

# Spaces not distinguishing various kinds of convergence of sequences of real functions

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New results in my talk are from a joint work with Adam Kwela:

*“Spaces not distinguishing ideal pointwise and  $\sigma$ -uniform convergence”*

which is available at arXiv (2308.09557).

## Redefinition

- Topological space = nonempty normal (a.k.a.  $T_4$ ) space
- Function = continuous real-valued function defined on  $X$  ( $f : X \rightarrow \mathbb{R}$ )

# Four kinds of convergence of sequences of functions

## Pointwise convergence

$$(f_n) \xrightarrow{P} 0 \text{ if } \forall_{\varepsilon > 0} \forall_{x \in X} \exists_k \forall_{n > k} (|f_n(x)| < \varepsilon)$$

## Equal convergence (1979 – Császár-Laczkovich)

### Quasi-normal convergence (1991 – Bukovská)

$$(f_n) \xrightarrow{QN} 0 \text{ if } \exists_{(\varepsilon_n) \rightarrow 0} \forall_{x \in X} \exists_k \forall_{n > k} (|f_n(x)| < \varepsilon_n)$$

## Uniform convergence

$$(f_n) \xrightarrow{U} 0 \text{ if } \forall_{\varepsilon > 0} \exists_k \forall_{n > k} \forall_{x \in X} (|f_n(x)| < \varepsilon)$$

## $\sigma$ -uniform convergence

$$(f_n) \xrightarrow{\sigma-U} 0 \text{ if there is a partition } X_1, X_2, \dots \text{ of } X \text{ such that}$$
$$\forall_{k \in \mathbb{N}} (f_n \upharpoonright X_k) \xrightarrow{U} 0$$

# Four kinds of convergence of sequences of functions

## Relationships

### Theorem (Easy)

$$(f_n) \xrightarrow{U} 0 \implies (f_n) \xrightarrow{\sigma-U} 0 \quad \text{and} \quad (f_n) \xrightarrow{QN} 0 \implies (f_n) \xrightarrow{P} 0$$

### Theorem (Császár-Laczkovich, 1979)

$$(f_n) \xrightarrow{\sigma-U} 0 \iff (f_n) \xrightarrow{QN} 0$$

### Question

$$(f_n) \xrightarrow{U} 0 \stackrel{?}{\iff} (f_n) \xrightarrow{\sigma-U} 0 \quad \text{and} \quad (f_n) \xrightarrow{QN} 0 \stackrel{?}{\iff} (f_n) \xrightarrow{P} 0$$

### Theorem (Easy)

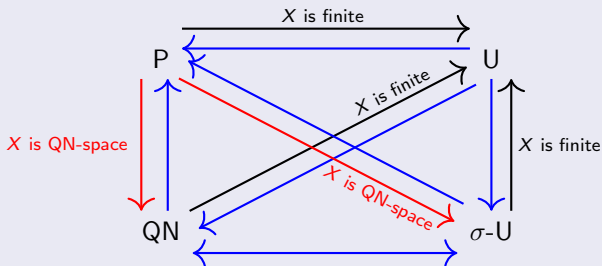
$$\forall (f_n: X \rightarrow \mathbb{R}) \left( (f_n) \xrightarrow{U} 0 \iff (f_n) \xrightarrow{\sigma-U} 0 \right) \iff X \text{ is finite}$$

### Definition (Bukovský-Reclaw-Repický, 1991)

$$X \text{ is a QN-space} \iff \forall (f_n: X \rightarrow \mathbb{R}) \left( (f_n) \xrightarrow{QN} 0 \iff (f_n) \xrightarrow{P} 0 \right)$$

# Four kinds of convergence of sequences of functions

Diagram of relationships



$$(f_n) \xrightarrow{U} 0 \implies (f_n) \xrightarrow{\sigma-U} 0, \dots$$

$$\forall (f_n: X \rightarrow \mathbb{R}) \left( (f_n) \xrightarrow{U} 0 \iff (f_n) \xrightarrow{\sigma-U} 0 \right) \iff X \text{ is finite}$$

$$X \text{ is a QN-spaces} \iff \forall (f_n: X \rightarrow \mathbb{R}) \left( (f_n) \xrightarrow{QN} 0 \iff (f_n) \xrightarrow{P} 0 \right)$$

## Definition

A family  $\mathcal{I} \subseteq \mathcal{P}(\mathbb{N})$  is an **ideal** on  $\mathbb{N}$  if

- 1  $A \subseteq B \in \mathcal{I} \implies A \in \mathcal{I}$ ,
- 2  $A, B \in \mathcal{I} \implies A \cup B \in \mathcal{I}$ ,
- 3  $\mathcal{I}$  contains all finite subsets of  $\mathbb{N}$  and  $\mathbb{N} \notin \mathcal{I}$ .

## Example

- 1  $\text{Fin} = \{A \subseteq \mathbb{N} : A \text{ is finite}\}$
- 2  $\mathcal{I}_{1/n} = \left\{ A \subseteq \mathbb{N} : \sum_{n \in A} \frac{1}{n} < \infty \right\}$
- 3  $\mathcal{I}_d = \left\{ A \subseteq \mathbb{N} : \lim_{n \rightarrow \infty} \frac{|A \cap \{1, \dots, n\}|}{n} = 0 \right\}$

# Ideal convergence of sequences of reals

## Definition

A sequence  $(x_n)$  of reals is **convergent to zero** if

$$\forall \varepsilon > 0 \quad \exists k \forall n > k \quad (|x_n| < \varepsilon)$$

equivalently:

$$\forall \varepsilon > 0 \quad \exists A \in \text{Fin} \quad \forall n \in \mathbb{N} \setminus A \quad (|x_n| < \varepsilon)$$

## Definition

Let  $\mathcal{I}$  be an ideal on  $\mathbb{N}$ .

A sequence  $(x_n)$  of reals is  **$\mathcal{I}$ -convergent to zero** if

$$\forall \varepsilon > 0 \quad \exists A \in \mathcal{I} \quad \forall n \in \mathbb{N} \setminus A \quad (|x_n| < \varepsilon).$$

—  
In symbols:  $(x_n) \xrightarrow{\mathcal{I}} 0$

# Ideal convergence of sequences of reals

## Examples

### Definition

A sequence  $(x_n)$  of reals is  **$\mathcal{I}$ -convergent to zero** ( $(x_n) \xrightarrow{\mathcal{I}} 0$ ) if

$$\forall \varepsilon > 0 \quad \exists A \in \mathcal{I} \quad \forall n \in \mathbb{N} \setminus A \quad (|x_n| < \varepsilon).$$

### Example

Since  $\text{Fin} \subseteq \mathcal{I}$ ,

$$\text{if } (x_n) \rightarrow 0, \text{ then } (x_n) \xrightarrow{\mathcal{I}} 0.$$

### Example

Let  $x_n = \begin{cases} 1 & \text{if } n = 1, 4, 9, \dots, k^2, \dots, \\ 0 & \text{otherwise.} \end{cases}$

Then  $(x_n)$  is a **divergent sequence**, but

$$(x_n) \xrightarrow{\mathcal{I}_{1/n}} 0,$$

because  $\sum_{k=1}^{\infty} \frac{1}{k^2} < \infty$ , so  $A = \{1, 4, 9, \dots, k^2, \dots\} \in \mathcal{I}_{1/n}$ .

# Ideal convergence of sequences of functions

Let  $\mathcal{I}$  be an ideal on  $\mathbb{N}$ . Let  $f_n : X \rightarrow \mathbb{R}$ ,  $n = 1, 2, \dots$

## Ideal pointwise convergence

$$(f_n) \xrightarrow{\mathcal{I}-P} 0 \text{ if } \forall_{\varepsilon > 0} \forall_{x \in X} \exists_{A \in \mathcal{I}} \forall_{n \in \mathbb{N} \setminus A} (|f_n(x)| < \varepsilon)$$

## Ideal quasi-normal convergence

$$(f_n) \xrightarrow{\mathcal{I}-QN} f \text{ if } \exists_{(\varepsilon_n) \xrightarrow{\mathcal{I}} 0} \forall_{x \in X} \exists_{A \in \mathcal{I}} \forall_{n \in \mathbb{N} \setminus A} (|f_n(x)| < \varepsilon_n)$$

## Ideal uniform convergence

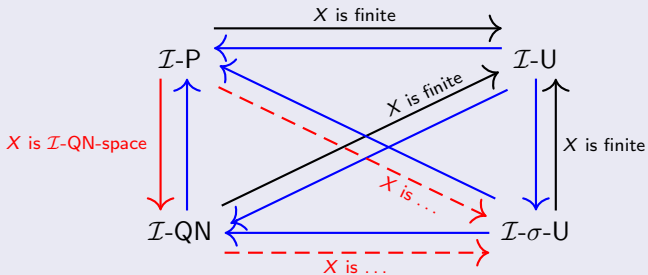
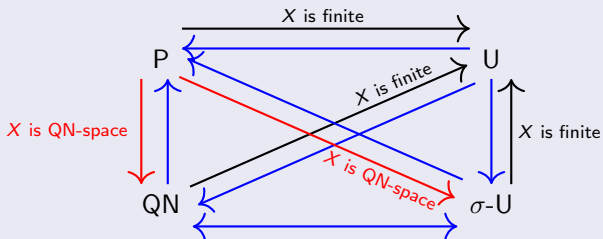
$$(f_n) \xrightarrow{\mathcal{I}-U} 0 \text{ if } \forall_{\varepsilon > 0} \exists_{A \in \mathcal{I}} \forall_{n \in \mathbb{N} \setminus A} \forall_{x \in X} (|f_n(x)| < \varepsilon)$$

## Ideal $\sigma$ -uniform convergence

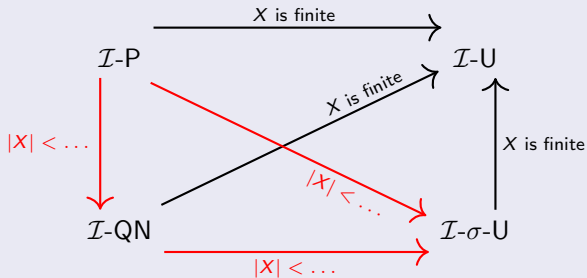
$$(f_n) \xrightarrow{\mathcal{I}-\sigma-U} 0 \text{ if there is a partition } X_1, X_2, \dots \text{ of } X \text{ such that}$$
$$\forall_{k \in \mathbb{N}} (f_n \upharpoonright X_k) \xrightarrow{\mathcal{I}-U} 0$$

# Ideal convergence of sequences of functions

Diagram of relationships



# Minimal size of a space which distinguishes convergence

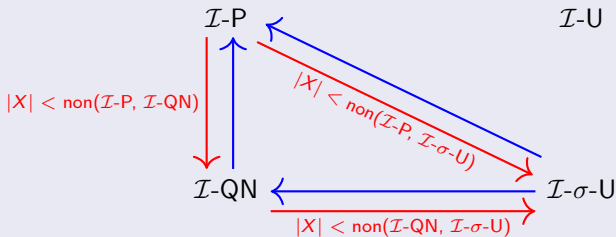


## Definition

- $\bullet$   $\text{non}(I-P, I-QN) = \min\{|X| : \exists f_n: X \rightarrow \mathbb{R} (f_n \xrightarrow{I-P} 0 \wedge f_n \not\xrightarrow{I-QN} 0)\}$
- $\bullet$   $\text{non}(I-QN, I-\sigma-U) = \min\{|X| : \exists f_n: X \rightarrow \mathbb{R} (f_n \xrightarrow{I-QN} 0 \wedge f_n \not\xrightarrow{I-\sigma-U} 0)\}$
- $\bullet$   $\text{non}(I-P, I-\sigma-U) = \min\{|X| : \exists f_n: X \rightarrow \mathbb{R} (f_n \xrightarrow{I-P} 0 \wedge f_n \not\xrightarrow{I-\sigma-U} 0)\}$

# Minimal size of a space which distinguishes convergence

## Relationship



## Theorem (Easy)

$$\text{non}(\mathcal{I}\text{-P}, \mathcal{I}\text{-}\sigma\text{-U}) = \min\{\text{non}(\mathcal{I}\text{-P}, \mathcal{I}\text{-QN}), \text{non}(\mathcal{I}\text{-QN}, \mathcal{I}\text{-}\sigma\text{-U})\}$$

## Proof

- If  $|X| < \text{non}(\mathcal{I}\text{-P}, \mathcal{I}\text{-}\sigma\text{-U})$ , then  $\dots$ , so  $|X| < \min\{\dots, \dots\}$
- If  $|X| < \min\{\dots, \dots\}$ , then  $\dots$ , so  $|X| < \text{non}(\mathcal{I}\text{-P}, \mathcal{I}\text{-}\sigma\text{-U})$

# Combinatorial characterizations of non(P, QN)

Theorem (Bukovský-Reclaw-Repický, 1991)

$$\text{non}(P, QN) = \text{non}(\text{Fin-P}, \text{Fin-QN}) = \mathfrak{b},$$

where  $\mathfrak{b}$  is the **bounding number** i.e. the minimal size of an unbounded family of sequences of natural numbers ordered by the relation  $\leq^*$  defined by

$$(a_n) \leq^* (b_n) \iff \exists k \forall n > k (a_n \leq b_n).$$

Definition

Let  $\mathfrak{b}_{\mathcal{I}}$  is the minimal size of an unbounded family of sequences of natural numbers ordered by the relation  $\leq^{\mathcal{I}}$  defined by

$$(a_n) \leq^{\mathcal{I}} (b_n) \iff \exists A \in \mathcal{I} \forall n \in \mathbb{N} \setminus A (a_n \leq b_n).$$

Remark

In general,

$$\text{non}(\mathcal{I}\text{-P}, \mathcal{I}\text{-QN}) \neq \mathfrak{b}_{\mathcal{I}}!$$

# Combinatorial characterization of non( $\mathcal{I}$ -P, $\mathcal{I}$ -QN)

Theorem (F.-Staniszewski, 2015)

$$\text{non}(\mathcal{I}\text{-P}, \mathcal{I}\text{-QN}) = \mathfrak{b}_S(\mathcal{I}),$$

where

$$\mathfrak{b}_S(\mathcal{I}) = \min \left\{ |\mathcal{E}| : \mathcal{E} \subseteq \mathcal{P}_{\mathcal{I}} \wedge \forall \{A_n\} \in \mathcal{P}_{\mathcal{I}} \exists \{E_n\} \in \mathcal{E} \bigcup_{n \in \mathbb{N}} (A_{n+1} \cap \bigcup_{i \leq n} E_i) \notin \mathcal{I} \right\}$$

and  $\mathcal{P}_{\mathcal{I}}$  is the family of all partitions  $\{A_n : n \in \mathbb{N}\}$  of  $\mathbb{N}$  such that  $A_n \in \mathcal{I}$  for each  $n \in \mathbb{N}$ .

# Combinatorial characterization of non( $\mathcal{I}$ -P, $\mathcal{I}$ - $\sigma$ -U)

Theorem (F.-Kwela, 2023)

$$\text{non}(\mathcal{I}\text{-P, } \mathcal{I}\text{-}\sigma\text{-U}) = \mathfrak{b}_\sigma(\mathcal{I}),$$

where

$$\mathfrak{b}_\sigma(\mathcal{I}) = \min\{|\mathcal{E}| : \mathcal{E} \subseteq \mathcal{M}_{\mathcal{I}} \wedge \forall (A_n)_{n \in \mathbb{N}} \in \mathcal{M}_{\mathcal{I}} \exists (E_n)_{n \in \mathbb{N}} \in \mathcal{E} \forall k \exists n > k (E_n \not\subseteq A_n)\}$$

and  $\mathcal{M}_{\mathcal{I}}$  is the family of all increasing sequences  $(A_n : n \in \mathbb{N})$  such that  $A_n \in \mathcal{I}$  for each  $n \in \mathbb{N}$ .

# Combinatorial characterization of non( $\mathcal{I}$ -QN, $\mathcal{I}$ - $\sigma$ -U)

Theorem (F.-Kwela, 2023)

$$\text{non}(\mathcal{I}\text{-QN}, \mathcal{I}\text{-}\sigma\text{-U}) = \text{add}_\omega(\mathcal{I}),$$

where

$$\text{add}_\omega(\mathcal{I}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \wedge \forall_{(B_n) \in \mathcal{I}^\omega} \exists_{A \in \mathcal{A}} \forall_n (A \not\subseteq B_n)\}$$

and  $\mathcal{I}^\omega$  is the family of all sequences  $(A_n : n \in \mathbb{N})$  such that  $A_n \in \mathcal{I}$  for each  $n \in \mathbb{N}$ .

Remark

- $\text{add}^*(\mathcal{I}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \wedge \forall_{B \in \mathcal{I}} \exists_{A \in \mathcal{A}} (A \not\subseteq^* B)\}$
- If  $\mathcal{I}$  is a P-ideal, then  $\text{add}_\omega(\mathcal{I}) = \text{add}^*(\mathcal{I})$ .
- There are non-P-ideals  $\mathcal{I}$  with  $\text{add}_\omega(\mathcal{I}) \neq \text{add}^*(\mathcal{I}) = \omega$ .

# Examples

## Corollary

$$\mathfrak{b}_\sigma(\mathcal{I}) = \min\{\mathfrak{b}_S(\mathcal{I}), \text{add}_\omega(\mathcal{I})\}$$

## Proof

$$\text{non}(\mathcal{I}\text{-P}, \mathcal{I}\text{-}\sigma\text{-U}) = \min\{\text{non}(\mathcal{I}\text{-P}, \mathcal{I}\text{-QN}), \text{non}(\mathcal{I}\text{-QN}, \mathcal{I}\text{-}\sigma\text{-U})\}$$

## Examples

$$\bullet \mathfrak{b}_\sigma(\text{Fin}) = \mathfrak{b}_S(\text{Fin}) = \mathfrak{b} < \infty = \text{add}_\omega(\text{Fin})$$

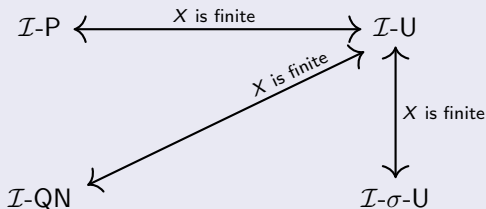
$$\bullet \mathfrak{b}_\sigma(\mathcal{I}_d) = \text{add}_\omega(\mathcal{I}_d) = \text{add}(\mathcal{N}) \leq \mathfrak{b} = \mathfrak{b}_S(\mathcal{I}_d)$$

$$\bullet \mathfrak{b}_\sigma(\text{Fin} \otimes \text{Fin}) = \mathfrak{b}_S(\text{Fin} \otimes \text{Fin}) = \text{add}_\omega(\text{Fin} \otimes \text{Fin}) = \mathfrak{b}$$

$$\bullet \mathfrak{b}_\sigma(\mathcal{I}_{AD}) = \mathfrak{b}_S(\mathcal{I}_{AD}) = \text{add}_\omega(\mathcal{I}_{AD}) = \omega_1$$

# Is distinguishing convergence a topological notion?

Sometimes no!

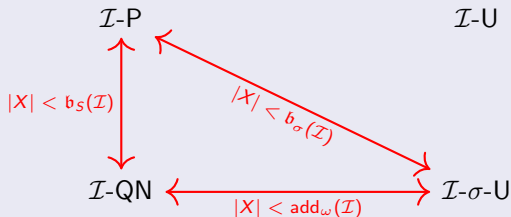


## Theorem

- $\forall (f_n: X \rightarrow \mathbb{R}) \left( f_n \xrightarrow{\mathcal{I}\text{-P}} 0 \iff f_n \xrightarrow{\mathcal{I}\text{-U}} 0 \right) \iff |X| < \omega$
- $\forall (f_n: X \rightarrow \mathbb{R}) \left( f_n \xrightarrow{\mathcal{I}\text{-QN}} 0 \iff f_n \xrightarrow{\mathcal{I}\text{-U}} 0 \right) \iff |X| < \omega$
- $\forall (f_n: X \rightarrow \mathbb{R}) \left( f_n \xrightarrow{\mathcal{I}\text{-}\sigma\text{-U}} 0 \iff f_n \xrightarrow{\mathcal{I}\text{-U}} 0 \right) \iff |X| < \omega$

# Is distinguishing convergence a topological notion?

Sometimes yes!



## Theorem (F.-Kwela, 2023)

- $|X| < b_S(\mathcal{I}) \quad \begin{matrix} \Rightarrow \\ \not\Leftarrow \end{matrix} \quad \forall (f_n: X \rightarrow \mathbb{R}) \quad (f_n \xrightarrow{\mathcal{I}\text{-P}} 0 \iff f_n \xrightarrow{\mathcal{I}\text{-QN}} 0)$
- $|X| < b_\sigma(\mathcal{I}) \quad \begin{matrix} \Rightarrow \\ \not\Leftarrow \end{matrix} \quad \forall (f_n: X \rightarrow \mathbb{R}) \quad (f_n \xrightarrow{\mathcal{I}\text{-P}} 0 \iff f_n \xrightarrow{\mathcal{I}\text{-}\sigma\text{-U}} 0)$
- $|X| < \text{add}_\omega(\mathcal{I}) \quad \begin{matrix} \Rightarrow \\ \not\Leftarrow \end{matrix} \quad \forall (f_n: X \rightarrow \mathbb{R}) \quad (f_n \xrightarrow{\mathcal{I}\text{-QN}} 0 \iff f_n \xrightarrow{\mathcal{I}\text{-}\sigma\text{-U}} 0)$

In fact, there exists a Hausdorff compact (hence normal) space  $X$  of arbitrary cardinality such that

$$\forall (f_n: X \rightarrow \mathbb{R}) \quad (f_n \xrightarrow{\mathcal{I}\text{-P}} 0 \iff f_n \xrightarrow{\mathcal{I}\text{-}\sigma\text{-U}} 0)$$

# Is distinguishing convergence a topological notion?

Separable spaces

## Theorem (F.-Kwela, 2023)

There exists a Hausdorff compact (hence normal) space  $X$  of arbitrary cardinality such that

$$\forall_{(f_n: X \rightarrow \mathbb{R})} (f_n \xrightarrow{\mathcal{I}\text{-P}} 0 \iff f_n \xrightarrow{\mathcal{I}\text{-}\sigma\text{-U}} 0)$$

## Remark

The above  $X$  is a one-point compactification of a discrete topological space of a given cardinality. This space is not separable (unless it is countable) and all but one points are isolated.

## Theorem (F.-Kwela, 2023)

There exists a Hausdorff separable, sequentially compact, compact (hence normal) space  $X$  of arbitrary cardinality up to  $\mathfrak{c}$  such that only countably many points of  $X$  are isolated and

$$\forall_{(f_n: X \rightarrow \mathbb{R})} (f_n \xrightarrow{\mathcal{I}\text{-P}} 0 \iff f_n \xrightarrow{\mathcal{I}\text{-}\sigma\text{-U}} 0)$$

## Remark

- Since

$$\text{non}(\mathcal{I}\text{-P}, \mathcal{I}\text{-}\sigma\text{-U}) = \mathfrak{b}_\sigma(\mathcal{I}),$$

there exists a topological space  $X$  such that

$$|X| = \mathfrak{b}_\sigma(\mathcal{I}) \quad \text{and} \quad \exists_{(f_n: X \rightarrow \mathbb{R})} (f_n \xrightarrow{\mathcal{I}\text{-P}} 0 \not\iff f_n \xrightarrow{\mathcal{I}\text{-}\sigma\text{-U}} 0)$$

- The space  $X$  that is constructed in the proof of the equality

$$\text{non}(\mathcal{I}\text{-P}, \mathcal{I}\text{-}\sigma\text{-U}) = \mathfrak{b}_\sigma(\mathcal{I})$$

is an uncountable discrete space. In particular,  $X$  is not a subspace of reals.

- To obtain  $X$  as a subset of reals we need one more characterization of  $\text{non}(\mathcal{I}\text{-P}, \mathcal{I}\text{-}\sigma\text{-U})$

# Subsets of reals distinguishing convergence

Bounding number of a binary relation

## Definition

- The set  $\mathcal{D}_{\mathcal{I}} \subseteq \mathbb{N}^{\mathbb{N}}$  is defined by

$$x \in \mathcal{D}_{\mathcal{I}} \iff x^{-1}[\{n\}] \in \mathcal{I} \text{ for every } n \in \mathbb{N}$$

- The binary relation  $\preceq_{\mathcal{I}}$  on  $\mathcal{D}_{\mathcal{I}}$  is defined by

$$x \preceq_{\mathcal{I}} y \iff \{m \in \omega : \exists k \in \omega (x(k) \leq m < y(k))\} \text{ is finite}$$

- The bounding number of the relation  $\preceq$ :

$$\begin{aligned} \mathfrak{b}(\preceq_{\mathcal{I}}) &= \min\{|B| : B \text{ is } \preceq_{\mathcal{I}}\text{-unbounded}\} \\ &= \min\{|B| : \forall x \in \mathcal{D}_{\mathcal{I}} \exists y \in B (y \not\preceq_{\mathcal{I}} x)\} \end{aligned}$$

## Theorem (F.-Kwela, 2023)

$$\text{non}(\mathcal{I}\text{-P}, \mathcal{I}\text{-}\sigma\text{-U}) = \mathfrak{b}(\preceq)$$

# Subsets of reals distinguishing convergence

Unbounded sets in  $\preceq$

## Corollary

$$\mathfrak{b}_\sigma(\mathcal{I}) = \text{non}(\mathcal{I}\text{-P}, \mathcal{I}\text{-}\sigma\text{-U}) = \mathfrak{b}(\preceq_{\mathcal{I}})$$

## Theorem (F.-Kwela, 2023)

If  $X \subseteq \mathbb{N}^{\mathbb{N}}$  is  $\preceq_{\mathcal{I}}$ -unbounded, then

$$\exists_{(f_n: X \rightarrow \mathbb{R})} (f_n \xrightarrow{\mathcal{I}\text{-P}} 0 \not\iff f_n \xrightarrow{\mathcal{I}\text{-}\sigma\text{-U}} 0)$$

## Corollary

There exists a subset  $X$  of **reals** such that  $|X| = \mathfrak{b}_\sigma(\mathcal{I})$  and

$$\exists_{(f_n: X \rightarrow \mathbb{R})} (f_n \xrightarrow{\mathcal{I}\text{-P}} 0 \not\iff f_n \xrightarrow{\mathcal{I}\text{-}\sigma\text{-U}} 0)$$

# Distinguishing spaces not distinguishing convergence

## Corollary

It is consistent that there exists  $X$  such that

- $\forall (f_n: X \rightarrow \mathbb{R}) (f_n \xrightarrow{\text{Fin-P}} 0 \iff f_n \xrightarrow{\text{Fin-}\sigma\text{-U}} 0)$
- $\exists (f_n: X \rightarrow \mathbb{R}) (f_n \xrightarrow{\mathcal{I}_d\text{-P}} 0 \not\iff f_n \xrightarrow{\mathcal{I}_d\text{-}\sigma\text{-U}} 0)$

## Proof

- $\text{non}(\mathcal{I}_d\text{-P}, \mathcal{I}_d\text{-}\sigma\text{-U}) = \mathfrak{b}_\sigma(\mathcal{I}_d) = \text{add}(\mathcal{N}) \leq \mathfrak{b} = \mathfrak{b}_\sigma(\text{Fin}) = \text{non}(\text{P}, \sigma\text{-U})$
- Consistently:  $\text{add}(\mathcal{N}) < \mathfrak{b}$
- Taking  $X$  such that  $|X| = \text{add}(\mathcal{N})$  and

$$\exists (f_n: X \rightarrow \mathbb{R}) (f_n \xrightarrow{\mathcal{I}_d\text{-P}} 0 \not\iff f_n \xrightarrow{\mathcal{I}_d\text{-}\sigma\text{-U}} 0),$$

we are done.

# Distinguishing spaces not distinguishing convergence

## Question

### Corollary

It is consistent that there exists  $X$  such that

- $\forall (f_n: X \rightarrow \mathbb{R}) \left( f_n \xrightarrow{\text{Fin-P}} 0 \iff f_n \xrightarrow{\text{Fin-}\sigma\text{-U}} 0 \right)$
- $\exists (f_n: X \rightarrow \mathbb{R}) \left( f_n \xrightarrow{\mathcal{I}_d\text{-P}} 0 \not\iff f_n \xrightarrow{\mathcal{I}_d\text{-}\sigma\text{-U}} 0 \right)$

### Question

Do there exist a space  $X$  and an ideal  $\mathcal{I}$  such that

- $\exists (f_n: X \rightarrow \mathbb{R}) \left( f_n \xrightarrow{\text{Fin-P}} 0 \not\iff f_n \xrightarrow{\text{Fin-}\sigma\text{-U}} 0 \right)$
- $\forall (f_n: X \rightarrow \mathbb{R}) \left( f_n \xrightarrow{\mathcal{I}\text{-P}} 0 \iff f_n \xrightarrow{\mathcal{I}\text{-}\sigma\text{-U}} 0 \right)$

### Remark

- For every ideal  $\mathcal{I}$ ,

$$\text{non}(\mathcal{I}\text{-P}, \mathcal{I}\text{-}\sigma\text{-U}) = \mathfrak{b}_\sigma(\mathcal{I}) \leq \mathfrak{b} = \mathfrak{b}_\sigma(\text{Fin}) = \text{non}(\text{P}, \sigma\text{-U})$$

- Hence, the cardinality argument does not work in this case.

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