

The Borel complexity of sets of ideal limit points

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The talk is based on a joint work with Adam Kwela and Paolo Leonetti

Redefinitions

$\omega = \mathbb{N}$ is the set of all natural numbers

X will stand for an **uncountable Polish space** (i.e. separable completely metrizable topological space)

Definition

A family $\mathcal{I} \subseteq \mathcal{P}(\omega)$ is an **ideal** on ω if

- 1 $\emptyset \in \mathcal{I}$ and $\omega \notin \mathcal{I}$,
- 2 $A \subseteq B \in \mathcal{I} \implies A \in \mathcal{I}$,
- 3 $A, B \in \mathcal{I} \implies A \cup B \in \mathcal{I}$,
- 4 \mathcal{I} contains all finite subsets of ω .

Example

- 1 $\text{Fin} = \{A \subseteq \omega : A \text{ is finite}\}$
- 2 $\mathcal{I}_{1/n} = \left\{ A \subseteq \omega : \sum_{n \in A} \frac{1}{n} < \infty \right\}$ — the summable ideal
- 3 $\mathcal{I}_d = \left\{ A \subseteq \omega : \lim_{n \rightarrow \infty} \frac{|A \cap \{1, \dots, n\}|}{n} = 0 \right\}$ — the density zero ideal

PART 1: Finding **one** convergent subsequence

Convergent subsequences

Theorem (Bolzano-Weierstrass)

For every sequence $(x_n)_{n \in \omega}$ in $[0, 1]$ there is an $A \notin \text{Fin}$ such that the subsequence $(x_n)_{n \in A}$ is convergent.

Theorem (Folklore)

For every sequence $(x_n)_{n \in \omega}$ in $[0, 1]$ there is an $A \notin \mathcal{I}_{1/n}$ such that the subsequence $(x_n)_{n \in A}$ is convergent.

Theorem (Fridy, 1993)

There exists a sequence $(x_n)_{n \in \omega}$ in $[0, 1]$ such that for every $A \notin \mathcal{I}_d$ the subsequence $(x_n)_{n \in A}$ is **not** convergent.

Finite Bolzano-Weierstrass property

Definition (F.-Mrozek-Reclaw-Szuca, 2007)

An ideal \mathcal{I} has **finite Bolzano-Weierstrass property** (**FinBW property**) if for every sequence $(x_n)_{n \in \omega}$ in $[0, 1]$ there is $A \notin \mathcal{I}$ such that the subsequence $(x_n)_{n \in A}$ is convergent.

- Fin and $\mathcal{I}_{1/n}$ have the FinBW property
- \mathcal{I}_d does not have the FinBW property

Definition

An **ideal \mathcal{I} is F_σ** if the set $\{\mathbf{1}_A : A \in \mathcal{I}\}$ is an F_σ subset of the Cantor space $2^\omega = \{0, 1\}^\omega$.

The same for $F_{\sigma\delta}$, Borel, analytic, and other topological properties.

Theorem (F.-Mrozek-Reclaw-Szuca, 2007)

Every **F_σ ideal** has FinBW property.

Finite Bolzano-Weierstrass property

Theorem (Folklore)

If X is **not** compact, then there is a sequence $(x_n)_{n \in \omega}$ in X such that for every $A \notin \mathcal{I}$ the subsequence $(x_n)_{n \in A}$ is **not** convergent in X .

Katětov order (Katětov, 1968)

$\mathcal{I} \leq_K \mathcal{J} \iff$ there exists $f : \omega \rightarrow \omega$ such that

$$\forall A \subseteq \omega (A \in \mathcal{I} \implies f^{-1}[A] \in \mathcal{J}).$$

The ideal *conv*

$\mathit{conv} = \{A \subseteq \mathbb{Q} : A \text{ has at most finitely many limit points in } \mathbb{R}\}$

Theorem (Meza-Alcántara, 2009)

If X is compact, then

$\mathit{conv} \not\leq_K \mathcal{I} \iff$ for every sequence $(x_n)_{n \in \omega}$ in X there is $A \notin \mathcal{I}$ such that the subsequence $(x_n)_{n \in A}$ is convergent in X .

PART 2: Finding **all** convergent subsequence

Sets of ideal limit points of a sequence

Set of ideal limit points of a sequence

$$\Lambda_{\mathcal{I}}((x_n)_{n \in \omega}) = \{p \in X : \exists A \notin \mathcal{I} ((x_n)_{n \in A} \rightarrow p)\}$$

Theorem (Folklore)

$\Lambda_{\mathbf{Fin}}(x_n)$ is a **closed** set for every sequence (x_n) .

Theorem (Balcerzak-Leonetti, 2019)

If an ideal \mathcal{I} is F_σ , then $\Lambda_{\mathcal{I}}(x_n)$ is **closed** for every sequence $(x_n)_{n \in \omega}$.

Theorem (Kostyrko-Mačaj-Šalát-Strauch, 2001)

- For every nonempty F_σ set $F \subseteq [0, 1]$ there exists a sequence $(x_n)_{n \in \omega}$ in $[0, 1]$ such that $F = \Lambda_{\mathcal{I}_d}(x_n)$.
- In particular, there is a sequence (x_n) such that $\Lambda_{\mathcal{I}_d}(x_n)$ is **not closed**.

PART 3: Borel complexity of ideal limit sets $\Lambda_{\mathcal{I}}(x)$

Family of all sets of ideal limit points of sequences

Recall: set of ideal limit points of a sequence

$$\Lambda_{\mathcal{I}}((x_n)_{n \in \omega}) = \{p \in X : \exists A \notin \mathcal{I} ((x_n)_{n \in A} \rightarrow p)\}$$

Family of all sets of ideal limit points of sequences

For a space X we write:

$$\Lambda_{\mathcal{I}}(X) = \{\Lambda((x_n)_{n \in \omega}) : \text{for each sequence } (x_n) \text{ in } X\}$$

Theorem (Folklore)

$$\Lambda_{\mathbf{Fin}}(X) = \Pi_1^0(X) \quad (\text{all closed subsets of } X)$$

Theorem (Folklore)

If \mathcal{I} is a **maximal** ideal, then

- 1 $\Lambda_{\mathcal{I}}(X) = \{ \{x\} : x \in X \} \cup \{ \emptyset \}$
- 2 $\Lambda_{\{\emptyset\} \otimes \mathcal{I}}(X) = \{ B : B \text{ is countable} \}$
- 3 $\Lambda_{\mathbf{Fin} \oplus (\{\emptyset\} \otimes \mathcal{I})}(X) = \{ A \cup B : A \text{ is closed and } B \text{ is countable} \}.$

Theorem

- 1 $\Lambda_{\mathcal{I}_{1/n}}(X) = \Pi_1^0(X)$ (Balcerzak-Leonetti, 2019)
- 2 $\Lambda_{\mathcal{I}_d}(X) = \Sigma_2^0(X)$ (Kostyrko-Mačaj-Šalát-Strauch, 2001)

Definition

A family $\{A_s : s \in 2^{<\omega}\}$ of subsets of ω is called an \mathcal{I} -scheme if for every $s \in 2^{<\omega}$

- 1 $A_s \notin \mathcal{I}$,
- 2 $A_{s \smallfrown 0} \cap A_{s \smallfrown 1} = \emptyset$,
- 3 $A_{s \smallfrown 0} \cup A_{s \smallfrown 1} \subseteq A_s$.

Definition

$$B_{\mathcal{I}}(\mathcal{A}) = \{x \in 2^\omega : \neg(\exists C \notin \mathcal{I} \forall n \in \omega |C \smallfrown n| < \omega)\}$$

Definition

- $\mathcal{I} \in P(\Pi_1^0)$ if there is an \mathcal{I} -scheme \mathcal{A} with $B_{\mathcal{I}}(\mathcal{A}) = \emptyset$
- $\mathcal{I} \in P(\Sigma_2^0)$ if there is an \mathcal{I} -scheme \mathcal{A} with $B_{\mathcal{I}}(\mathcal{A}) = \{(0, 0, \dots)\}$
- $\mathcal{I} \in P(\Pi_3^0)$ if there is an \mathcal{I} -scheme \mathcal{A} with $B_{\mathcal{I}}(\mathcal{A}) = \mathbb{Q}(2^\omega)$.

Open and closed sets

Theorem

(1) The following conditions are equivalent.

- 1 $\mathcal{I} \in P(\Pi_1^0)$.
- 2 $\Pi_1^0(X) \subseteq \Lambda_{\mathcal{I}}(X)$.
- 3 $\Lambda_{\mathcal{I}}(X)$ contains an analytic set which is **not** countable.

(2)

- $\mathcal{I} \in P(\Pi_1^0) \setminus P(\Sigma_2^0)$ for every F_σ ideal \mathcal{I}
- $\mathcal{I} \in P(\Pi_1^0)$ for every ideal \mathcal{I} with the Baire property.
- There exists an ideal $\mathcal{I} \in P(\Pi_1^0)$ which does not have the Baire property (at least under CH).

Remark

The inclusion $\Pi_1^0(X) \subseteq \Lambda_{\mathcal{I}}(X)$ was earlier proved for

- F_σ ideals by Balcerzak-Leonetti (2019)
- ideals with Baire property by He-Zang-Zang (2022)

Theorem (Folklore)

$\Lambda_{\mathcal{I}}(X) \neq \Sigma_1^0(X)$ for any ideal \mathcal{I} .

Theorem

(1) $\Pi_1^0(X) \subseteq \Lambda_{\mathcal{I}}(X) \iff \mathcal{I} \in P(\Pi_1^0)$,

(2) If \mathcal{I} is *coanalytic* (e.g. \mathcal{I} is Borel), then

- 1 $\Lambda_{\mathcal{I}}(X) \subseteq \Pi_1^0(X) \iff \mathcal{I} \notin P(\Sigma_2^0)$,
- 2 $\Lambda_{\mathcal{I}}(X) = \Pi_1^0(X) \iff \mathcal{I} \in P(\Pi_1^0) \setminus P(\Sigma_2^0)$.

Theorem

(1) The following conditions are equivalent.

- 1 $\mathcal{I} \in P(\Sigma_2^0)$.
- 2 $\Sigma_2^0(X) \subseteq \Lambda_{\mathcal{I}}(X)$.
- 3 $\Sigma_1^0(X) \subseteq \Lambda_{\mathcal{I}}(X)$.
- 4 $\Lambda_{\mathcal{I}}(X)$ contains an analytic set which is **not** the union of a closed set and a countable set.

(2) $\mathcal{I} \in P(\Sigma_2^0)$ for ideals with the hereditary Baire prop. which aren't P^+ .

In particular,

- $\{\emptyset\} \otimes \text{Fin} \in P(\Sigma_2^0)$
- $\mathcal{I} \in P(\Sigma_2^0)$ for every analytic P-ideal \mathcal{I} which is not F_σ .

Theorem

(1) $\Lambda_{\mathcal{I}}(X) \neq \Pi_2^0(X)$ for any ideal \mathcal{I} .

(2) $\Sigma_2^0(X) \subseteq \Lambda_{\mathcal{I}}(X) \iff \mathcal{I} \in P(\Sigma_2^0)$,

(3) If \mathcal{I} is *coanalytic* (e.g. \mathcal{I} is Borel), then

① $\Lambda_{\mathcal{I}}(X) \subseteq \Sigma_2^0(X) \iff \mathcal{I} \notin P(\Pi_3^0)$,

② $\Lambda_{\mathcal{I}}(X) = \Sigma_2^0(X) \iff \mathcal{I} \in P(\Sigma_2^0) \setminus P(\Pi_3^0)$.

PART 4: Borel complexity of ideal vs. Borel complexity of $\Lambda_{\mathcal{I}}(x)$

Borel complexity of ideals

Theorem

(1) For each $\alpha \geq 3$ there is an ideal $\mathcal{I} \in \Sigma_\alpha^0 \setminus \Pi_\alpha^0$ such that

$$\Lambda_{\mathcal{I}}(X) = \Pi_1^0(X).$$

(2) If \mathcal{I} is Σ_2^0 , then $\Lambda_{\mathcal{I}}(X) = \Pi_1^0(X)$.

(3) If \mathcal{I} is Π_3^0 , then one of the following items holds.

- 1 $\Lambda_{\mathcal{I}}(X) = \Pi_1^0(X)$.
- 2 $\Lambda_{\mathcal{I}}(X) = \Sigma_2^0(X)$.
- 3 $\Lambda_{\mathcal{I}}(X) = \Sigma_1^1(X)$.

Conjecture

If \mathcal{I} is Π_3^0 , then one of the following items holds.

- 1 $\Lambda_{\mathcal{I}}(X) = \Pi_1^0(X)$.
- 2 $\Lambda_{\mathcal{I}}(X) = \Sigma_2^0(X)$.

Borel complexity of ideals

Definition

An ideal \mathcal{I} is called a **Farah ideal** if there is a family of compact hereditary sets $\{C_n : n < \omega\}$ such that $\mathcal{I} = \{A \subseteq \omega : \forall n < \omega \exists m < \omega (A \setminus [0, m) \in C_n)\}$.

It is known that every Farah ideal is Π_3^0 .

Theorem (He-Zang-Zang, 2022)

If \mathcal{I} is a Farah ideal, then $\Lambda_{\mathcal{I}}(X) \subseteq \Sigma_2^0(X)$

Corollary

If \mathcal{I} is a Farah ideal, then one of the following items holds.

- 1 $\Lambda_{\mathcal{I}}(X) = \Pi_1^0(X)$.
- 2 $\Lambda_{\mathcal{I}}(X) = \Sigma_2^0(X)$.

Question (Farah, 2004)

Is every Π_3^0 ideal a Farah ideal?

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