

Spaces not distinguishing various kinds of convergence of sequences of real functions

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New results in my talk are from a joint work with **Adam Kwela**:

“Spaces not distinguishing ideal pointwise and σ -uniform convergence”

which is available at arXiv (2308.09557).

Redefinition

- Topological space = nonempty normal (a.k.a. T_4) space
- Function = continuous real-valued function defined on X ($f : X \rightarrow \mathbb{R}$)

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Four kinds of convergence of sequences of functions

Pointwise convergence

$$(f_n) \xrightarrow{P} 0 \text{ if } \forall \varepsilon > 0 \forall x \in X \exists k \forall n > k (|f_n(x)| < \varepsilon)$$

Equal convergence (1979 – Császár-Laczkovich)

Quasi-normal convergence (1991 – Bukovská)

$$(f_n) \xrightarrow{QN} 0 \text{ if } \exists (\varepsilon_n) \rightarrow 0 \forall x \in X \exists k \forall n > k (|f_n(x)| < \varepsilon_n)$$

Uniform convergence

$$(f_n) \xrightarrow{U} 0 \text{ if } \forall \varepsilon > 0 \exists k \forall n > k \forall x \in X (|f_n(x)| < \varepsilon)$$

σ -uniform convergence

$$(f_n) \xrightarrow{\sigma-U} 0 \text{ if there is a partition } X_1, X_2, \dots \text{ of } X \text{ such that}$$
$$\forall k \in \mathbb{N} \quad (f_n \upharpoonright X_k) \xrightarrow{U} 0$$

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Relationships

Theorem (Easy)

$$(f_n) \xrightarrow{U} 0 \implies (f_n) \xrightarrow{\sigma-U} 0 \quad \text{and} \quad (f_n) \xrightarrow{QN} 0 \implies (f_n) \xrightarrow{P} 0$$

Theorem (Császár-Laczkovich, 1979)

$$(f_n) \xrightarrow{\sigma-U} 0 \iff (f_n) \xrightarrow{QN} 0$$

Question

$$(f_n) \xrightarrow{U} 0 \stackrel{?}{\iff} (f_n) \xrightarrow{\sigma-U} 0 \quad \text{and} \quad (f_n) \xrightarrow{QN} 0 \stackrel{?}{\iff} (f_n) \xrightarrow{P} 0$$

Theorem (Easy)

$$\forall (f_n: X \rightarrow \mathbb{R}) ((f_n) \xrightarrow{U} 0 \iff (f_n) \xrightarrow{\sigma-U} 0) \iff X \text{ is finite}$$

Definition (Bukovský-Reclaw-Repický, 1991)

$$X \text{ is a QN-space} \iff \forall (f_n: X \rightarrow \mathbb{R}) ((f_n) \xrightarrow{QN} 0 \iff (f_n) \xrightarrow{P} 0)$$

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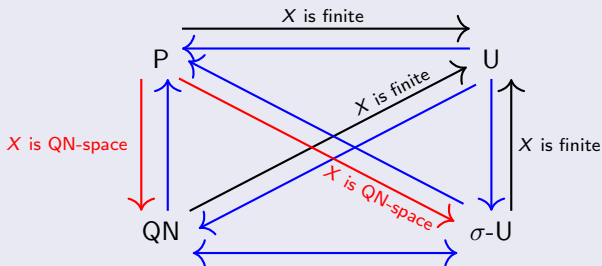
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Diagram of relationships



$$(f_n) \xrightarrow{U} 0 \implies (f_n) \xrightarrow{\sigma-U} 0, \dots$$

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Definition

A family $\mathcal{I} \subseteq \mathcal{P}(\mathbb{N})$ is an **ideal** on \mathbb{N} if

- 1 $A \subseteq B \in \mathcal{I} \implies A \in \mathcal{I}$,
- 2 $A, B \in \mathcal{I} \implies A \cup B \in \mathcal{I}$,
- 3 \mathcal{I} contains all finite subsets of \mathbb{N} and $\mathbb{N} \notin \mathcal{I}$.

Example

- 1 $\text{Fin} = \{A \subseteq \mathbb{N} : A \text{ is finite}\}$
- 2 $\mathcal{I}_{1/n} = \left\{ A \subseteq \mathbb{N} : \sum_{n \in A} \frac{1}{n} < \infty \right\}$
- 3 $\mathcal{I}_d = \left\{ A \subseteq \mathbb{N} : \lim_{n \rightarrow \infty} \frac{|A \cap \{1, \dots, n\}|}{n} = 0 \right\}$

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Ideal convergence of sequences of reals

Definition

A sequence (x_n) of reals is **convergent to zero** if

$$\forall \varepsilon > 0 \quad \exists k \quad \forall n > k \quad (|x_n| < \varepsilon)$$

equivalently:

$$\forall \varepsilon > 0 \quad \exists A \in \text{Fin} \quad \forall n \in \mathbb{N} \setminus A \quad (|x_n| < \varepsilon)$$

Definition

Let \mathcal{I} be an ideal on \mathbb{N} .

A sequence (x_n) of reals is **\mathcal{I} -convergent to zero** if

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A sequence (x_n) of reals is **\mathcal{I} -convergent to zero** ($(x_n) \xrightarrow{\mathcal{I}} 0$) if

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Example

Since $\text{Fin} \subseteq \mathcal{I}$,

$$\text{if } (x_n) \rightarrow 0, \text{ then } (x_n) \xrightarrow{\mathcal{I}} 0.$$

Example

Let $x_n = \begin{cases} 1 & \text{if } n = 1, 4, 9, \dots, k^2, \dots, \\ 0 & \text{otherwise.} \end{cases}$

Then (x_n) is a **divergent sequence**, but

$$(x_n) \xrightarrow{\mathcal{I}_{1/n}} 0,$$

because $\sum_{k=1}^{\infty} \frac{1}{k^2} < \infty$, so $A = \{1, 4, 9, \dots, k^2, \dots\} \in \mathcal{I}_{1/n}$.

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Let \mathcal{I} be an ideal on \mathbb{N} . Let $f_n : X \rightarrow \mathbb{R}$, $n = 1, 2, \dots$

Ideal pointwise convergence

$$(f_n) \xrightarrow{P} 0 \text{ if } \forall_{\varepsilon > 0} \forall_{x \in X} \exists_k \forall_{n > k} (|f_n(x)| < \varepsilon)$$

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$$(f_n) \xrightarrow{\mathcal{I}-P} 0 \text{ if } \forall_{\varepsilon > 0} \forall_{x \in X} \exists_{A \in \mathcal{I}} \forall_{n \in \mathbb{N} \setminus A} (|f_n(x)| < \varepsilon)$$

Ideal quasi-normal convergence

$$(f_n) \xrightarrow{\mathcal{I}-QN} f \text{ if } \exists_{(\varepsilon_n) \xrightarrow{\mathcal{I}} 0} \forall_{x \in X} \exists_{A \in \mathcal{I}} \forall_{n \in \mathbb{N} \setminus A} (|f_n(x)| < \varepsilon_n)$$

Ideal uniform convergence

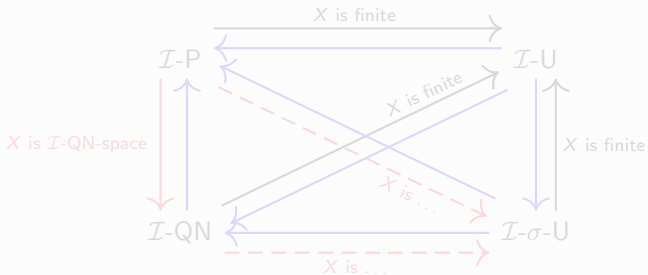
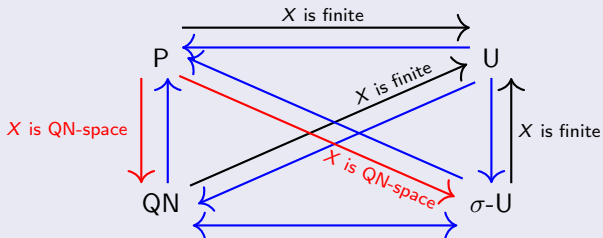
$$(f_n) \xrightarrow{\mathcal{I}-U} 0 \text{ if } \forall_{\varepsilon > 0} \exists_{A \in \mathcal{I}} \forall_{n \in \mathbb{N} \setminus A} \forall_{x \in X} (|f_n(x)| < \varepsilon)$$

Ideal σ -uniform convergence

$$(f_n) \xrightarrow{\mathcal{I}-\sigma-U} 0 \text{ if there is a partition } X_1, X_2, \dots \text{ of } X \text{ such that}$$
$$\forall_{k \in \mathbb{N}} (f_n \upharpoonright X_k) \xrightarrow{\mathcal{I}-U} 0$$

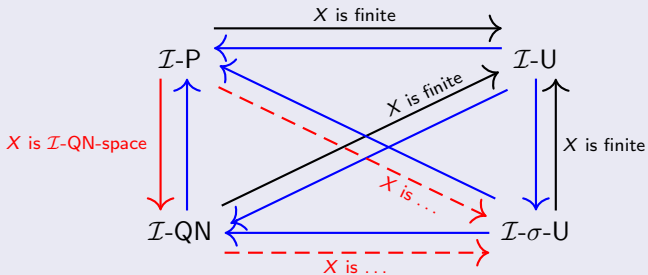
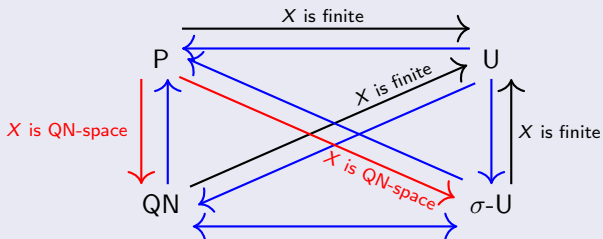
Ideal convergence of sequences of functions

Diagram of relationships

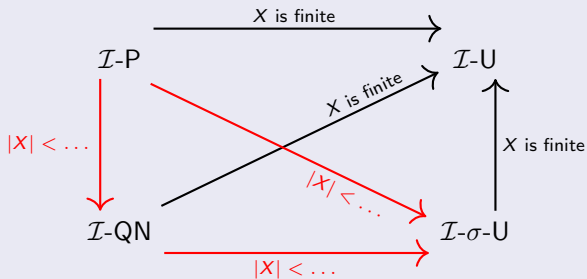


Ideal convergence of sequences of functions

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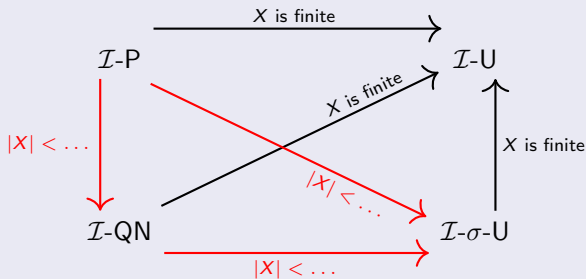
Minimal size of a space which distinguishes convergence



Definition

- $\text{non}(I-P, I-QN) = \min\{|X| : \exists f_n: X \rightarrow \mathbb{R} (f_n \xrightarrow{I-P} 0 \wedge f_n \not\xrightarrow{I-QN} 0)\}$
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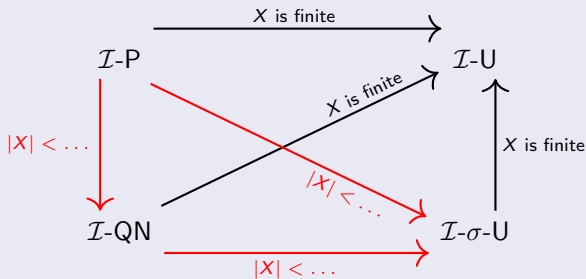
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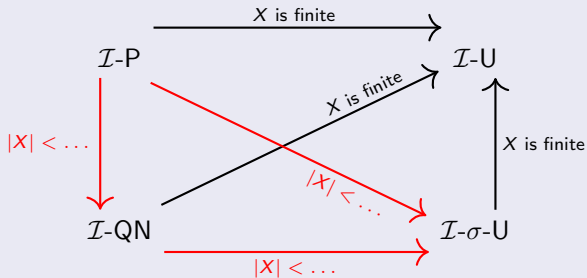
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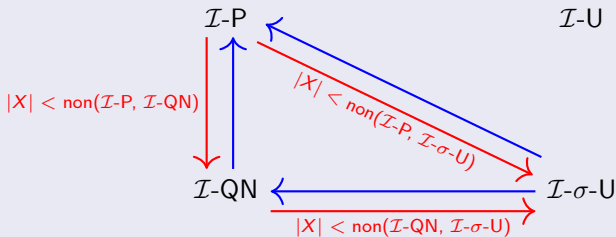


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Minimal size of a space which distinguishes convergence

Relationship



Theorem (Easy)

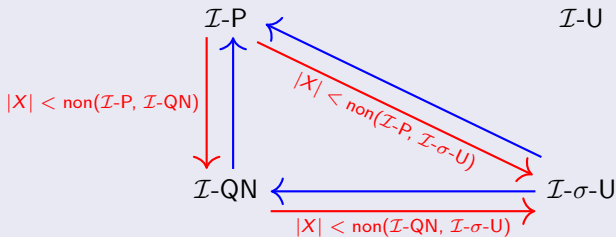
$$\text{non}(\mathcal{I}\text{-P}, \mathcal{I}\text{-}\sigma\text{-U}) = \min\{\text{non}(\mathcal{I}\text{-P}, \mathcal{I}\text{-QN}), \text{non}(\mathcal{I}\text{-QN}, \mathcal{I}\text{-}\sigma\text{-U})\}$$

Proof

- If $|X| < \text{non}(\mathcal{I}\text{-P}, \mathcal{I}\text{-}\sigma\text{-U})$, then \dots , so $|X| < \min\{\dots, \dots\}$
- If $|X| < \min\{\dots, \dots\}$, then \dots , so $|X| < \text{non}(\mathcal{I}\text{-P}, \mathcal{I}\text{-}\sigma\text{-U})$

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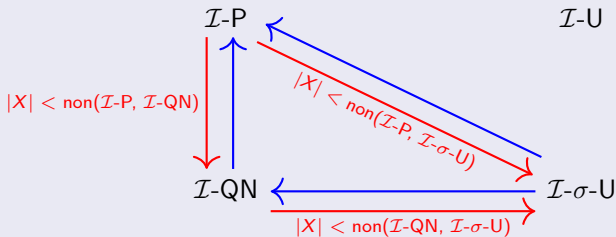
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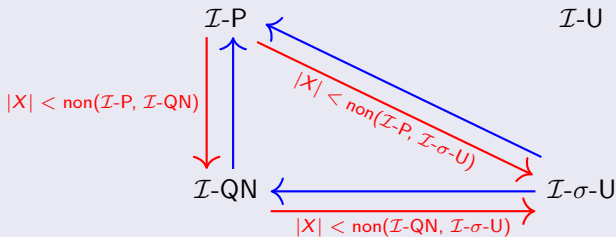
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Combinatorial characterizations of non(P, QN)

Theorem (Bukovský-Reclaw-Repický, 1991)

$$\text{non}(P, QN) = \text{non}(\text{Fin-P}, \text{Fin-QN}) = \mathfrak{b},$$

where \mathfrak{b} is the **bounding number** i.e. the minimal size of an unbounded family of sequences of natural numbers ordered by the relation \leq^* defined by

$$(a_n) \leq^* (b_n) \iff \exists k \forall n > k (a_n \leq b_n).$$

Definition

Let $\mathfrak{b}_{\mathcal{I}}$ is the minimal size of an unbounded family of sequences of natural numbers ordered by the relation $\leq^{\mathcal{I}}$ defined by

$$(a_n) \leq^{\mathcal{I}} (b_n) \iff \exists A \in \mathcal{I} \forall n \in \mathbb{N} \setminus A (a_n \leq b_n).$$

Remark

In general,

$$\text{non}(\mathcal{I}\text{-P}, \mathcal{I}\text{-QN}) \neq \mathfrak{b}_{\mathcal{I}}!$$

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Combinatorial characterization of non(\mathcal{I} -P, \mathcal{I} -QN)

Theorem (F.-Staniszewski, 2015)

$$\text{non}(\mathcal{I}\text{-P, } \mathcal{I}\text{-QN}) = \mathfrak{b}_S(\mathcal{I}),$$

where

$$\mathfrak{b}_S(\mathcal{I}) = \min \left\{ |\mathcal{E}| : \mathcal{E} \subseteq \mathcal{P}_{\mathcal{I}} \wedge \forall \{A_n\} \in \mathcal{P}_{\mathcal{I}} \exists \{E_n\} \in \mathcal{E} \bigcup_{n \in \mathbb{N}} (A_{n+1} \cap \bigcup_{i \leq n} E_i) \notin \mathcal{I} \right\}$$

and $\mathcal{P}_{\mathcal{I}}$ is the family of all partitions $\{A_n : n \in \mathbb{N}\}$ of \mathbb{N} such that $A_n \in \mathcal{I}$ for each $n \in \mathbb{N}$.

Combinatorial characterization of non(\mathcal{I} -P, \mathcal{I} - σ -U)

Theorem (F.-Kwela, 2023)

$$\text{non}(\mathcal{I}\text{-P, } \mathcal{I}\text{-}\sigma\text{-U}) = \mathbf{b}_\sigma(\mathcal{I}),$$

where

$$\mathbf{b}_\sigma(\mathcal{I}) = \min\{|\mathcal{E}| : \mathcal{E} \subseteq \mathcal{M}_{\mathcal{I}} \wedge \forall (A_n)_{n \in \mathbb{N}} \in \mathcal{M}_{\mathcal{I}} \exists (E_n)_{n \in \mathbb{N}} \in \mathcal{E} \forall k \exists n > k (E_n \not\subseteq A_n)\}$$

and $\mathcal{M}_{\mathcal{I}}$ is the family of all increasing sequences $(A_n : n \in \mathbb{N})$ such that $A_n \in \mathcal{I}$ for each $n \in \mathbb{N}$.

Combinatorial characterization of non(\mathcal{I} -QN, \mathcal{I} - σ -U)

Theorem (F.-Kwela, 2023)

$$\text{non}(\mathcal{I}\text{-QN}, \mathcal{I}\text{-}\sigma\text{-U}) = \text{add}_\omega(\mathcal{I}),$$

where

$$\text{add}_\omega(\mathcal{I}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \wedge \forall_{(B_n) \in \mathcal{I}^\omega} \exists_{A \in \mathcal{A}} \forall_n (A \not\subseteq B_n)\}$$

and \mathcal{I}^ω is the family of all sequences $(A_n : n \in \mathbb{N})$ such that $A_n \in \mathcal{I}$ for each $n \in \mathbb{N}$.

Remark

- $\text{add}^*(\mathcal{I}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \wedge \forall_{B \in \mathcal{I}} \exists_{A \in \mathcal{A}} (A \not\subseteq^* B)\}$
- If \mathcal{I} is a P-ideal, then $\text{add}_\omega(\mathcal{I}) = \text{add}^*(\mathcal{I})$.
- There are non-P-ideals \mathcal{I} with $\text{add}_\omega(\mathcal{I}) \neq \text{add}^*(\mathcal{I}) = \omega$.

Combinatorial characterization of non(\mathcal{I} -QN, \mathcal{I} - σ -U)

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Examples

Corollary

$$b_\sigma(\mathcal{I}) = \min\{b_S(\mathcal{I}), \text{add}_\omega(\mathcal{I})\}$$

Proof

$$\text{non}(\mathcal{I}\text{-P}, \mathcal{I}\text{-}\sigma\text{-U}) = \min\{\text{non}(\mathcal{I}\text{-P}, \mathcal{I}\text{-QN}), \text{non}(\mathcal{I}\text{-QN}, \mathcal{I}\text{-}\sigma\text{-U})\}$$

Examples

$$\bullet b_\sigma(\text{Fin}) = b_S(\text{Fin}) = b < \infty = \text{add}_\omega(\text{Fin})$$

$$\bullet b_\sigma(\mathcal{I}_d) = \text{add}_\omega(\mathcal{I}_d) = \text{add}(\mathcal{N}) \leq b = b_S(\mathcal{I}_d)$$

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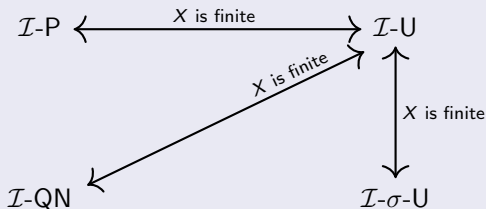
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Is distinguishing convergence a topological notion?

Sometimes no!

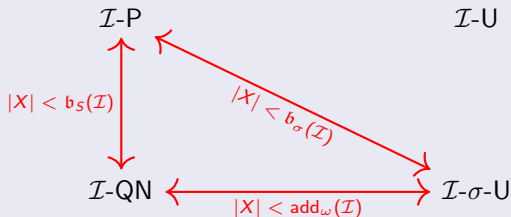


Theorem

- $\forall (f_n: X \rightarrow \mathbb{R}) \left(f_n \xrightarrow{\mathcal{I}\text{-P}} 0 \iff f_n \xrightarrow{\mathcal{I}\text{-U}} 0 \right) \iff |X| < \omega$
- $\forall (f_n: X \rightarrow \mathbb{R}) \left(f_n \xrightarrow{\mathcal{I}\text{-QN}} 0 \iff f_n \xrightarrow{\mathcal{I}\text{-U}} 0 \right) \iff |X| < \omega$
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Theorem (F.-Kwela, 2023)

- $|X| < b_S(I) \quad \begin{matrix} \Rightarrow \\ \not\Leftarrow \end{matrix} \quad \forall (f_n: X \rightarrow \mathbb{R}) \quad (f_n \xrightarrow{I-P} 0 \iff f_n \xrightarrow{I-QN} 0)$
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In fact, there exists a Hausdorff compact (hence normal) space X of arbitrary cardinality such that

$$\forall (f_n: X \rightarrow \mathbb{R}) \quad (f_n \xrightarrow{I-P} 0 \iff f_n \xrightarrow{I-\sigma-U} 0)$$

Is distinguishing convergence a topological notion?

Separable spaces

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Remark

The above X is a one-point compactification of a discrete topological space of a given cardinality. This space is not separable (unless it is countable) and all but one points are isolated.

Theorem (F.-Kwela, 2023)

There exists a Hausdorff separable, sequentially compact, compact (hence normal) space X of arbitrary cardinality up to \mathfrak{c} such that only countably many points of X are isolated and

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Remark

- Since

$$\text{non}(\mathcal{I}\text{-P}, \mathcal{I}\text{-}\sigma\text{-U}) = \mathfrak{b}_\sigma(\mathcal{I}),$$

there exists a topological space X such that

$$|X| = \mathfrak{b}_\sigma(\mathcal{I}) \quad \text{and} \quad \exists_{(f_n: X \rightarrow \mathbb{R})} (f_n \xrightarrow{\mathcal{I}\text{-P}} 0 \not\iff f_n \xrightarrow{\mathcal{I}\text{-}\sigma\text{-U}} 0)$$

- The space X that is constructed in the proof of the equality

$$\text{non}(\mathcal{I}\text{-P}, \mathcal{I}\text{-}\sigma\text{-U}) = \mathfrak{b}_\sigma(\mathcal{I})$$

is an uncountable discrete space. In particular, X is not a subspace of reals.

- To obtain X as a subset of reals we need one more characterization of $\text{non}(\mathcal{I}\text{-P}, \mathcal{I}\text{-}\sigma\text{-U})$

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- The space X that is constructed in the proof of the equality

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is an uncountable discrete space. In particular, X is not a subspace of reals.

- To obtain X as a subset of reals we need one more characterization of $\text{non}(\mathcal{I}\text{-P}, \mathcal{I}\text{-}\sigma\text{-U})$

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Subsets of reals distinguishing convergence

Bounding number of a binary relation

Definition

- The set $\mathcal{D}_{\mathcal{I}} \subseteq \mathbb{N}^{\mathbb{N}}$ is defined by

$$x \in \mathcal{D}_{\mathcal{I}} \iff x^{-1}[\{n\}] \in \mathcal{I} \text{ for every } n \in \mathbb{N}$$

- The binary relation $\preceq_{\mathcal{I}}$ on $\mathcal{D}_{\mathcal{I}}$ is defined by

$$x \preceq_{\mathcal{I}} y \iff \{m \in \omega : \exists k \in \omega (x(k) \leq m < y(k))\} \text{ is finite}$$

- The bounding number of the relation \preceq :

$$\begin{aligned} b(\preceq_{\mathcal{I}}) &= \min\{|B| : B \text{ is } \preceq_{\mathcal{I}}\text{-unbounded}\} \\ &= \min\{|B| : \forall x \in \mathcal{D}_{\mathcal{I}} \exists y \in B (y \not\preceq_{\mathcal{I}} x)\} \end{aligned}$$

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Subsets of reals distinguishing convergence

Unbounded sets in \preceq

Corollary

$$\mathfrak{b}_\sigma(\mathcal{I}) = \text{non}(\mathcal{I}\text{-P}, \mathcal{I}\text{-}\sigma\text{-U}) = \mathfrak{b}(\preceq_{\mathcal{I}})$$

Theorem (F.-Kwela, 2023)

If $X \subseteq \mathbb{N}^{\mathbb{N}}$ is $\preceq_{\mathcal{I}}$ -unbounded, then

$$\exists_{(f_n: X \rightarrow \mathbb{R})} (f_n \xrightarrow{\mathcal{I}\text{-P}} 0 \not\iff f_n \xrightarrow{\mathcal{I}\text{-}\sigma\text{-U}} 0)$$

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There exists a subset X of reals such that $|X| = \mathfrak{b}_\sigma(\mathcal{I})$ and

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Distinguishing spaces not distinguishing convergence

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It is consistent that there exists X such that

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- $\exists (f_n: X \rightarrow \mathbb{R}) (f_n \xrightarrow{\mathcal{I}_d\text{-P}} 0 \not\iff f_n \xrightarrow{\mathcal{I}_d\text{-}\sigma\text{-U}} 0)$

Proof

- $\text{non}(\mathcal{I}_d\text{-P}, \mathcal{I}_d\text{-}\sigma\text{-U}) = \mathfrak{b}_\sigma(\mathcal{I}_d) = \text{add}(\mathcal{N}) \leq \mathfrak{b} = \mathfrak{b}_\sigma(\text{Fin}) = \text{non}(\text{P}, \sigma\text{-U})$
- Consistently: $\text{add}(\mathcal{N}) < \mathfrak{b}$
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Distinguishing spaces not distinguishing convergence

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Do there exist a space X and an ideal \mathcal{I} such that

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Remark

- For every ideal \mathcal{I} ,

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- Hence, the cardinality argument does not work in this case.



Distinguishing spaces not distinguishing convergence

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