where $\phi_R(x)$ is the orbit of the gradient flow ϕ_t of f through x, $\bar{A}$ is the closure of A. For sufficiently small ϵ, r_0 and for $r \le r_0$ all the neighbourhoods $U(\epsilon, r)$ are compact. This follows because for sufficiently small ϵ all the flow lines passing through points $S_r \cap f^{-1}(0)$ cross the hypersurfaces $f^{-1}(\pm\epsilon)$. Let $T_r = U(\epsilon, r) \cap f^{-1}(-\epsilon)$.

Instead of (and more general than) analyticity assume that the singularity has the property that there exists r_1 such that for all $r \le r_1$ the intersection of the sphere S_r and $f^{-1}(0)$ is transverse. Then $\bar{B}_{r_1} \cap f^{-1}(0)$ has a conical structure and for such a small ϵ and $r \le r_1$ all pairs $(U(\epsilon, r), T_r)$ have the same homeomorphism type. Using an invertible cobordism argument one may prove that T_r has the same homotopy type as F_- which also agrees with the homotopy type of the sublink of the singularity $f^{-1}(-\infty, 0] \cap S_{r_1}$. Of course, the sets T_r form a cofinal system of neighbourhoods of the set of points in $f^{-1}(-\epsilon)$ which are attracted by the origin and this set is in bijection with the set of trajectories having the origin as ω-limit. This clearly implies the claimed result.

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