

The Borel complexity of sets of ideal limit points

—
properties of ideals *inspired* by limit points of sequences
(Part I)

Rafał Filipów



Inspirations in Real Analysis II (2024)

The talk is based on a joint work with Adam Kwela and Paolo Leonetti

Redefinitions

- $\omega = \mathbb{N}$ is the set of all natural numbers
- X will stand for an **uncountable Polish space** (i.e. separable completely metrizable topological space)

Definition

A family $\mathcal{I} \subseteq \mathcal{P}(\omega)$ is an **ideal** on ω if

- 1 $\emptyset \in \mathcal{I}$ and $\omega \notin \mathcal{I}$,
- 2 $A \subseteq B \in \mathcal{I} \implies A \in \mathcal{I}$,
- 3 $A, B \in \mathcal{I} \implies A \cup B \in \mathcal{I}$,
- 4 \mathcal{I} contains all finite subsets of ω .

Example

- 1 $\text{Fin} = \{A \subseteq \omega : A \text{ is finite}\}$
- 2 $\mathcal{I}_{1/n} = \left\{ A \subseteq \omega : \sum_{n \in A} \frac{1}{n} < \infty \right\}$ — the summable ideal
- 3 $\mathcal{I}_d = \left\{ A \subseteq \omega : \lim_{n \rightarrow \infty} \frac{|A \cap \{1, \dots, n\}|}{n} = 0 \right\}$ — the density zero ideal

Convergent subsequences

Theorem (Bolzano-Weierstrass)

For every sequence $(x_n)_{n \in \omega}$ in $[0, 1]$ there is an $A \notin \text{Fin}$ such that the subsequence $(x_n)_{n \in A}$ is convergent.

Theorem (Folklore)

For every sequence $(x_n)_{n \in \omega}$ in $[0, 1]$ there is an $A \notin \mathcal{I}_{1/n}$ such that the subsequence $(x_n)_{n \in A}$ is convergent.

Theorem (Fridy, 1993)

There exists a sequence $(x_n)_{n \in \omega}$ in $[0, 1]$ such that for every $A \notin \mathcal{I}_d$ the subsequence $(x_n)_{n \in A}$ is **not** convergent.

Finite Bolzano-Weierstrass property

Definition (F.-Mrożek-Reclaw-Szuca, 2007)

An ideal \mathcal{I} has **finite Bolzano-Weierstrass property** (**FinBW property**) if for every sequence $(x_n)_{n \in \omega}$ in $[0, 1]$ there is $A \notin \mathcal{I}$ such that the subsequence $(x_n)_{n \in A}$ is convergent.

- Fin and $\mathcal{I}_{1/n}$ have the FinBW property
- \mathcal{I}_d does not have the FinBW property

Theorem (F.-Mrożek-Reclaw-Szuca, 2007)

Every F_σ ideal has finite Bolzano-Weierstrass property.

Definition

An ideal \mathcal{I} is F_σ if the set $\{\mathbf{1}_A : A \in \mathcal{I}\}$ is an F_σ subset of the Cantor space $2^\omega = \{0, 1\}^\omega$.

The same for $F_{\sigma\delta}$, Borel, analytic, and other topological properties.

Topologised finite Bolzano-Weierstrass property

Definition (Kwela, 2023)

For a fixed ideal \mathcal{I} ,

$FinBW(\mathcal{I})$

denote the class of all topological spaces X such that for every sequence $(x_n)_{n \in \omega}$ in X there is $A \notin \mathcal{I}$ such that the subsequence $(x_n)_{n \in A}$ is convergent in X .

- $[0, 1] \in FinBW(\text{Fin})$ and $[0, 1] \in FinBW(\mathcal{I}_{1/n})$
- $[0, 1] \notin FinBW(\mathcal{I}_d)$

Corollary

$X \in FinBW(\mathcal{I}) \xRightarrow{\quad} \text{is sequentially compact} \quad (\not\Leftarrow) \iff X \text{ is compact}$

Topologised finite Bolzano-Weierstrass property

Corollary

If X is **not** compact, then $X \notin \text{FinBW}(\mathcal{I})$.

Theorem (Meza-Alcántara, 2009)

If X is compact, then $X \notin \text{FinBW}(\mathcal{I}) \iff \text{conv} \leq_K \mathcal{I}$.

Katětov order (Katětov, 1968)

$\mathcal{I} \leq_K \mathcal{J} \iff$ there exists $f : \omega \rightarrow \omega$ such that

$$\forall A \subseteq \omega (A \in \mathcal{I} \implies f^{-1}[A] \in \mathcal{J}).$$

The ideal *conv*

$\text{conv} = \{A \subseteq \mathbb{Q} : A \text{ has at most finitely many limit points in } \mathbb{R}\}$

Sets of limit and cluster points

Set of limit point of a sequence

$$\begin{aligned}\Lambda((x_n)_{n \in \omega}) &= \{p \in X : \exists A \notin \text{Fin} ((x_n)_{n \in A} \rightarrow p)\} \\ &= \{p \in X : \exists A \notin \text{Fin} \forall U \ni p \underset{\text{open}}{\forall^\infty} n \in A (x_n \in U)\}\end{aligned}$$

Set of cluster points of a sequence

$$\Gamma((x_n)_{n \in \omega}) = \{p \in X : \forall U \ni p \underset{\text{open}}{\exists} A \notin \text{Fin} \forall n \in A (x_n \in U)\}.$$

Theorem (Folklore)

- $\Lambda(x_n) = \Gamma(x_n)$.
- $\Lambda(x_n)$ and $\Gamma(x_n)$ are **closed**.
- For **every nonempty closed set** F there is a sequence $(x_n)_{n \in \omega}$ such that

$$F = \Gamma(x_n).$$

Ideal sets of limit and cluster points

Ideal set of limit point of a sequence

$$\begin{aligned}\Lambda_{\mathcal{I}}((x_n)_{n \in \omega}) &= \{p \in X : \exists A \notin \mathcal{I} ((x_n)_{n \in A} \rightarrow p)\} \\ &= \{p \in X : \exists A \notin \mathcal{I} \forall U \ni p \underset{\text{open}}{\forall^\infty n \in A} (x_n \in U)\}\end{aligned}$$

Set of cluster points of a sequence

$$\Gamma_{\mathcal{I}}((x_n)_{n \in \omega}) = \{p \in X : \forall U \ni p \exists A \notin \mathcal{I} \forall n \in A (x_n \in U)\}.$$

open

Theorem (Kostyrko-Šalát-Wilczyński, 2001)

- $\Lambda_{\mathcal{I}}(x_n) \subseteq \Gamma_{\mathcal{I}}(x_n)$.
- $\Lambda_{\mathcal{I}}(x_n)$ and $\Gamma_{\mathcal{I}}(x_n)$ are closed.
- T.F.A.E.
 - For every nonempty closed set F , there is a sequence $(x_n)_{n \in \omega}$ such that $F = \Gamma(x_n)$.
 - There exists an infinite partition of ω into sets which are not in \mathcal{I} .

ideal limit set \neq ideal cluster set

- $\Lambda_{\mathcal{I}}(x_n) \subseteq \Gamma_{\mathcal{I}}(x_n)$.

Theorem (Fridy, 1993)

There exists a sequence $(x_n)_{n \in \omega}$ in $[0, 1]$ such that

$$\Lambda_{\mathcal{I}_d}(x_n) \neq \Gamma_{\mathcal{I}_d}(x_n).$$

Theorem (He-Zang-Zang, 2022)

T.F.A.E.

- $\Lambda_{\mathcal{I}}(x_n) = \Gamma_{\mathcal{I}}(x_n)$ for every sequence $(x_n)_{n \in \omega}$.
- \mathcal{I} is a P^+ ideal.

P-like properties of ideals

$\mathcal{I} \in P^+$ if for every sequence $A_1 \supseteq A_2 \supseteq \dots$ such that $A_n \notin \mathcal{I}$

$$\exists A \notin \mathcal{I} \forall n (A \setminus A_n \text{ is finite}).$$

$\mathcal{I} \in P^-$ if for every sequence $A_1 \supseteq A_2 \supseteq \dots$ such that $A_n \notin \mathcal{I}$ and $A_n \setminus A_{n+1} \in \mathcal{I}$ for every n

$$\exists A \notin \mathcal{I} \forall n (A \setminus A_n \text{ is finite}).$$

$\mathcal{I} \in P^!$ if for every sequence $A_1 \supseteq A_2 \supseteq \dots$ such that $A_n \notin \mathcal{I}$ and $A_n \setminus A_{n+1} \notin \mathcal{I}$ for every n

$$\exists A \notin \mathcal{I} \forall n (A \setminus A_n \text{ is finite}).$$

Fact

$$\mathcal{I} \in P^+ \iff \mathcal{I} \in P^- \text{ and } \mathcal{I} \in P^!.$$

ideal limit set versus ideal cluster set

- $\Lambda_{\mathcal{I}}(x_n) \subseteq \Gamma_{\mathcal{I}}(x_n)$.
- $\Lambda_{\mathcal{I}}(x_n) = \Gamma_{\mathcal{I}}(x_n)$ for each $(x_n) \iff \mathcal{I}$ is a P^+ ideal.

For $A \subseteq X$, we write

$A^|$ to denote the **derived set** of A i.e. the set of all limit points of A

A^- to denote the set of all **isolated** points of A

Theorem (F.-Kwela-Leonetti, 2023)

T.F.A.E.

- $\Lambda_{\mathcal{I}}(x_n) \supseteq (\Gamma_{\mathcal{I}}(x_n))^|$ for every sequence $(x_n)_{n \in \omega}$.
- \mathcal{I} is $P^|$.

Theorem (F.-Kwela-Leonetti, 2023)

T.F.A.E.

- $\Lambda_{\mathcal{I}}(x_n) \supseteq (\Gamma_{\mathcal{I}}(x_n))^-$ for every sequence $(x_n)_{n \in \omega}$.
- \mathcal{I} is P^- .

Ideal limit set is not closed

Theorem (Kostyrko-Šalát-Wilczyński, 2001)

- $\Lambda_{\mathcal{I}}(x_n)$ and $\Gamma_{\mathcal{I}}(x_n)$ are closed.

Theorem (Balcerzak-Leonetti, 2019)

If an ideal \mathcal{I} is F_{σ} , then $\Lambda_{\mathcal{I}}(x_n)$ is closed for every sequence $(x_n)_{n \in \omega}$.

Theorem (Kostyrko-Mačaj-Šalát-Strauch, 2001)

- For every nonempty F_{σ} set $F \subseteq [0, 1]$ there exists a sequence $(x_n)_{n \in \omega}$ in $[0, 1]$ such that $F = \Lambda_{\mathcal{I}_d}(x_n)$.
- There exists a sequence (x_n) such that $\Lambda_{\mathcal{I}_d}(x_n)$ is not closed.

Question

What properties of ideals characterize Borel complexity of $\Lambda_{\mathcal{I}}$ sets?

Answer

You have to wait for Part II.

- M. Balcerzak, P. Leonetti, *On the relationship between ideal cluster points and ideal limit points*, Topology Appl. 252 (2019), 178–190.
- R. Filipów, N. Mrożek, I. Reclaw and P. Szuca, *Ideal convergence of bounded sequences*, J. Symbolic Logic 72 (2007), no. 2, 501–512.
- A. Fridy, *Statistical limit points*, Proc. Amer. Math. Soc. 118 (1993), no. 4, 1187–1192.
- Xi He, Hang Zhang, Shuguo Zhang, *The Borel complexity of ideal limit points*, Topology Appl. 312 (2022), Paper No. 108061, 12.
- M. Katětov, *Products of filters*, Comment. Math. Univ. Carolinae 9 (1968), 173–189
- P. Kostyrko, M. Mačaj, T. Šalát, O. Strauch, *On statistical limit points*, Proc. Amer. Math. Soc. 129 (2001), no. 9, 2647–2654.
- P. Kostyrko, T. Šalát, W. Wilczyński, *I-convergence*, Real Anal. Exchange 26 (2000/01), no. 2, 669–685.
- A. Kwela, *Unboring ideals*, Fund. Math. 261 (2023), 235–272.
- D. Meza-Alcátara, *Ideals and filters on countable set*, Ph.D. thesis, Universidad Nacional Autonoma de Mexico, 2009