

Two \mathfrak{b} or not two \mathfrak{b} ?

—

cardinal numbers *inspired* by convergence of sequences of functions

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Inspirations in Real Analysis (2022)

The talk is based on a joint work with **Adam Kwela**.

Quasinormal convergence of sequences of functions

$$f, f_n : X \rightarrow \mathbb{R}, n = 1, 2, \dots$$

Pointwise convergence

$$(f_n) \xrightarrow{P} f \iff \forall \varepsilon > 0 \forall x \in X \exists k \forall n > k (|f_n(x) - f(x)| < \varepsilon)$$

Quasinormal convergence

$$(f_n) \xrightarrow{QN} f \iff \exists (\varepsilon_n) \rightarrow 0 \forall x \in X \exists k \forall n > k (|f_n(x) - f(x)| < \varepsilon_n)$$

Relationship between pointwise and quasinormal convergence

- $(f_n) \xrightarrow{QN} f \implies (f_n) \xrightarrow{P} f$ [always “yes”]
- $(f_n) \xrightarrow{P} f \not\implies (f_n) \xrightarrow{QN} f$ [in general “no”]

E.g. if $X = \mathbb{R}$, there are even continuous functions f_n and f such that $(f_n) \xrightarrow{P} f$ but $(f_n) \not\xrightarrow{QN} f$

Definition (Bukovský-Reclaw-Repický, 1991)

X is a **QN-space** if

$$(f_n) \xrightarrow{P} f \implies (f_n) \xrightarrow{QN} f$$

for any continuous functions $f, f_n : X \rightarrow \mathbb{R}$

Theorem (Bukovský-Reclaw-Repický, 1991)

- continuous image of a QN-space is a QN-space
- QN-spaces are not hereditary,
- ... many properties of QN-spaces

Minimal size of non-QN-spaces

Definition

X is a QN-space if

$$(f_n) \xrightarrow{P} f \implies (f_n) \xrightarrow{QN} f$$

for any continuous functions $f, f_n : X \rightarrow \mathbb{R}$

Definition (Bukovský-Reclaw-Repický, 1991)

$$\text{non(QN-space)} = \min\{|X| : X \text{ in } \underline{\text{not}} \text{ a QN-space}\}$$

Theorem

$$\aleph_0 < \text{non(QN-space)} \leq \mathfrak{c}$$

Purely combinatorial characterization of non(QN-space)

$$\text{non}(\text{QN-space}) = \min\{|X| : X \text{ in } \underline{\text{not}} \text{ a QN-space}\}$$

Theorem (Bukovska (\geq); Bukovský-Reclaw-Repický (\leq); 1991)

$$\text{non}(\text{QN-space}) = \mathfrak{b}$$

Definition (The bounding number)

$$\begin{aligned} \mathfrak{b} &= \min \left\{ |\mathcal{F}| : \mathcal{F} \subseteq \mathbb{N}^{\mathbb{N}} \wedge \mathcal{F} \text{ is unbounded from above} \right. \\ &\quad \left. \text{with respect to the ordering } \leq^* \right\} \\ &= \min \left\{ |\mathcal{F}| : \neg \left(\exists (b_n) \in \mathbb{N}^{\mathbb{N}} \forall (a_n) \in \mathcal{F} ((a_n) \leq^* (b_n)) \right) \right\}, \end{aligned}$$

where $(a_n) \leq^* (b_n) \iff \exists k \forall n > k (a_n \leq b_n)$.

Definition

A family $\mathcal{I} \subseteq \mathcal{P}(\mathbb{N})$ is an **ideal** on \mathbb{N} if

- 1 $\emptyset \in \mathcal{I}$ and $\mathbb{N} \notin \mathcal{I}$,
- 2 $A \subseteq B \in \mathcal{I} \implies A \in \mathcal{I}$,
- 3 $A, B \in \mathcal{I} \implies A \cup B \in \mathcal{I}$,
- 4 \mathcal{I} contains all finite subsets of \mathbb{N} .

Example

- 1 $\text{Fin} = \{A \subseteq \mathbb{N} : A \text{ is finite}\}$
- 2 $\mathcal{I}_{1/n} = \left\{ A \subseteq \mathbb{N} : \sum_{n \in A} \frac{1}{n} < \infty \right\}$
- 3 $\mathcal{I}_d = \left\{ A \subseteq \mathbb{N} : \lim_{n \rightarrow \infty} \frac{|A \cap \{1, \dots, n\}|}{n} = 0 \right\}$

Ideal convergence

Definition

A sequence (x_n) of reals is **convergent** to $a \in \mathbb{R}$ if

$$\forall \varepsilon > 0 \exists k \forall n > k (|x_n - a| < \varepsilon)$$

equivalently:

$$\forall \varepsilon > 0 \exists A \in \text{Fin} \forall n \in \mathbb{N} \setminus A (|x_n - a| < \varepsilon).$$

Definition

Let \mathcal{I} be an ideal on \mathbb{N} .

A sequence (x_n) of reals is **\mathcal{I} -convergent** to $a \in \mathbb{R}$ if

$$\forall \varepsilon > 0 \exists A \in \mathcal{I} \forall n \in \mathbb{N} \setminus A (|x_n - a| < \varepsilon).$$

In symbols: $(x_n) \xrightarrow{\mathcal{I}} a$

Ideal pointwise and quasinormal convergence

Let $\mathcal{I}, \mathcal{J}, \mathcal{K}$ be ideals on \mathbb{N} .

Let $f, f_n : X \rightarrow \mathbb{R}$, $n = 1, 2, \dots$

Ideal pointwise convergence

$$(f_n) \xrightarrow{\mathcal{K}} f \iff \forall x \in \mathbb{R} \left((f_n(x)) \xrightarrow{\mathcal{K}} f(x) \right) \iff$$

$$\forall x \in X \left(\forall \varepsilon > 0 \exists A \in \mathcal{K} \forall n \in \mathbb{N} \setminus A (|f_n(x) - f(x)| < \varepsilon) \right)$$

Ideal quasinormal convergence

$$(f_n) \xrightarrow{QN(\mathcal{I}, \mathcal{J})} f \iff$$

$$\exists (\varepsilon_n) \xrightarrow{\mathcal{J}} 0 \forall x \in X \exists A \in \mathcal{I} \forall n \in \mathbb{N} \setminus A (|f_n(x) - f(x)| < \varepsilon_n)$$

In general there are no relations:

- $(f_n) \xrightarrow{QN(\mathcal{I}, \mathcal{J})} f \not\Rightarrow (f_n) \xrightarrow{\mathcal{K}} f$
- $(f_n) \xrightarrow{\mathcal{K}} f \not\Rightarrow (f_n) \xrightarrow{QN(\mathcal{I}, \mathcal{J})} f$

When does ideal quasinormal convergence imply pointwise?

Theorem (Staniszewski, 2017)

The following conditions are equivalent:

- 1 $(f_n) \xrightarrow{QN(\mathcal{I}, \mathcal{J})} f \implies (f_n) \xrightarrow{\mathcal{K}} f$
- 2 $\mathcal{I} \subseteq \mathcal{K}$ and $\mathcal{J} \subseteq \mathcal{K}$.

- The above characterization is expressed without mentioning the cardinality of X .
- A characterization of the reversed implication:

$$(f_n) \xrightarrow{\mathcal{K}} f \implies (f_n) \xrightarrow{QN(\mathcal{I}, \mathcal{J})} f$$

will depend on the cardinality of X as it was in the ordinary case.

Minimal size of non ideal-QN-space

Definition

X is a $\text{QN}(\mathcal{I}, \mathcal{J}, \mathcal{K})$ -space if

$$(f_n) \xrightarrow{\mathcal{K}} f \implies (f_n) \xrightarrow{\text{QN}(\mathcal{I}, \mathcal{J})} f$$

for any continuous functions $f, f_n : X \rightarrow \mathbb{R}$

Definition

$\text{non}(\text{QN}(\mathcal{I}, \mathcal{J}, \mathcal{K})\text{-space}) =$

$$\min(\{|X| : X \text{ in } \underline{\text{not}} \text{ a } \text{QN}(\mathcal{I}, \mathcal{J}, \mathcal{K})\text{-space}\} \cup \{\infty\})$$

Theorem (Staniszewski, 2017)

There are ideals for $\mathcal{I}, \mathcal{J}, \mathcal{K}$ with $\text{non}(\text{QN}(\mathcal{I}, \mathcal{J}, \mathcal{K})\text{-space}) = \infty$
i.e. for any space X every \mathcal{K} -pointwise convergent sequence
 $f_n : X \rightarrow \mathbb{R}$ is automatically $\text{QN}(\mathcal{I}, \mathcal{J})$ -convergent.

Theorem (Staniszewski, 2017)

- If $\mathcal{K} \subseteq \mathcal{I}$, then

$$\text{non}(\text{QN}(\mathcal{I}, \mathcal{J}, \mathcal{K})\text{-space}) = \mathfrak{b}_S(\mathcal{I}, \mathcal{J}, \mathcal{K}),$$

where $\mathfrak{b}_S(\mathcal{I}, \mathcal{J}, \mathcal{K})$ is a cardinal defined in a purely combinatorial manner (we'll see the definition in a moment).

- If $\mathcal{K} \not\subseteq \mathcal{I}$, there are some partial results about $\text{non}(\text{QN}(\mathcal{I}, \mathcal{J}, \mathcal{K}))$ but mainly for ideals with the hereditary Baire property (thus: there is still something to explore).

The cardinal $\mathfrak{b}_S(\mathcal{I}, \mathcal{J}, \mathcal{K})$

Notation

$\mathcal{P}_{\mathcal{I}}$ is the family of all partitions $\{A_n : n \in \mathbb{N}\}$ of \mathbb{N} such that $A_n \in \mathcal{I}$ for each $n \in \omega$.

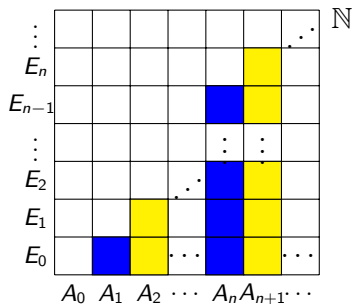
Definition (Staniszewski, 2017)

$$\mathfrak{b}_S(\mathcal{I}, \mathcal{J}, \mathcal{K}) = \min \left\{ |\mathcal{E}| : \mathcal{E} \subseteq \mathcal{P}_{\mathcal{K}} \wedge \forall \{A_n\} \in \mathcal{P}_{\mathcal{J}} \exists \{E_n\} \in \mathcal{E} \right. \\ \left. \bigcup_{n \in \mathbb{N}} \left(A_{n+1} \cap \bigcup_{i \leq n} E_i \right) \notin \mathcal{I} \right\}.$$

At the next slide we have a picture that can help us to grasp the definition ...

The cardinal $\mathfrak{b}_S(\mathcal{I}, \mathcal{J}, \mathcal{K})$

$$\mathfrak{b}_S(\mathcal{I}, \mathcal{J}, \mathcal{K}) = \min \left\{ |\mathcal{E}| : \mathcal{E} \subseteq \mathcal{P}_{\mathcal{K}} \wedge \forall \{A_n\} \in \mathcal{P}_{\mathcal{J}} \exists \{E_n\} \in \mathcal{E} \right. \\ \left. \bigcup_{n \in \mathbb{N}} \left(A_{n+1} \cap \bigcup_{i \leq n} E_i \right) \notin \mathcal{I} \right\}.$$



$$\bigcup_{n \in \mathbb{N}} \left(A_{n+1} \cap \bigcup_{i \leq n} E_i \right) = \\ A_1 \cap E_0 \cup \\ A_2 \cap (E_0 \cup E_1) \cup \\ \dots \\ A_n \cap (E_0 \cup \dots \cup E_{n-1}) \cup \\ A_{n+1} \cap (E_0 \cup \dots \cup E_{n-1} \cup E_n) \cup \dots$$

Is $\mathfrak{b}_S(\mathcal{I}, \mathcal{J}, \mathcal{K})$ a bounding number, anyway?

Theorem (Bukovska (\geq); Bukovský-Reclaw-Repický (\leq); 1991)

$$\text{non}(\text{QN-space}) = \mathfrak{b},$$

where

$$\mathfrak{b} = \min\{|\mathcal{F}| : \mathcal{F} \subseteq \mathbb{N}^{\mathbb{N}} \wedge \mathcal{F} \text{ is unbounded from above wrt } \leq^*\}.$$

Theorem (Staniszewski, 2017)

- If $\mathcal{K} \subseteq \mathcal{I}$, then

$$\text{non}(\text{QN}(\mathcal{I}, \mathcal{J}, \mathcal{K})\text{-space}) = \mathfrak{b}_S(\mathcal{I}, \mathcal{J}, \mathcal{K}),$$

where

$$\mathfrak{b}_S(\mathcal{I}, \mathcal{J}, \mathcal{K}) = \min \left\{ |\mathcal{E}| : \mathcal{E} \subseteq \mathcal{P}_{\mathcal{K}} \wedge \forall \{A_n\} \in \mathcal{P}_{\mathcal{J}} \exists \{E_n\} \in \mathcal{E} \right. \\ \left. \bigcup_{n \in \mathbb{N}} \left(A_{n+1} \cap \bigcup_{i \leq n} E_i \right) \notin \mathcal{I} \right\}.$$

Yes, $b_S(\mathcal{I}, \mathcal{J}, \mathcal{K})$ is, indeed, a bounding number!

Theorem (F.-Kwela)

$$b_S(\mathcal{I}, \mathcal{J}, \mathcal{K}) = b(\geq_{\mathcal{I}} \cap (\mathcal{D}_{\mathcal{K}} \times \mathcal{D}_{\mathcal{J}}))$$

Definition (Vojtáš, 1993)

For a relation $R \subseteq X \times Y$,

$$b(R) = \min(\{|B| : B \subseteq \text{dom}(R) \wedge B \text{ is } R\text{-unbounded}\} \cup \{\infty\}),$$

where $B \subseteq \text{dom}(R)$ is R -unbounded if

$$\neg(\exists y \in \text{ran}(R) \forall x \in B ((x, y) \in R))$$

Example

$$b(\leq^*) = b,$$

where $\leq^* = \{((a_n), (b_n)) \in \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} : (a_n) \leq^* (b_n)\} \subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$.

$\mathfrak{b}(\leq^{\mathcal{I}})$ and $\mathfrak{b}(\geq^{\mathcal{I}})$

Definition

For an ideal \mathcal{I} and sequences $(a_n), (b_n) \in \mathbb{N}^{\mathbb{N}}$,

$$(a_n) \leq^{\mathcal{I}} (b_n) \iff \exists A \in \mathcal{I} \forall n \in \mathbb{N} \setminus A (a_n \leq b_n).$$

Theorem (Farkas-Soukup, 2009)

If \mathcal{I} has the Baire property, $\mathfrak{b}(\leq^{\mathcal{I}}) = \mathfrak{b}$.

Definition

For an ideal \mathcal{I} and sequences $(a_n), (b_n) \in \mathbb{N}^{\mathbb{N}}$,

$$(a_n) \geq^{\mathcal{I}} (b_n) \iff \exists A \in \mathcal{I} \forall n \in \mathbb{N} \setminus A (a_n \geq b_n).$$

Then $\mathfrak{b}(\geq^{\mathcal{I}}) = \infty$, as every set is $\geq^{\mathcal{I}}$ -bounded by the sequence $(b_n) = (0, 0, 0, \dots)$

The relation $\geq_{\mathcal{I}} \cap (\mathcal{D}_{\mathcal{K}} \times \mathcal{D}_{\mathcal{J}})$

Theorem (F.-Kwela)

$$\mathfrak{b}_S(\mathcal{I}, \mathcal{J}, \mathcal{K}) = \mathfrak{b}(\geq_{\mathcal{I}} \cap (\mathcal{D}_{\mathcal{K}} \times \mathcal{D}_{\mathcal{J}}))$$

Definition

For an ideal \mathcal{I} ,

$$\begin{aligned} \mathcal{D}_{\mathcal{I}} &= \{(a_n) \in \mathbb{N}^{\mathbb{N}} : (a_n) \text{ is constant only on sets from } \mathcal{I}\} \\ &= \{(a_n) \in \mathbb{N}^{\mathbb{N}} : \forall c \in \mathbb{N} (\{n \in \mathbb{N} : a_n = c\} \in \mathcal{I})\} \end{aligned}$$

Definition

$$\geq_{\mathcal{I}} \cap (\mathcal{D}_{\mathcal{K}} \times \mathcal{D}_{\mathcal{J}}) = \{((a_n), (b_n)) \in \mathcal{D}_{\mathcal{K}} \times \mathcal{D}_{\mathcal{J}} : (a_n) \geq^{\mathcal{I}} (b_n)\}$$

$$\text{non}(\text{QN}(\mathcal{I}, \mathcal{J}, \mathcal{K})\text{-space}) = \mathfrak{b}(\geq_{\mathcal{I}} \cap (\mathcal{D}_{\mathcal{K}} \times \mathcal{D}_{\mathcal{J}}))$$

$$\mathcal{D}_{\mathcal{I}} = \{(a_n) \in \mathbb{N}^{\mathbb{N}} : (a_n) \text{ is constant only on sets from } \mathcal{I}\}$$

$$\geq_{\mathcal{I}} \cap (\mathcal{D}_{\mathcal{K}} \times \mathcal{D}_{\mathcal{J}}) = \{((a_n), (b_n)) \in \mathcal{D}_{\mathcal{K}} \times \mathcal{D}_{\mathcal{J}} : (a_n) \geq^{\mathcal{I}} (b_n)\}$$

Theorem (Staniszewski)

$$\text{If } \mathcal{K} \subseteq \mathcal{I}, \text{non}(\text{QN}(\mathcal{I}, \mathcal{J}, \mathcal{K})\text{-space}) = \mathfrak{b}_S(\mathcal{I}, \mathcal{J}, \mathcal{K})$$

Theorem (F.-Kwela)

$$\mathfrak{b}_S(\mathcal{I}, \mathcal{J}, \mathcal{K}) = \mathfrak{b}(\geq_{\mathcal{I}} \cap (\mathcal{D}_{\mathcal{K}} \times \mathcal{D}_{\mathcal{J}}))$$

Corollary (independently, Repický)

$$\text{If } \mathcal{K} \subseteq \mathcal{I}, \text{non}(\text{QN}(\mathcal{I}, \mathcal{J}, \mathcal{K})\text{-space}) = \mathfrak{b}(\geq_{\mathcal{I}} \cap (\mathcal{D}_{\mathcal{K}} \times \mathcal{D}_{\mathcal{J}}))$$

Let's examine $b_S(\mathcal{I}, \mathcal{J}, \mathcal{K})$

Question 1

What are the possible values of $b_S(\mathcal{I}, \mathcal{J}, \mathcal{K})$?

Question 2

What are relationships between $b_S(\mathcal{I}, \mathcal{J}, \mathcal{K})$ for various triples $(\mathcal{I}, \mathcal{J}, \mathcal{K})$?

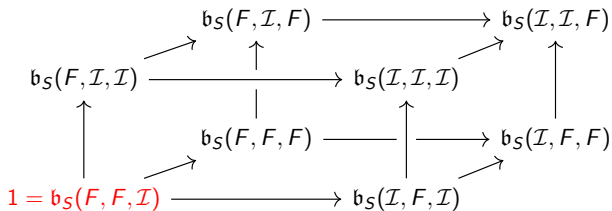
To make life easier we considered only triples such that at most one ideal is not equal Fin i.e. we have 8 triples:

$$(F, F, F), (F, F, \mathcal{I}), (F, \mathcal{I}, F), (\mathcal{I}, F, F),$$

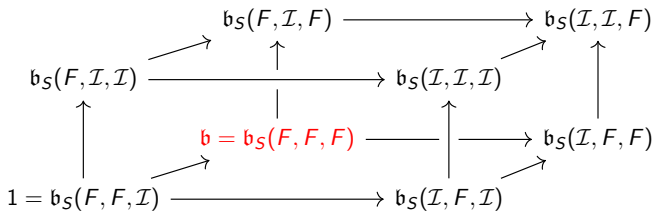
$$(F, \mathcal{I}, \mathcal{I}), (\mathcal{I}, F, \mathcal{I}), (\mathcal{I}, \mathcal{I}, F), (\mathcal{I}, \mathcal{I}, \mathcal{I}),$$

where $F = \text{Fin}$.

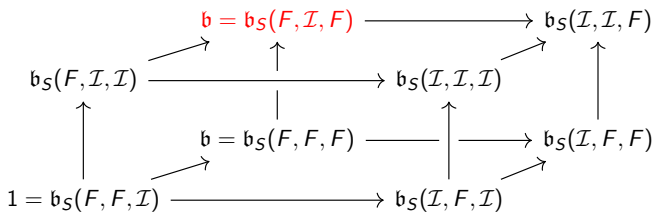
Known values in the diagram: $b_S(F, F, \mathcal{I})$



Known values in the diagram: $b_S(F, F, F)$

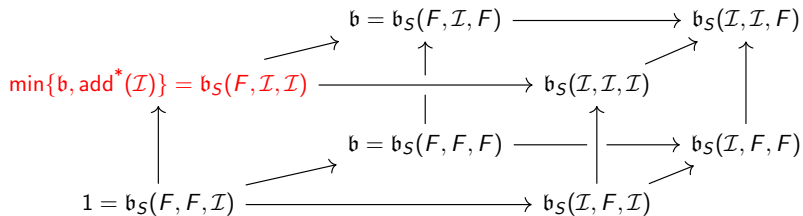


Known values in the diagram: $b_S(F, \mathcal{I}, F)$



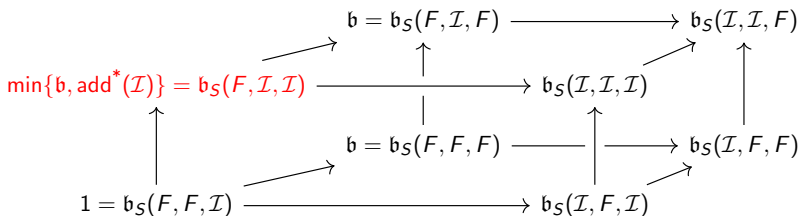
Known values in the diagram: $b_S(F, \mathcal{I}, \mathcal{I})$

Theorem (F.-Kwela)



$$\text{add}^*(\mathcal{I}) = \min(\{|\mathcal{F}| : \mathcal{F} \subseteq \mathcal{I} \wedge \neg[\exists A \in \mathcal{I} \forall F \in \mathcal{F} (F \setminus A \in \text{Fin})]\})$$

Known values in the diagram: $\mathfrak{b}_S(F, \mathcal{I}, \mathcal{I})$



$$\text{add}^*(\mathcal{I}) = \min(\{|\mathcal{F}| : \mathcal{F} \subseteq \mathcal{I} \wedge \neg[\exists A \in \mathcal{I} \forall F \in \mathcal{F} (F \setminus A \in \text{Fin})]\})$$

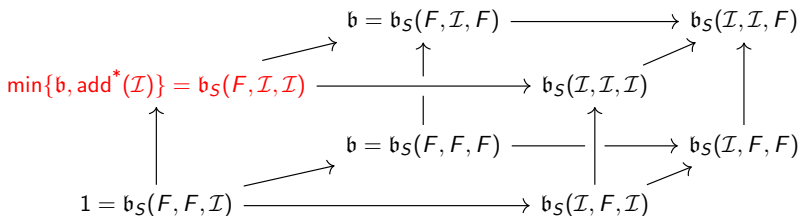
Corollary

$\mathfrak{b}_S(F, \mathcal{I}, \mathcal{I}) = \aleph_0$ for **non** P-ideals.

\mathcal{I} is a **P-ideal** if $\forall A_1, A_2, \dots \in \mathcal{I} \exists B \in \mathcal{I} \forall n (A_n \setminus B \in \text{Fin})$.

Proof: For non-P-ideals, $\text{add}^*(\mathcal{I}) = \aleph_0 < \mathfrak{b}$.

Known values in the diagram: $\mathfrak{b}_S(F, \mathcal{I}, \mathcal{I})$



$$\text{add}^*(\mathcal{I}) = \min(\{|\mathcal{F}| : \mathcal{F} \subseteq \mathcal{I} \wedge \neg[\exists A \in \mathcal{I} \forall F \in \mathcal{F} (|F \setminus A| < \aleph_0)]\})$$

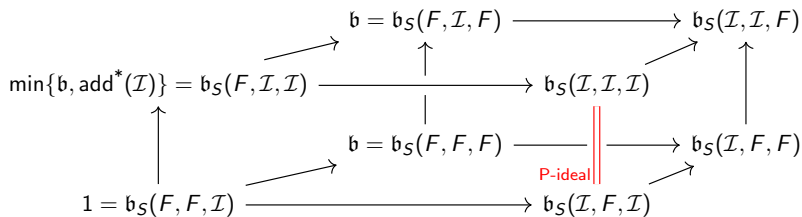
Corollary

$$\mathfrak{b}_S(\text{Fin}, \emptyset \otimes \text{Fin}, \emptyset \otimes \text{Fin}) = \mathfrak{b}$$

$$\emptyset \otimes \text{Fin} = \{A \subseteq \mathbb{N} \times \mathbb{N} : \forall n \{k : (n, k) \in A\} \in \text{Fin}\}.$$

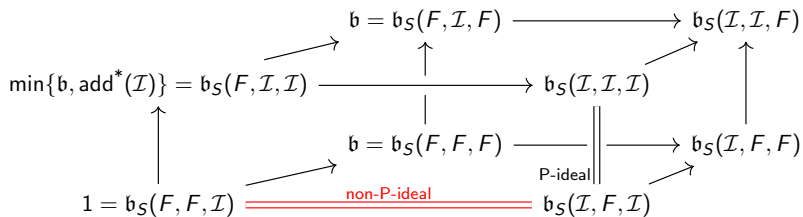
Proof: It's known that $\text{add}^*(\emptyset \otimes \text{Fin}) = \mathfrak{b}$.

Known values in the diagram: $b_S(\mathcal{I}, F, \mathcal{I})$



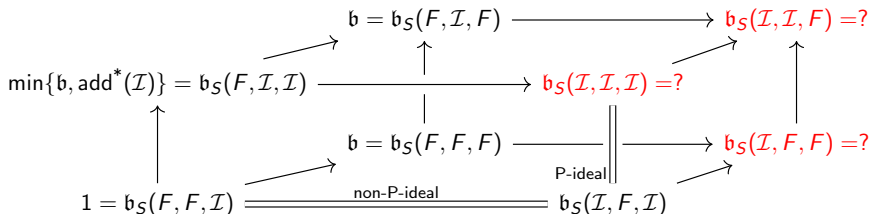
\mathcal{I} is a **P-ideal** if $\forall A_1, A_2, \dots \in \mathcal{I} \exists B \in \mathcal{I} \forall n (A_n \setminus B \in \text{Fin})$.

Known values in the diagram: $b_S(\mathcal{I}, F, \mathcal{I})$



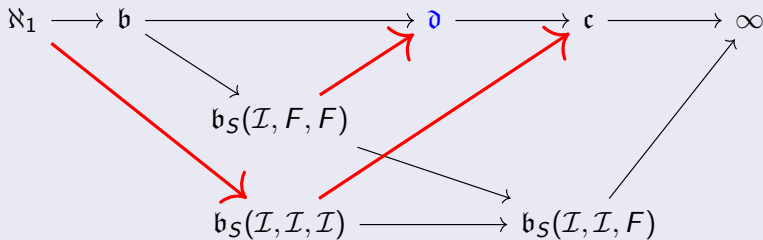
\mathcal{I} is a **P-ideal** if $\forall A_1, A_2, \dots \in \mathcal{I} \exists B \in \mathcal{I} \forall n (A_n \setminus B \in \text{Fin})$.

Difficult to get to know values in the diagram



The remaining three cardinals in the diagram

Theorem (F.-Kwela-Staniszewski)



$$\begin{aligned}\mathfrak{d} &= \min \left\{ |\mathcal{F}| : \mathcal{F} \subseteq \mathbb{N}^{\mathbb{N}} \wedge \mathcal{F} \text{ is cofinal in } (\mathbb{N}^{\mathbb{N}}, \leq^*) \right\} \\ &= \min \left\{ |\mathcal{F}| : \forall (b_n) \in \mathbb{N}^{\mathbb{N}} \exists (a_n) \in \mathcal{F} ((b_n) \leq^* (a_n)) \right\}.\end{aligned}$$

The remaining three cardinals for P-ideals

\mathcal{I} is a P-ideal if $\forall A_1, A_2, \dots \in \mathcal{I} \exists B \in \mathcal{I} \forall n (A_n \setminus B \in \text{Fin})$

Theorem (F.-Kwela)

(1) If \mathcal{I} is a P-ideal,

$$\mathfrak{b} \leq \mathfrak{b}_S(\mathcal{I}, F, F) = \mathfrak{b}_S(\mathcal{I}, \mathcal{I}, \mathcal{I}) = \mathfrak{b}_S(\mathcal{I}, \mathcal{I}, F) \leq \mathfrak{d}$$

(2) If \mathcal{I} is a P-ideal with the Baire property (e.g. $\mathcal{I} = \mathcal{I}_{1/n}, \mathcal{I}_d$),

$$\mathfrak{b} = \mathfrak{b}_S(\mathcal{I}, F, F) = \mathfrak{b}_S(\mathcal{I}, \mathcal{I}, \mathcal{I}) = \mathfrak{b}_S(\mathcal{I}, \mathcal{I}, F) \leq \mathfrak{d}$$

(3) If $\mathfrak{b} < cf(\mathfrak{d})$ and $\mathfrak{d} = \mathfrak{c}$, then there exists a P-ideal \mathcal{I} such that

$$\mathfrak{b} < \mathfrak{b}_S(\mathcal{I}, F, F) = \mathfrak{b}_S(\mathcal{I}, \mathcal{I}, \mathcal{I}) = \mathfrak{b}_S(\mathcal{I}, \mathcal{I}, F) = cf(\mathfrak{d}) \leq \mathfrak{d}$$

The remaining three cardinals for weak P-ideals

Definition

\mathcal{I} is a **P-ideal** if $\forall A_1, A_2, \dots \in \mathcal{I} \exists B \in \mathcal{I} \forall n (A_n \setminus B \in \text{Fin})$

Equivalently (taking $C = \mathbb{N} \setminus B$): \mathcal{I} is a **P-ideal** if

$$\forall A_1, A_2, \dots \in \mathcal{I} \exists C \in \mathcal{I}^* \forall n (A_n \cap C \in \text{Fin})$$

where $\mathcal{I}^* = \{A \subseteq \mathbb{N} : \mathbb{N} \setminus A \notin \mathcal{I}\}$

Definition

\mathcal{I} is a **weak P-ideal** if

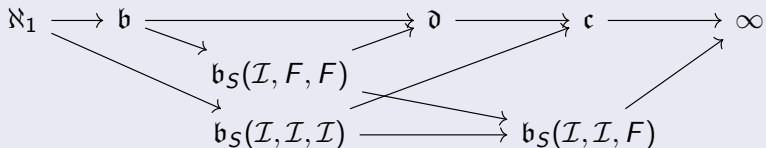
$$\forall A_1, A_2, \dots \in \mathcal{I} \exists C \in \mathcal{I}^+ \forall n (A_n \cap C \in \text{Fin}),$$

where $\mathcal{I}^+ = \{A \subseteq \mathbb{N} : A \notin \mathcal{I}\}$

Note: $\mathcal{I}^* \subseteq \mathcal{I}^+$.

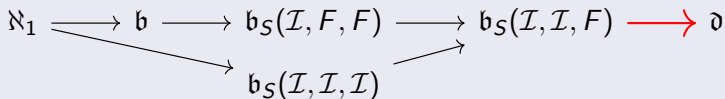
The remaining three cardinals for weak P-ideals

Diagram for an arbitrary ideal \mathcal{I}



Theorem (F.-Kwela)

If \mathcal{I} is a **weak P-ideal**,



The remaining three cardinals for weak P-ideals

Theorem (F.-Kwela)

If \mathcal{I} is a **weak P-ideal**,

$$\aleph_1 \begin{array}{l} \longrightarrow \mathfrak{b} \longrightarrow \mathfrak{b}_S(\mathcal{I}, F, F) \longrightarrow \mathfrak{b}_S(\mathcal{I}, \mathcal{I}, F) \longrightarrow \mathfrak{d} \\ \searrow \longrightarrow \mathfrak{b}_S(\mathcal{I}, \mathcal{I}, \mathcal{I}) \nearrow \end{array}$$

(1) If $\mathcal{I} = \mathcal{ED} = \{A \subseteq \mathbb{N} \times \mathbb{N} : \exists M \forall^\infty n (|\{k : (n, k) \in A\}| \leq M)\}$,

$$\aleph_1 = \mathfrak{b}_S(\mathcal{I}, \mathcal{I}, \mathcal{I}) \leq \mathfrak{b} = \mathfrak{b}_S(\mathcal{I}, F, F) = \mathfrak{b}_S(\mathcal{I}, \mathcal{I}, F)$$

(2) If $\mathcal{I} = \text{nwd} = \{A \subseteq \mathbb{Q} : A \text{ is nowhere dense in } \mathbb{R}\}$,

$$\aleph_1 \leq \mathfrak{b}_S(\mathcal{I}, \mathcal{I}, \mathcal{I}) = \text{add}(\mathcal{M}) \leq \mathfrak{b} = \mathfrak{b}_S(\mathcal{I}, F, F) = \mathfrak{b}_S(\mathcal{I}, \mathcal{I}, F)$$

(3) If $\mathfrak{b} < \text{cf}(\mathfrak{d})$, there exists \mathcal{I} with

$$\aleph_1 \leq \mathfrak{b}_S(\mathcal{I}, \mathcal{I}, \mathcal{I}) \leq \mathfrak{b} = \mathfrak{b}_S(\mathcal{I}, F, F) < \mathfrak{b}_S(\mathcal{I}, \mathcal{I}, F) = \text{cf}(\mathfrak{d})$$

Question for weak P-ideals

If \mathcal{I} is a **weak P-ideal**,

$$\aleph_1 \longrightarrow \mathfrak{b} \longrightarrow \mathfrak{b}_S(\mathcal{I}, F, F) \longrightarrow \mathfrak{b}_S(\mathcal{I}, \mathcal{I}, F) \longrightarrow \mathfrak{d}$$

\searrow
 $\mathfrak{b}_S(\mathcal{I}, \mathcal{I}, \mathcal{I}) \nearrow$

Question

Is it consistent that there is a weak P-ideal \mathcal{I} such that

$$\mathfrak{b}_S(\mathcal{I}, \mathcal{I}, \mathcal{I}) < \mathfrak{b}_S(\mathcal{I}, F, F) < \mathfrak{b}_S(\mathcal{I}, \mathcal{I}, F)?$$

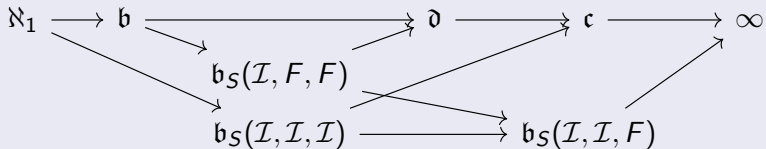
Question

Is it consistent that there is a weak P-ideal \mathcal{I} such that

$$\mathfrak{b}_S(\mathcal{I}, F, F) < \mathfrak{b}_S(\mathcal{I}, \mathcal{I}, \mathcal{I}) < \mathfrak{b}_S(\mathcal{I}, \mathcal{I}, F)?$$

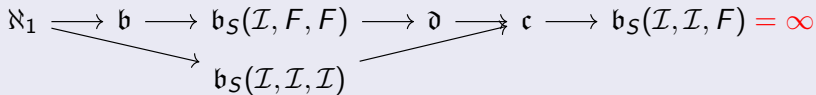
The remaining three cardinals for **non** weak P-ideals

Diagram for an arbitrary ideal \mathcal{I}



Theorem (F.-Kwela)

If \mathcal{I} is **not** a weak P-ideal,



Can $\mathfrak{b}_S(\mathcal{I}, \mathcal{I}, \mathcal{I})$ be bigger than \mathfrak{d} ?

If \mathcal{I} is a **weak P-ideal**,

$$\aleph_1 \longrightarrow \mathfrak{b} \longrightarrow \mathfrak{b}_S(\mathcal{I}, \mathcal{F}, \mathcal{F}) \longrightarrow \mathfrak{b}_S(\mathcal{I}, \mathcal{I}, \mathcal{F}) \longrightarrow \mathfrak{d}$$

\searrow $\mathfrak{b}_S(\mathcal{I}, \mathcal{I}, \mathcal{I})$ \nearrow

If \mathcal{I} is **not** a weak P-ideal,

$$\aleph_1 \longrightarrow \mathfrak{b} \longrightarrow \mathfrak{b}_S(\mathcal{I}, \mathcal{F}, \mathcal{F}) \longrightarrow \mathfrak{d} \longrightarrow \mathfrak{c} \longrightarrow \mathfrak{b}_S(\mathcal{I}, \mathcal{I}, \mathcal{F}) = \infty$$

\searrow $\mathfrak{b}_S(\mathcal{I}, \mathcal{I}, \mathcal{I})$ \nearrow

In all previous examples we had $\mathfrak{b}_S(\mathcal{I}, \mathcal{I}, \mathcal{I}) \leq \mathfrak{d}$.

Theorem (F.-Kwela)

If \mathcal{I} is an analytic ideal, then $\mathfrak{b}_S(\mathcal{I}, \mathcal{I}, \mathcal{I}) \leq \mathfrak{d}$.

Can $b_S(\mathcal{I}, \mathcal{I}, \mathcal{I})$ be bigger than δ ?

If \mathcal{I} is a **weak P-ideal**,

$$\aleph_1 \longrightarrow \mathfrak{b} \longrightarrow b_S(\mathcal{I}, F, F) \longrightarrow b_S(\mathcal{I}, \mathcal{I}, F) \longrightarrow \delta$$

\searrow \nearrow
 $b_S(\mathcal{I}, \mathcal{I}, \mathcal{I})$

If \mathcal{I} is **not** a weak P-ideal,

$$\aleph_1 \longrightarrow \mathfrak{b} \longrightarrow b_S(\mathcal{I}, F, F) \longrightarrow \delta \longrightarrow \mathfrak{c} \longrightarrow b_S(\mathcal{I}, \mathcal{I}, F) = \infty$$

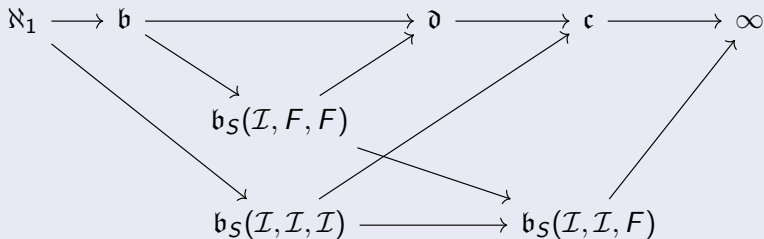
\searrow \nearrow
 $b_S(\mathcal{I}, \mathcal{I}, \mathcal{I})$

Theorem (F.-Kwela)

If $\delta < cf(\mathfrak{c}) \leq \mathfrak{u} = \mathfrak{c} = \mathfrak{c}^\delta$, then there exists an ideal \mathcal{I} with

$$b_S(\mathcal{I}, \mathcal{I}, \mathcal{I}) > \delta.$$

The remaining three cardinals for nice ideals: Baire



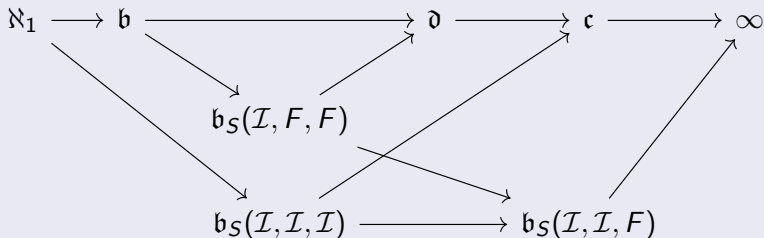
Theorem (F.-Kwela)

If \mathcal{I} has the **Baire property**,

$$\mathfrak{b} = \mathfrak{b}_S(\mathcal{I}, F, F)$$

- (1) If $\mathcal{I} = \mathcal{I}_d$, $\mathfrak{b}_S(\mathcal{I}, \mathcal{I}, \mathcal{I}) = \mathfrak{b}_S(\mathcal{I}, F, F) = \mathfrak{b}_S(\mathcal{I}, \mathcal{I}, F) = \mathfrak{b}$
- (2) If $\mathcal{I} = \mathcal{E}\mathcal{D}$, $\mathfrak{b}_S(\mathcal{I}, \mathcal{I}, \mathcal{I}) = \aleph_1 \leq \mathfrak{b}_S(\mathcal{I}, F, F) = \mathfrak{b}_S(\mathcal{I}, \mathcal{I}, F) = \mathfrak{b}$

The remaining three cardinals for nice ideals: Borel



Theorem (F.-Kwela)

If \mathcal{I} is Π_4^0 (e.g. F_σ or $F_{\sigma\delta}$ ideal),

$$\mathfrak{b}_S(\mathcal{I}, \mathcal{I}, \mathcal{I}) \leq \mathfrak{b} = \mathfrak{b}_S(\mathcal{I}, F, F) = \mathfrak{b}_S(\mathcal{I}, \mathcal{I}, F)$$

- (1) If $\mathcal{I} = \mathcal{I}_d$, $\mathfrak{b}_S(\mathcal{I}, \mathcal{I}, \mathcal{I}) = \mathfrak{b}_S(\mathcal{I}, F, F) = \mathfrak{b}_S(\mathcal{I}, \mathcal{I}, F) = \mathfrak{b}$
- (2) If $\mathcal{I} = \mathcal{E}\mathcal{D}$, $\mathfrak{b}_S(\mathcal{I}, \mathcal{I}, \mathcal{I}) = \aleph_1 \leq \mathfrak{b}_S(\mathcal{I}, F, F) = \mathfrak{b}_S(\mathcal{I}, \mathcal{I}, F) = \mathfrak{b}$

Definition

(λ, κ) **gap** in a poset $(P, <)$ is a pair of sequences

$$\{p_\alpha : \alpha < \lambda\} \quad \text{and} \quad \{q_\beta : \beta < \kappa\}$$

of elements of P such that

$$p_0 < p_1 < \dots < p_\alpha < \dots < q_\beta < \dots < q_1 < q_0$$

and there is no element r with

$$p_\alpha < r < q_\beta \quad \text{for all } \alpha \text{ and } \beta.$$

Example of (ω, ω) gap in $(\mathbb{Q}, <)$

p_n – increasing sequence of rationals with limit $\sqrt{2}$

q_n – decreasing sequence of rationals with limit $\sqrt{2}$

Gaps in $(\mathcal{P}(\mathbb{N}), \subseteq^*)$

Theorem (Hadamard, 1894)

There is no (ω, ω) gap in $(\mathcal{P}(\mathbb{N}), \subseteq^*)$, where

$$A \subseteq^* B \iff A \setminus B \in \text{Fin.}$$

Theorem (Hausdorff, 1909)

There exists (ω_1, ω_1) gap in $(\mathcal{P}(\mathbb{N}), \subseteq^*)$.

Theorem (Rothberger, 1941)

There exists (ω, \mathfrak{b}) gap in $(\mathcal{P}(\mathbb{N}), \subseteq^*)$ and there is no (ω, λ) gap for $\lambda < \mathfrak{b}$.

$$\mathfrak{b} = \min\{\lambda : \exists(\omega, \lambda) \text{ gap in } (\mathcal{P}(\mathbb{N}), \subseteq^*)\}$$

The Rothberger numbers

$$\mathfrak{b} = \min\{\lambda : \exists(\omega, \lambda) \text{ gap in } (\mathcal{P}(\mathbb{N}), \subseteq^*)\}$$

Definition (Brendle-Mejía, 2014)

$$\mathfrak{b}_R(\mathcal{I}) = \min\left(\{\lambda : \exists(\omega, \lambda) \text{ gap in } (\mathcal{P}(\mathbb{N}), \subseteq^{\mathcal{I}})\} \cup \{\infty\}\right),$$

where

$$A \subseteq^{\mathcal{I}} B \iff A \setminus B \in \mathcal{I}.$$

Example

If \mathcal{I} is a maximal ideal, $\mathfrak{b}_R(\mathcal{I}) = \infty$.

Theorem (Todorčević, 1998)

If \mathcal{I} is an F_σ ideal, then $\mathfrak{b}_R(\mathcal{I}) \leq \mathfrak{b}$.

Characterization of Rothberger numbers á la Staniszewski

$\mathcal{P}_{\mathcal{I}}$ is the family of all partitions $\{A_n : n \in \mathbb{N}\}$ of \mathbb{N} such that $A_n \in \mathcal{I}$ for each n .

Definition (Staniszewski, 2017)

$$b_S(\mathcal{I}, \mathcal{I}, \mathcal{I}) = \min \left\{ |\mathcal{E}| : \mathcal{E} \subseteq \mathcal{P}_{\mathcal{I}} \wedge \forall \{A_n\} \in \mathcal{P}_{\mathcal{I}} \exists \{E_n\} \in \mathcal{E} \right. \\ \left. \bigcup_{n \in \mathbb{N}} \left(A_{n+1} \cap \bigcup_{i \leq n} E_i \right) \notin \mathcal{I} \right\}.$$

Theorem (F.-Kwela)

$$b_R(\mathcal{I}) = \min \left\{ |\mathcal{E}| : \mathcal{E} \subseteq \mathcal{P}_{\mathcal{I}}^* \wedge \forall \{A_n\} \in \mathcal{P}_{\mathcal{I}} \exists \{E_n\} \in \mathcal{E} \right. \\ \left. \bigcup_{n \in \mathbb{N}} \left(A_{n+1} \cap \bigcup_{i \leq n} E_i \right) \notin \mathcal{I} \right\}.$$

where $\mathcal{P}_{\mathcal{I}}^*$ is somehow defined subfamily of $\mathcal{P}_{\mathcal{I}}$, and consequently:

$$b_S(\mathcal{I}, \mathcal{I}, \mathcal{I}) \leq b_R(\mathcal{I})$$

Two \mathfrak{b} or not two \mathfrak{b} ?

Question: Two \mathfrak{b} or not two \mathfrak{b} ?

$$b_S(\mathcal{I}, \mathcal{I}, \mathcal{I}) \stackrel{?}{=} b_R(\mathcal{I})$$

Answer

In general, two \mathfrak{b} .

Example

If \mathcal{I} is a maximal ideal,

$$b_S(\mathcal{I}, \mathcal{I}, \mathcal{I}) \leq \mathfrak{c} < \infty = b_R(\mathcal{I}).$$

Theorem (F.-Kwela)

It is consistent that there exists \mathcal{I} with

$$b_S(\mathcal{I}, \mathcal{I}, \mathcal{I}) < b_R(\mathcal{I}) \leq \mathfrak{c}$$

Two \mathfrak{b} or not two \mathfrak{b} in Borel realm?

Question: Two \mathfrak{b} or not two \mathfrak{b} ?

$$\mathfrak{b}_S(\mathcal{I}, \mathcal{I}, \mathcal{I}) \stackrel{?}{=} \mathfrak{b}_R(\mathcal{I}) \quad \text{for Borel ideals}$$

Answer

We do not know, but examples suggest: not two \mathfrak{b} .

Theorem (Brendle-Mejía, 2014)

If \mathcal{I} is an analytic P-ideal, then $\mathfrak{b}_R(\mathcal{I}) = \mathfrak{b}$.

Consequently, $\mathfrak{b}_S(\mathcal{I}, \mathcal{I}, \mathcal{I}) = \mathfrak{b}_R(\mathcal{I})$ for analytic P-ideals.

Theorem (Brendle-Mejía, 2014)

If $\mathcal{I} = \mathcal{ED}_{\text{Fin}} = \{A \subseteq \Delta : \exists M \forall n (|\{k : (n, k) \in A\}| \leq M)\}$, then $\mathfrak{b}_R(\mathcal{I}) = \aleph_1$.

Consequently, $\mathfrak{b}_S(\mathcal{ED}_{\text{Fin}}, \mathcal{ED}_{\text{Fin}}, \mathcal{ED}_{\text{Fin}}) = \mathfrak{b}_R(\mathcal{ED}_{\text{Fin}})$.

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