The Borel complexity of sets of ideal limit points

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The talk is based on a join work with Adam Kwela and Paolo Leonetti

Redefinitions

$\omega = \mathbb{N}$ is the set of all natural numbers

X will stand for an uncountable Polish space (i.e. separable completely metrizable topological space)

Ideals on ω

Definition

A family $\mathcal{I} \subseteq \mathcal{P}(\omega)$ is an ideal on ω if

1. $\emptyset \in \mathcal{I}$ and $\omega \notin \mathcal{I}$,
2. $A \subseteq B \in \mathcal{I} \implies A \in \mathcal{I}$,
3. $A, B \in \mathcal{I} \implies A \cup B \in \mathcal{I}$,
4. $\mathcal{I}$ contains all finite subsets of ω.

Example

1. $\mathrm{Fin} = \{A \subseteq \omega : A \text{ is finite}\}$
2. $\mathcal{I}_{1/n} = \left\{A \subseteq \omega : \sum_{n \in A} \frac{1}{n} < \infty\right\}$ — the summable ideal
3. $\mathcal{I}_d = \left\{A \subseteq \omega : \lim_{n \to \infty} \frac{|A \cap \{1,\ldots,n\}|}{n} = 0\right\}$ — the density zero ideal

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PART 1: Finding one convergent subsequence

Convergent subsequences

Theorem (Bolzano-Weierstrass)

For every sequence $(x_n)_{n\in\omega}$ in $[0,1]$ there is an $A \notin \mathrm{Fin}$ such that the subsequence $(x_n)_{n\in A}$ is convergent.

Theorem (Folklore)

For every sequence $(x_n)_{n\in\omega}$ in $[0,1]$ there is an $A \notin \mathcal{I}_{1/n}$ such that the subsequence $(x_n)_{n\in A}$ is convergent.

Theorem (Fridy, 1993)

There exists a sequence $(x_n)_{n\in\omega}$ in $[0,1]$ such that for every $A \notin \mathcal{I}_d$ the subsequence $(x_n)_{n\in A}$ is not convergent.

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Finite Bolzano-Weierstrass property

Definition (F.-Mrożek-Recław-Szuca, 2007)

An ideal $\mathcal{I}$ has finite Bolzano-Weierstrass property (FinBW property) if for every sequence $(x_n)_{n\in\omega}$ in $[0,1]$ there is $A \notin \mathcal{I}$ such that the subsequence $(x_n)_{n\in A}$ is convergent.

- Fin and $\mathcal{I}_{1/n}$ have the FinBW property
- $\mathcal{I}_d$ does not have the FinBW property

Definition

An ideal $\mathcal{I}$ is F_σ if the set $\{\mathbf{1}_A : A \in \mathcal{I}\}$ is an F_σ subset of the Cantor space $2^\omega = \{0,1\}^\omega$.

The same for $F_{\sigma\delta}$, Borel, analytic, and other topological properties.

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Every F_σ ideal has FinBW property.

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Topologised finite Bolzano-Weierstrass property

Definition (Kwela, 2023)

For a fixed ideal $\mathcal{I}$,

$$FinBW(\mathcal{I})$$

denote the class of all topological spaces X such that for every sequence $(x_n)_{n\in\omega}$ in X there is $A \notin \mathcal{I}$ such that the subsequence $(x_n)_{n\in A}$ is convergent in X.

- $[0,1] \in FinBW(\mathrm{Fin})$ and $[0,1] \in FinBW(\mathcal{I}_{1/n})$
- $[0,1] \notin FinBW(\mathcal{I}_d)$

Corollary

$$X \in FinBW(\mathcal{I}) \overset{\Longrightarrow}{\not\Longleftarrow} \text{ is sequentially compact } (\iff X \text{ is compact})$$

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If X is not compact, then $X \notin FinBW(\mathcal{I})$.

Katětov order (Katětov, 1968)

$\mathcal{I} \leq_K \mathcal{J} \iff$ there exists $f : \omega \to \omega$ such that

$$\forall A \subseteq \omega \, (A \in \mathcal{I} \implies f^{-1}[A] \in \mathcal{J}).$$

The ideal *conv*

$conv = \{A \subseteq \mathbb{Q} : A \text{ has at most finitely many limit points in } \mathbb{R}\}$

Theorem (Meza-Alcántara, 2009)

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PART 2: Finding all convergent subsequence

Sets of limit and cluster points

Set of limit points of a sequence

$$\begin{aligned}\Lambda((x_n)_{n\in\omega}) &= \{p \in X : \exists\, A \notin \mathrm{Fin}\ ((x_n)_{n\in A} \to p)\}\\ &= \{p \in X : \exists\, A \notin \mathrm{Fin}\ \forall \underset{\text{open}}{U} \ni p\ \forall^\infty n \in A\ (x_n \in U)\}\end{aligned}$$

Set of cluster points of a sequence

$$\Gamma((x_n)_{n\in\omega}) = \{p \in X : \forall \underset{\text{open}}{U} \ni p\ \exists A \notin \mathrm{Fin}\ \forall n \in A\ (x_n \in U)\}.$$

Theorem (Folklore)

- $\Lambda(x_n) = \Gamma(x_n)$.
- $\Lambda(x_n)$ and $\Gamma(x_n)$ are closed.
- For every nonempty closed set F there is a sequence $(x_n)_{n\in\omega}$ such that

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Theorem (Kostyrko-Šalát-Wilczyński, 2001)

- $\Lambda_{\mathcal{I}}(x_n) \subseteq \Gamma_{\mathcal{I}}(x_n)$.
- ~~$\Lambda_{\mathcal{I}}(x_n)$, and~~ $\Gamma_{\mathcal{I}}(x_n)$ ~~are~~ is closed.
- T.F.A.E.
  - For every nonempty closed set F, there is a sequence $(x_n)_{n\in\omega}$ such that $F = \Gamma_{\mathcal{I}}(x_n)$.
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P-like properties of ideals

$\mathcal{I} \in P^+$ if for every sequence $A_1 \supseteq A_2 \supseteq \dots$ such that $A_n \notin \mathcal{I}$

$$\exists A \notin \mathcal{I}\ \forall n\ (A \setminus A_n \text{ is finite}).$$

$\mathcal{I} \in P^-$ if for every sequence $A_1 \supseteq A_2 \supseteq \dots$ such that $A_n \notin \mathcal{I}$ and $A_n \setminus A_{n+1} \in \mathcal{I}$ for every n

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$$\mathcal{I} \in P^+ \iff \mathcal{I} \in P^- \text{ and } \mathcal{I} \in P^|.$$

P-like properties of ideals

$\mathcal{I} \in P^+$ if for every sequence $A_1 \supseteq A_2 \supseteq \dots$ such that $A_n \notin \mathcal{I}$

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ideal limit set versus ideal cluster set

- $\Lambda_{\mathcal{I}}(x_n) \subseteq \Gamma_{\mathcal{I}}(x_n)$.
- $\Lambda_{\mathcal{I}}(x_n) = \Gamma_{\mathcal{I}}(x_n)$ for each (x_n) $\iff$ $\mathcal{I}$ is a P^+ ideal.

For $A \subseteq X$, we write
$A^{|}$ to denote the derived set of A i.e. the set of all limit points of A
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Theorem

(1) T.F.A.E.

- $\Lambda_{\mathcal{I}}(x_n) \supseteq (\Gamma_{\mathcal{I}}(x_n))^{|}$ for every sequence $(x_n)_{n\in\omega}$.
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Ideal limit set is not closed

Theorem (Kostyrko-Šalát-Wilczyński, 2001)

- ~~$\Lambda_{\mathcal{I}}(x_n)$ and~~ $\Gamma_{\mathcal{I}}(x_n)$ ~~are~~ is closed.

Theorem (Balcerzak-Leonetti, 2019)

If an ideal $\mathcal{I}$ is F_σ, then $\Lambda_{\mathcal{I}}(x_n)$ is closed for every sequence $(x_n)_{n\in\omega}$.

Theorem (Kostyrko-Mačaj-Šalát-Strauch, 2001)

- For every nonempty F_σ set $F \subseteq [0,1]$ there exists a sequence $(x_n)_{n\in\omega}$ in $[0,1]$ such that $F = \Lambda_{\mathcal{I}_d}(x_n)$.
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PART 3: Borel complexity of ideal limit sets $\Lambda_{\mathcal{I}}(x)$

Family of all sets of ideal limit points of sequences

Recall: set of ideal limit points of a sequence

$$\Lambda_{\mathcal{I}}((x_n)_{n\in\omega}) = \{p \in X : \exists\, A \notin \mathcal{I}\ ((x_n)_{n\in A} \to p)\}$$

Family of all sets of ideal limit points of sequences

For a space X we write:

$$\Lambda_{\mathcal{I}}(X) = \{\Lambda((x_n)_{n\in\omega}) : \text{ for each sequence } (x_n) \text{ in } X\}$$

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Recall (Meza-Alcántara, 2009)

- If X is compact, then $X \notin FinBW(\mathcal{I}) \iff conv \leq_K \mathcal{I}$.
- If X is not compact, then $X \notin FinBW(\mathcal{I})$.

Corollary

- If X is compact, then $\emptyset \in \Lambda_{\mathcal{I}}(X) \iff conv \leq_K \mathcal{I}$.
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Writing equality

$$\Lambda_{\mathcal{I}}(X) = \mathcal{F}$$

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Theorem (Folklore)

$\Lambda_{\mathrm{Fin}}(X) = \Pi^0_1(X)$ (all closed subsets of X)

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If $\mathcal{I}$ is a maximal ideal, then

1. $\Lambda_{\mathcal{I}}(X) = \{\{x\} : x \in X\} \cup \{\emptyset\}$
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Tools for characterizations

Definition

A family $\{A_s : s \in 2^{<\omega}\}$ of subsets of ω is called an $\mathcal{I}$-scheme if for every $s \in 2^{<\omega}$

1. $A_s \notin \mathcal{I}$,
2. $A_{s^\frown 0} \cap A_{s^\frown 1} = \emptyset$,
3. $A_{s^\frown 0} \cup A_{s^\frown 1} \subseteq A_s$.

Definition

$B_{\mathcal{I}}(\mathcal{A}) = \{x \in 2^\omega : \neg(\exists C \notin \mathcal{I}\ \forall n \in \omega\ |C \setminus A_{x \upharpoonright n}| < \omega)\}$

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- $\mathcal{I} \in P(\Pi^0_1)$ if there is an $\mathcal{I}$-scheme $\mathcal{A}$ with $B_{\mathcal{I}}(\mathcal{A}) = \emptyset$
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Open and closed sets

Theorem

(1) The following conditions are equivalent.

1. $\mathcal{I} \in P(\Pi^0_1)$.
2. $\Pi^0_1(X) \subseteq \Lambda_{\mathcal{I}}(X)$.
3. $\Lambda_{\mathcal{I}}(X)$ contains an analytic set which is not countable.

Remark

If $\mathcal{I}$ is maximal, then

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The inclusion $\Pi^0_1(X) \subseteq \Lambda_{\mathcal{I}}(X)$ was earlier proved for

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$\Lambda_{\mathcal{I}}(X) \neq \Sigma^0_1(X)$ for any ideal $\mathcal{I}$.

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F_σ and G_δ sets

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Remark

If $\mathcal{I}$ is maximal, then

- $\Lambda_{\mathrm{Fin}\oplus(\{\emptyset\}\otimes\mathcal{I})}(X) = \{A \cup B : A \text{ is closed and } B \text{ is countable}\}$.
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$\mathrm{Fin}^2 \in P(\Pi^0_3)$. Consequently, $\mathrm{Fin}^2 \oplus Max \in P(\Pi^0_3)$.

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But wait, we haven't defined the property $P(\Sigma^0_4)$!

PART 4: Borel complexity of ideal vs. Borel complexity of $\Lambda_{\mathcal{I}}(x)$

Borel complexity of ideals

Theorem

(1) For each $\alpha \geq 3$ there is an ideal $\mathcal{I} \in \Sigma^0_\alpha \setminus \Pi^0_\alpha$ such that

$$\Lambda_{\mathcal{I}}(X) = \Pi^0_1(X).$$

(2) If $\mathcal{I}$ is Σ^0_2, then $\Lambda_{\mathcal{I}}(X) = \Pi^0_1(X)$.

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Borel complexity of ideals

Definition

An ideal $\mathcal{I}$ is called a Farah ideal if there is a family of compact hereditary sets $\{C_n : n < \omega\}$ such that $\mathcal{I} = \{A \subseteq \omega : \forall n < \omega \, \exists m < \omega \, (A \setminus [0, m) \in C_n)\}$.

It is known that every Farah ideal is Π^0_3.

Theorem (He-Zang-Zang, 2022)

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