

If I were a rich density

Rafał Filipów



Set-theoretic methods in topology and real functions theory
(2019)

The talk is based on a joint work with **Jacek Tryba** published in a paper “*Densities for sets of natural numbers vanishing on a given family*”, J. of Number Theory 211 (2020), 371-382

Abstract upper density

Examples of upper densities

- Asymptotic density: $\bar{d}(A) = \limsup_{n \rightarrow \infty} \frac{|A \cap \{1, \dots, n\}|}{n}$
- Logarithmic density: $\bar{\delta}(A) = \limsup_{n \rightarrow \infty} \frac{\sum_{k \in A \cap \{1, \dots, n\}} \frac{1}{k}}{\sum_{k \leq n} \frac{1}{k}}$
- Uniform density (aka Banach density):
 $\bar{u}(A) = \limsup_{n \rightarrow \infty} \max_{k \in \mathbb{N}} \frac{|A \cap \{k+1, \dots, k+n\}|}{n}$

Definition

An **abstract upper density** on \mathbb{N} is a function $\delta : \mathcal{P}(\mathbb{N}) \rightarrow [0, 1]$ that satisfies the following properties:

- 1 $\delta(\mathbb{N}) = 1$,
- 2 if $F \subseteq \mathbb{N}$ is finite then $\delta(F) = 0$,
- 3 if $A \subseteq B$ then $\delta(A) \leq \delta(B)$,
- 4 $\delta(A \cup B) \leq \delta(A) + \delta(B)$.

Proposition

If $\delta : \mathcal{P}(\mathbb{N}) \rightarrow [0, 1]$ is an abstract upper density, then

$$\mathcal{Z}_\delta = \{A \subseteq \mathbb{N} : \delta(A) = 0\}$$

is an **ideal on \mathbb{N}** i.e.

- 1 if $A, B \in \mathcal{Z}_\delta$ then $A \cup B \in \mathcal{Z}_\delta$,
- 2 if $A \subseteq B$ and $B \in \mathcal{Z}_\delta$ then $A \in \mathcal{Z}_\delta$,
- 3 \mathcal{Z}_δ contains all finite subsets of \mathbb{N} ,
- 4 $\mathbb{N} \notin \mathcal{Z}_\delta$.

Abstract upper densities and ideals

Proposition

Let \mathcal{I} be an ideal. The function $\delta : \mathcal{P}(\mathbb{N}) \rightarrow [0, 1]$ given by

$$\delta(A) = \begin{cases} 0 & \text{if } A \in \mathcal{I}, \\ 1 & \text{otherwise} \end{cases}$$

is an abstract upper density and $\mathcal{I} = \mathcal{Z}_\delta$.

Proof

Straightforward.

Question (G. Gerkos, 2013)

Let \mathcal{I} be an ideal. Does there is a “nice” abstract upper density δ such that $\mathcal{Z}_\delta = \mathcal{I}$, where “nice” would mean the properties of the familiar densities consider in number theory?

Nice = translation invariance

Definition

- **Translation invariant density:** $\delta(A + k) = \delta(A)$ for all A and k
- **Translation invariant ideal:** $A + k \in \mathcal{I}$ for all $A \in \mathcal{I}$ and k

Proposition

Let \mathcal{I} be a translation invariant ideal. The function

$$\delta(A) = \begin{cases} 0 & \text{if } A \in \mathcal{I}, \\ 1 & \text{otherwise} \end{cases}$$

is a translation invariant abstract upper density and $\mathcal{I} = \mathcal{Z}_\delta$.

Proof

Straightforward.

Definition

Rich density: for every $r \in [0, 1]$ there is $A \subseteq \mathbb{N}$ with $\delta(A) = r$.

Theorem (M. Di Nasso–R. Jin, 2018)

If \mathcal{I} is a summable ideal then there is a rich abstract upper density δ with $\mathcal{I} = \mathcal{Z}_\delta$.

Definition of a summable ideal

There is $f : \mathbb{N} \rightarrow [0, \infty)$ such that

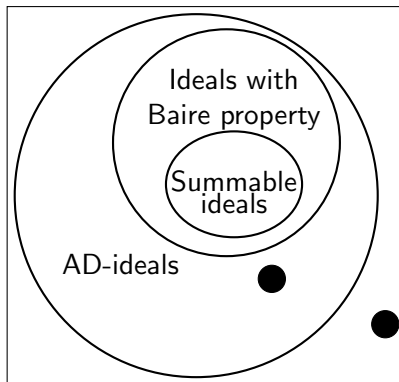
$$\mathcal{I} = \{A \subseteq \mathbb{N} : \sum_{n \in A} f(n) < \infty\}.$$

$\text{Fin} = \{A : A \text{ is finite}\}$ and $\mathcal{I}_{1/n} = \{A : \sum_{n \in A} \frac{1}{n} < \infty\}$ are summable ideals.

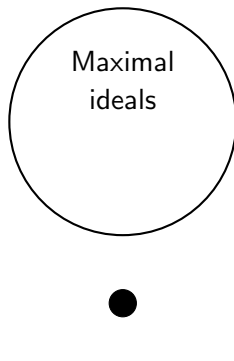
Nice = richness

What we know

THERE IS A RICH DENSITY



THERE IS NO RICH DENSITY



Nice = richness; case of AD-ideals

Definition

$\mathcal{A} \subseteq \mathcal{P}(\mathbb{N})$ is **\mathcal{I} almost disjoint family** (**\mathcal{I} -AD family**) if $A \notin \mathcal{I}$ and $A \cap B \in \mathcal{I}$ for any distinct $A, B \in \mathcal{A}$.

Theorem

If there exists an \mathcal{I} -AD family of cardinality \mathfrak{c} , then there is a rich abstract upper density δ such that $\mathcal{Z}_\delta = \mathcal{I}$.

Sketch of the proof

- Extend \mathcal{A} to a maximal \mathcal{I} -AD-family.
- Enumerate: $\mathcal{A} = \{A_\alpha : \alpha < \mathfrak{c}\}$ and $(0, 1) = \{r_\alpha : \alpha < \mathfrak{c}\}$.
- Define: $\delta(A) = \sup\{r_\alpha : A_\alpha \cap A \notin \mathcal{I}\}$
- δ is an abstract upper density
- δ is rich [use \mathcal{I} -almost disjointness of \mathcal{A}]
- $\mathcal{Z}_\delta = \mathcal{I}$ [use maximality of \mathcal{A}]

Corollary

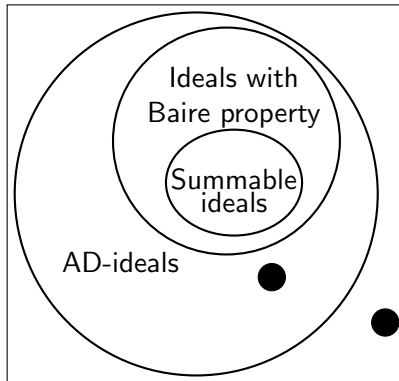
If \mathcal{I} has the Baire property, then there is a rich abstract upper density δ such that $\mathcal{Z}_\delta = \mathcal{I}$. In particular, summable ideals.

Sketch of the proof

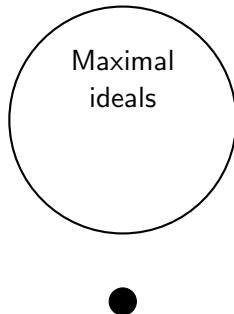
- Talagrand: $\exists k_1 < k_2 < \dots (\exists_n^\infty [k_n, k_{n+1}) \subseteq A \implies A \notin \mathcal{I}$
- Let: $I_n = [k_n, k_{n+1})$
- Take: Fin-AD family $\mathcal{A} \subseteq \mathcal{P}(\mathbb{N})$ of cardinality \mathfrak{c} .
- Define: $C_A = \bigcup_{n \in A} I_n$ for every $A \in \mathcal{A}$
- Let: $\mathcal{C} = \{C_A : A \in \mathcal{A}\}$
- \mathcal{C} is of cardinality \mathfrak{c} [because \mathcal{A} has cardinality \mathfrak{c}]
- $C_A \in \mathcal{I}^+$ for every $A \in \mathcal{A}$ [use Talagrand's characterization]
- \mathcal{C} is \mathcal{I} -AD family [use \mathcal{I} -almost disjointness of \mathcal{A}]

Nice = richness: what we know

THERE IS A RICH DENSITY



THERE IS NO RICH DENSITY



Example

Let

- \mathcal{I} be a maximal ideal,
- $\mathcal{I} = \{\emptyset\} \otimes \mathcal{J}$,
- $\delta(A) = \sum_{A_n \notin \mathcal{J}} \frac{1}{2^n}$.

Then

- \mathcal{I} is an ideal,
- δ is an abstract upper density,
- δ is rich [use binary expansion],
- $\mathcal{Z}_\delta = \mathcal{I}$,
- there is no \mathcal{I} -AD family of cardinality \mathfrak{c} (even uncountable) [use maximality of \mathcal{J}].

Example

Let

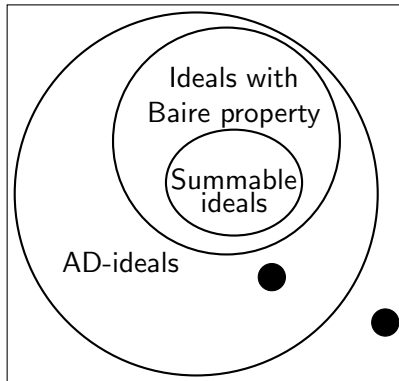
- \mathcal{J} be a maximal ideal,
- $\mathcal{I} = \text{Fin} \otimes \mathcal{J}$.

Then

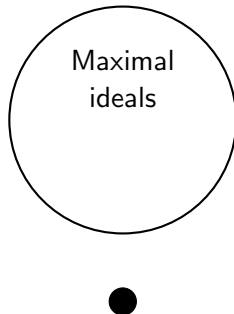
- \mathcal{I} is an ideal,
- there is \mathcal{I} -AD family of cardinality \mathfrak{c} [use Fin -AD family of cardinality \mathfrak{c}]
- \mathcal{I} does not have the Baire property [use maximality of \mathcal{J} and Plewik theorem saying that intersection of countably many maximal ideals does not have the Baire property]

Nice = richness: what we know

THERE IS A RICH DENSITY



THERE IS NO RICH DENSITY



Theorem

If \mathcal{I} is a maximal ideal, then there is no rich abstract upper density δ with $\mathcal{Z}_\delta = \mathcal{I}$.

Proof

- If $A \in \mathcal{I}$ then $\delta(A) = 0$.
- If $A \notin \mathcal{I}$, then $\mathbb{N} \setminus A \in \mathcal{I}$ [use maximality]
- hence

$$1 = \delta(\mathbb{N}) \leq \delta(A) + \delta(\mathbb{N} \setminus A) = \delta(A) + 0 = \delta(A) \leq 1$$

- so $\delta(A) = 1$.
- Thus δ takes only 2 values, so it is not rich.

Example

Let

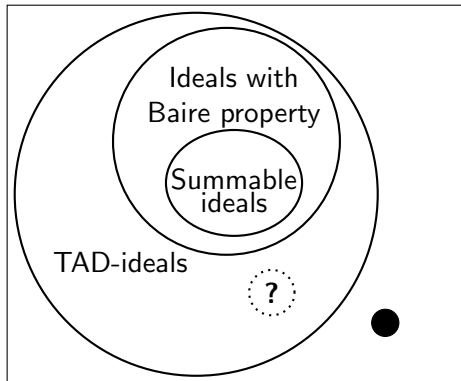
- $\mathcal{I}_1, \mathcal{I}_2$ be maximal ideals,
- $\mathcal{I} = \mathcal{I}_1 \oplus \mathcal{I}_2$.

Then

- \mathcal{I} is an ideal
- \mathcal{I} is non-maximal [since $\mathcal{P}(\mathbb{N}) \oplus \mathcal{I}_2$ is a larger ideal]
- If $A \notin \mathcal{I}$, then $A \cap (\{1\} \times \mathbb{N}) \notin \mathcal{I}_1$ or $A \cap (\{2\} \times \mathbb{N}) \notin \mathcal{I}_2$.
- Say $A \cap (\{1\} \times \mathbb{N}) \notin \mathcal{I}_1$.
- Then $\delta(A \cap (\{1\} \times \mathbb{N})) = \delta(\{1\} \times \mathbb{N})$ [since \mathcal{I}_1 is maximal]
- Since $\delta(A) \geq \delta(A \cap (\{1\} \times \mathbb{N})) = \delta(\{1\} \times \mathbb{N}) \neq 0$, so
- $\text{ran}(\delta) = \{0\} \cup [\delta(\{1\} \times \mathbb{N}), 1]$.
- Thus δ is not rich.

Nice = richness + translation invariance: what we (don't) know

THERE IS A RICH TRANSLATION
INVARIANT DENSITY



THERE IS NO RICH TRANS.
INVARIANT DENSITY



Nice = richness + translation invariance: non TAD-ideal with density

Example

Let

- Let \mathcal{J} be a maximal ideal,
- $\delta(A) = \mathcal{J}\text{-}\lim \frac{|A \cap \{1, \dots, n\}|}{n}$

Then

- δ is well defined [since \mathcal{J} is maximal]
- δ is an abstract upper density (it is finitely additive measure)
- δ is translation invariant.
- δ is rich [since asymptotic density is so]
- Let $\mathcal{I} = \mathcal{Z}_\delta$.
- There is no \mathcal{I} -AD family of cardinality \mathfrak{c} [since finite measures satisfy ccc]