The Katětov order in-between $\mathcal{ED}$ and $\mathrm{Fin} \otimes \mathrm{Fin}$

Jacek Tryba
University of Gdańsk
jacek.tryba@ug.edu.pl

Uniwersytet Gdański

Ideals generated by sequences of functions

Definition 1. [1] For a family $\mathcal{F} \subseteq \omega^\omega$ we define the ideal $\mathcal{I}(\mathcal{F})$ on $\omega \times \omega$ to be the ideal generated by the family

$$\{A \subseteq \omega \times \omega : \exists f \in \mathcal{F} \, \forall^\infty n \, (|\{k : (n,k) \in A\}| \leq f(n))\}.$$

Examples

Some well known ideals are of the form $\mathcal{I}(\mathcal{F})$:

- $\mathcal{I}(\mathcal{F}) = \mathrm{Fin} \otimes \{\emptyset\}$ for $\mathcal{F} = \{(0, 0, \ldots)\}$,
- $\mathcal{I}(\mathcal{F}) = \mathcal{ED}$ for $\mathcal{F} = \{f \in \omega^\omega : f \text{ is constant}\}$,
- $\mathcal{I}(\mathcal{F}) = \mathrm{Fin} \otimes \mathrm{Fin}$ for $\mathcal{F} = \omega^\omega$.

Properties

Proposition 1. *If there exists $f \in \mathcal{F}$ such that $f(n) \neq 0$ for all but finitely many $n \in \omega$, then $\mathcal{I}(\mathcal{F})$ is tall, i.e. for every infinite $A \subseteq \omega \times \omega$ there exists infinite $B \subseteq A$ such that $B \in \mathcal{I}(\mathcal{F})$.*

Proposition 2. *For every $\mathcal{F} \subseteq \omega^\omega$, $\mathcal{I}(\mathcal{F}) \subseteq \mathrm{Fin} \otimes \mathrm{Fin}$. Moreover, if there exists $f \in \mathcal{F}$ such that $f(n) \neq 0$ for all but finitely many $n \in \omega$, then $\mathcal{ED} \subseteq \mathcal{I}(\mathcal{F})$.*

Proposition 3. *The ideal $\mathcal{I}(\mathcal{F})$ is not a P-ideal, i.e. there is a sequence $A_0, A_1, \ldots \in \mathcal{I}(\mathcal{F})$ such that for every $A \in \mathcal{I}(\mathcal{F})$ there is $n \in \omega$ such that $|A_n \setminus A| = \omega$.*

Proposition 4. *Ideals of the form $\mathcal{I}(\mathcal{F})$ have the following topological properties:*

- *Every ideal $\mathcal{I}(\mathcal{F})$ has the Baire property.*
- *If $\mathcal{F}$ is σ-compact, then $\mathcal{I}(\mathcal{F})$ is a σ-compact (hence, F_σ) ideal.*
- *If $\mathcal{F}$ is countable, then $\mathcal{I}(\mathcal{F})$ is an F_σ ideal.*
- *If $|\mathcal{F}| < \mathfrak{b}$, then $\mathcal{I}(\mathcal{F})$ is contained in an F_σ ideal.*
- *If $\mathcal{F}$ is a Borel (or even analytic) set, then $\mathcal{I}(\mathcal{F})$ is an analytic ideal.*
- *There are Borel ideals of the form $\mathcal{I}(\mathcal{F})$ of arbitrarily high Borel complexity i.e. for every $\alpha < \omega_1$ there exists $\mathcal{F} \subseteq \omega^\omega$ such that the ideal $\mathcal{I}(\mathcal{F})$ is Borel but not in $\mathbf{\Sigma}^0_\alpha$.*
- *There exists $\mathcal{F} \subseteq \omega^\omega$ such that the ideal $\mathcal{I}(\mathcal{F})$ is not Borel.*

Cardinal characteristics

Hernández-Hernández and Hrušák introduced in [2] modified versions of classical cardinal characteristics that are more suitable for tall ideals on countable sets.

$$\mathrm{add}^*(\mathcal{I}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \wedge \neg \exists B \in \mathcal{I} \, \forall A \in \mathcal{A} \, (A \subset^* B)\},$$
$$\mathrm{cov}^*(\mathcal{I}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \wedge \neg \exists B \in \mathcal{P}(\omega) \setminus \mathrm{Fin}^* \, \forall A \in \mathcal{A} \, (A \subset^* B)\},$$
$$\mathrm{non}^*(\mathcal{I}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{P}(\omega) \setminus \mathrm{Fin}^* \wedge \forall B \in \mathcal{I} \, \exists A \in \mathcal{A} \, (B \subset^* A)\},$$
$$\mathrm{cof}^*(\mathcal{I}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \wedge \forall B \in \mathcal{I} \, \exists A \in \mathcal{A} \, (B \subset^* A)\}.$$

Meza-Alcántara have calculated these characteristics for $\mathcal{ED}$ and $\mathrm{Fin} \otimes \mathrm{Fin}$ in [3]. These results can be generalized in the following way.

Proposition 5. $\mathrm{add}^*(\mathcal{I}(\mathcal{F})) = \aleph_0$ *and* $\mathrm{non}^*(\mathcal{I}(\mathcal{F})) = \aleph_0$ *for all tall ideals* $\mathcal{I}(\mathcal{F})$.

Proposition 6. $\mathfrak{b} \leq \mathrm{cov}^*(\mathcal{I}(\mathcal{F})) \leq \mathrm{non}(\mathcal{M})$ *for each tall ideal* $\mathcal{I}(\mathcal{F})$.

Theorem 1. *Let $\mathcal{I}(\mathcal{F})$ be a tall ideal. If $|\mathcal{F}| < \mathfrak{b}$, then $\mathrm{cov}^*(\mathcal{I}(\mathcal{F})) = \mathrm{non}(\mathcal{M})$.*

Proposition 7. $\mathrm{cof}^*(\mathcal{I}(\mathcal{F})) \geq \mathfrak{d}$ *if and only if* $\mathcal{I}(\mathcal{F}) \neq \mathrm{Fin} \otimes \{\emptyset\}$.

Theorem 2. $\mathrm{cof}^*(\mathcal{I}(\{k \cdot f : k \in \omega\})) = \mathfrak{c}$ *for each $f \in \omega^\omega$ such that $f \neq^* 0$.*

Katětov order

Definition 2. For ideals $\mathcal{I}, \mathcal{J}$ on X and Y respectively, we write

- $\mathcal{I} \leq_K \mathcal{J}$ if there is a function $f : Y \to X$ such that $f^{-1}[A] \in \mathcal{J}$ for every $A \in \mathcal{I}$ ($\leq_K$ is called the *Katětov order*);
- $\mathcal{I} \leq_{KB} \mathcal{J}$ if there is a finite to one function $f : Y \to X$ such that $f^{-1}[A] \in \mathcal{J}$ for every $A \in \mathcal{I}$ ($\leq_{KB}$ is called the *Katětov-Blass order*).

How many $\mathcal{I}(\mathcal{F})$ ideals are there?

Theorem 3. *There is an order embedding of the family of all ideals, ordered by the Katětov-Blass order, into the family of ideals of the form $\mathcal{I}(\mathcal{F})$, ordered by the Katětov-Blass order.*

Theorem 4. *There are $2^\mathfrak{c}$ pairwise $\leq_K$-incomparable ideals of the form $\mathcal{I}(\mathcal{F})$.*

Structure of Borel ideals of the form $\mathcal{I}(\mathcal{F})$

By using ideals of the form $\mathcal{I}(\mathcal{F})$, one can show that the structure of ideals in-between the ideals $\mathcal{ED}$ and $\mathrm{Fin} \otimes \mathrm{Fin}$ in the Katětov order is quite complicated.

Theorem 5. *There is an order embedding of $\mathcal{P}(\omega)/\mathrm{Fin}$, ordered by $\subseteq^*$, into the family of Borel (in fact $\mathbf{\Sigma}^0_4$) ideals of the form $\mathcal{I}(\mathcal{F})$ which are in-between the ideals $\mathcal{ED}$ and $\mathrm{Fin} \otimes \mathrm{Fin}$, ordered by the Katětov (or equivalently Katětov-Blass) order. In particular,*

- *there is a $\leq_K$-antichain of cardinality $\mathfrak{c}$ of Borel ideals of the form $\mathcal{I}(\mathcal{F})$ (in particular, there are $\mathfrak{c}$ pairwise nonisomorphic Borel ideals of the form $\mathcal{I}(\mathcal{F})$);*
- *there are increasing and decreasing $\leq_{KB}$-chains of length $\mathfrak{b}$ of Borel ideals of the form $\mathcal{I}(\mathcal{F})$.*

Theorem 6. *There are $\mathfrak{c}$ pairwise nonisomorphic F_σ ideals of the form $\mathcal{I}(\mathcal{F})$.*

Open problems

Problem 1. Is there an order embedding of $\mathcal{P}(\omega)/\mathrm{Fin}$, ordered by $\subseteq^*$, into the family of F_σ ideals of the form $\mathcal{I}(\mathcal{F})$, ordered by the Katětov order?

Problem 2. Are there uncountable increasing (decreasing, resp.) $\leq_K$-chains of F_σ ideals of the form $\mathcal{I}(\mathcal{F})$?

Problem 3. Are there uncountable $\leq_K$-antichains of F_σ ideals of the form $\mathcal{I}(\mathcal{F})$?

Problem 4. Does $\mathrm{cof}^*(\mathcal{I}(\mathcal{F})) = \mathfrak{c}$ for each countable family $\mathcal{F}$ such that $\mathcal{I}(\mathcal{F}) \neq \mathrm{Fin} \otimes \{\emptyset\}$?

References

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