Differentially compact spaces

Krzysztof Kowitz
University of Gdańsk

Krzysztof.Kowitz@phdstud.ug.edu.pl

Introduction

Examples of F_σ ideals are:

$$\mathcal{W} = \{A \subseteq \omega : A \text{ is not an AP-set}\}$$

and

$$\mathcal{F} = \{A \subseteq \omega : A \text{ is not an IP}_{rich}\text{-set}\}.$$

An AP-set is the set of natural numbers, which contains arithmetic progressions of arbitrary length. An IP$_{rich}$-set A is the set of natural numbers such that

$$\underset{n\in\omega}{\forall}\ \underset{B\subseteq\omega}{\exists} (|B| = n \wedge FS(B) \subseteq A),$$

where $FS(B) = \{\sum \alpha : \alpha \subseteq B \text{ and } \alpha \neq \emptyset \text{ is finite}\}$.

A topological space X is called van der Waerden (Folkman) if for every sequence $(x_n)_{n\in\omega}$ in X there exists a converging subsequence $(x_{n_k})_{k\in\omega}$ so that $\{n_k : k \in \omega\}$ is an AP-set (IP$_{rich}$-set) [3].

A topological space X is called an $\mathcal{I}$-space [3] if for every sequence $(x_n)_{n\in\omega}$ in X there exists a converging subsequence $(x_n)_{n\in A}$ with $A \notin \mathcal{I}$. Note that $\mathcal{I}$-spaces were earlier defined and examined in [1], where the authors used the term "$(X, \mathcal{I})$ has $FinBW$ property" instead of "X is an $\mathcal{I}$-space". In particular, van der Waerden (Folkman) spaces coincide with $\mathcal{W}$-spaces ($\mathcal{F}$-spaces).

A set $A \subseteq \omega$ is an DP-set if there exists an infinite set $S \subseteq \omega$ so that the difference set $D(S) \subseteq A$, where $D(S) = \{m - n : m > n; m, n \in S\}$. The ideal $\mathcal{D}$ is a family of sets A such that $\mathcal{D} = \{A \subseteq \omega : A \text{ is not a DP-set}\}$.

Suppose that $S \subseteq \omega$ is infinite. A DP-sequence $(x_n)_{n\in D(S)}$ in a topological space X DP-converges [9] to a point $x \in X$ if for every neighborhood U of x there exists $m \in \omega$ so that $\{x_n : n \in D(S \setminus m)\} \subseteq U$.

A topological space X is differentially compact [9] if for every sequence $(x_n)_{n\in\omega}$ in X there exists an infinite set $S \subseteq \omega$ so that $(x_n)_{n\in D(S)}$ is DP-converges to some $x \in X$. Differentially compact spaces were independently introduced and examined in [2], where the author used the name "R-space" instead of "differentially compact space".

Motivation

In 2002, Kojman and Shelah [6] showed (assuming the continuum hypothesis) that there is a van der Waerden space that is not a Hindman space. A year later, Lingsheng Shi [9] in his doctorate constructed a van der Waerden space that is not a differentially compact space. In the following year Jones [4] replaced the continuum hypothesis with the Martin's axiom in the construction of Kojman and Shelah. Assuming Martin's axiom, we will generalize the Shi result to F_σ ideals. Assuming the continuum hypothesis, we extend this result to P^+-ideals.

Mrówka spaces are not differentially compact spaces

Let $\mathcal{A}$ be a pairwise almost disjoint family of infinite subsets of ω. Define a topological space $\Psi(\mathcal{A})$ as follows: the underlying set of $\Psi(\mathcal{A})$ is $\omega \cup \mathcal{A}$, the points of ω are isolated and a basic neighborhood of $A \in \mathcal{A}$ has the form $\{A\} \cup (A \setminus F)$ with F finite. (The space $\Psi(\mathcal{A})$ was introduced in [8].)

Let $\Phi(\mathcal{A}) = \Psi(\mathcal{A}) \cup \{\infty\}$ be the one-point compactification of $\Psi(\mathcal{A})$. (Recall that open neighborhoods of ∞ are of the form $\Phi(\mathcal{A}) \setminus K$ for compact sets $K \subseteq \Psi(\mathcal{A})$.)

If $\mathcal{A}$ is a *mad family on* ω (i.e. infinite maximal pairwise almost disjoint family of infinite subsets of ω), then the space $\Psi(\mathcal{A})$ is called a *Mrówka space defined by* $\mathcal{A}$.

Theorem 1 (KK, [7]). *No Mrówka space is a differentially compact space.*

$\mathcal{I}$-spaces which are not differentially compact spaces

Lemma 1 (KK, [7]). *Let $\mathcal{I}$ be a P^+-ideal, a set $A \in \mathcal{I}^+$ ($A \subseteq \omega$) and $f : \omega \to \omega$. There exists a set $C \subseteq A$ and $C \in \mathcal{I}^+$ such that either*
1) f is constant on C,
or 2) f is finite-to-one on C.

Lemma 2 (KK, [7]). *Suppose CH. Let $\mathcal{I}$ be a P^+-ideal. There exists a maximal almost disjoint family $\mathcal{A} \subseteq [\omega]^\omega$ such that for each $\mathcal{I}^+$-set $B \subseteq \omega$ and each finite-to-one function $f : B \to \omega$ there is an $\mathcal{I}^+$-set $C \subseteq B$ and $A \in \mathcal{A}$ so that $f[C] \subseteq A$.*

Lemma 3 (KK, [7]). *Suppose $MA_{\sigma-centered}$ holds. Let $\mathcal{I}$ be an F_σ ideal. There exists a maximal almost disjoint family $\mathcal{A} \subseteq [\omega]^\omega$ such that for each $\mathcal{I}^+$-set $B \subseteq \omega$ and each finite-to-one function $f : B \to \omega$ there is an $\mathcal{I}^+$-set $C \subseteq B$ and $A \in \mathcal{A}$ so that $f[C] \subseteq A$.*

Theorem 2 (KK, [7]). *Suppose $MA_{\sigma-centered}$ (CH, resp.) holds. If $\mathcal{I}$ is an F_σ ideal (P^+-ideal, resp.), then there exists a Mrówka space which is an $\mathcal{I}$-space, but not a differentially compact space.*

Differentially compact spaces that are $\mathcal{I}$-spaces

Definition 1 ([5]). For ideals $\mathcal{I}, \mathcal{J}$ we say that $\mathcal{I}$ is below $\mathcal{J}$ in the *Katětov order* if there is a function $f : \omega \to \omega$ such that $f^{-1}[A] \in \mathcal{J}$ for every $A \in \mathcal{I}$. We denote it by $\mathcal{I} \leq_K \mathcal{J}$. When $\mathcal{I}$ is not below $\mathcal{J}$ in Katětov order, we denote it by $\mathcal{I} \not\leq_K \mathcal{J}$.

Theorem 3 (KK, [7]). *If $\mathcal{I} \leqslant_K \mathcal{D}$ and $\mathcal{I}$ is a P^+-ideal, then every differentially compact space is an $\mathcal{I}$-space. In particularly, if $\mathcal{I}$ is an F_σ ideal and $\mathcal{I} \leqslant_K \mathcal{D}$ then every differentially compact space is an $\mathcal{I}$-space.*

Open problems

Question 1. Is it consistent that there is a differentially compact space that is not a Folkman space ($\mathcal{F}$-space)?

Question 2. Is it consistent that there is a differentially compact space that is not a van der Waerden space ($\mathcal{W}$-space)?

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