Determinacy and generic absoluteness for the definable powerset of the universally Baire sets

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joint with Grigor Sargsyan

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- Universally Baire sets

- Generic absoluteness

- The uB-powerset

Main results: Determinacy and generic absoluteness (“Sealing”) for the uB-powerset of the universally Baire sets.

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Universally Baire sets: the definition

Definition (Schilling-Vaught, Feng-Magidor-Woodin)

A subset A of a topological space Y is *universally Baire* if for every topological space X and continuous $f\colon X \to Y$,

$$f^{-1}\text{''}A \text{ has the property of Baire in } X.$$

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- Every analytic set $A \subseteq {}^{\omega}\omega$ is universally Baire.

But this is not the definition we want to use.

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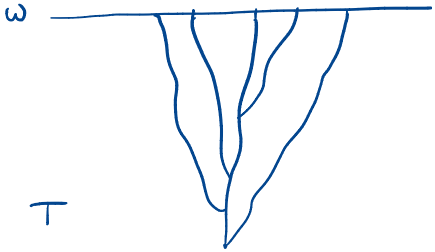

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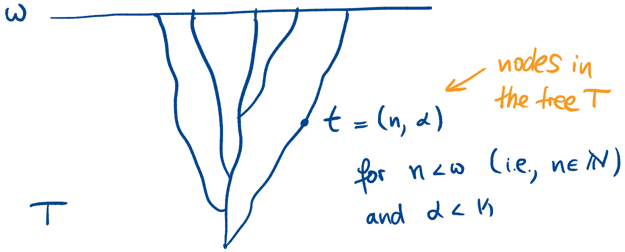

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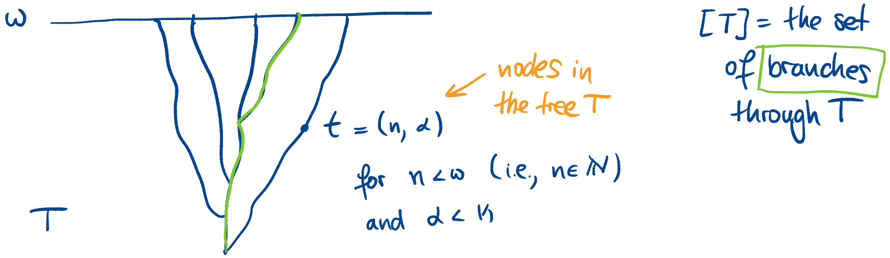

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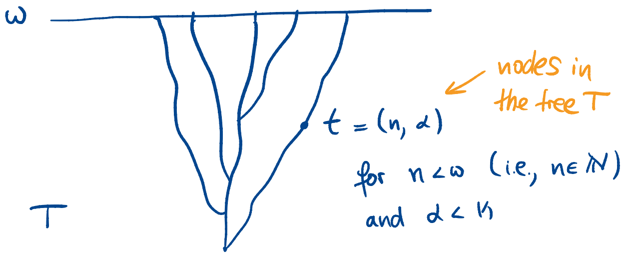

$[T]$ = the set of branches through T

$x \in p[T]$ iff $x = (n_0, n_1, n_2, \dots)$ and there is a branch $b \in [T]$ with $(n_i, \alpha_i) \in b$ for some $\alpha_i < \kappa$ for all $i < \omega$

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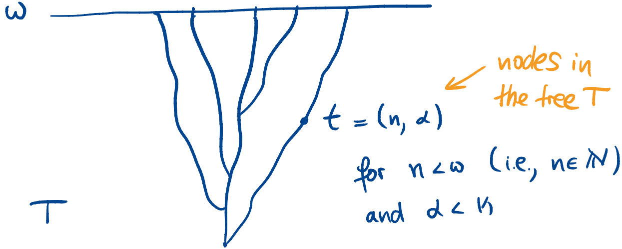

for $n < \omega$ (i.e., $n \in \mathbb{N}$) and $\alpha < \kappa$

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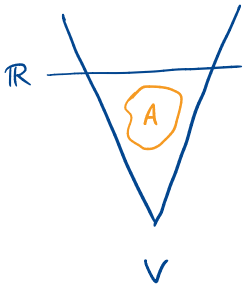
A set $A \subseteq \mathbb{N}^{\mathbb{N}}$ is <u>Suslin</u> iff $A = p[T]$ for some tree T.

Universally Baire sets: the set-theoretic picture

- Being universally Baire is a strengthening of being Suslin.



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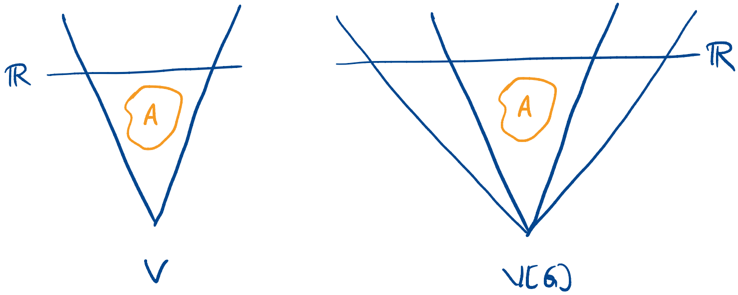

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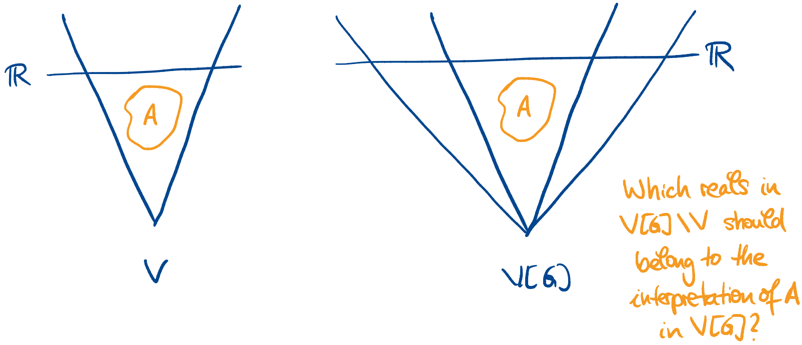

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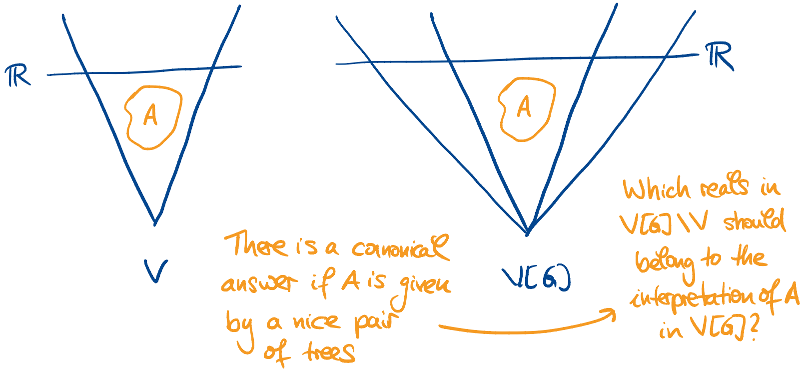

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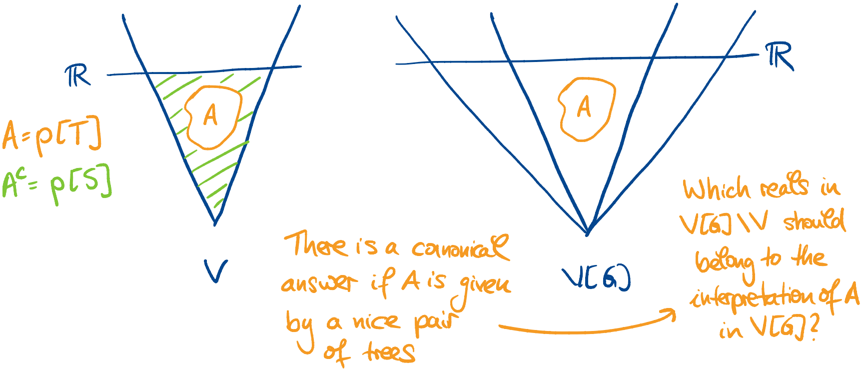

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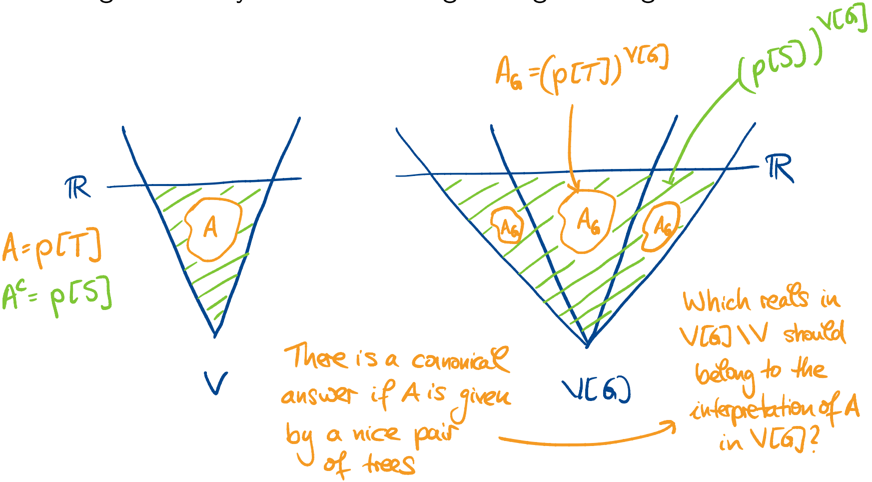

Universally Baire sets: the useful definition (in set theory)

Definition

Let (S, T) be trees on $\omega \times \kappa$ for some ordinal κ and let Z be any set. We say (S, T) *is* Z*-absolutely complementing* iff

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Definition (Feng-Magidor-Woodin)

A set of reals A is *universally Baire (uB)* if for every Z, there are Z-absolutely complementing trees (S, T) with

$$p[S] = A.$$

Universally Baire sets in inner model theory

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- Universally Baire iteration strategies have this property.

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Main results: Determinacy and generic absoluteness ("Sealing") for the uB-powerset of the universally Baire sets.

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Theorem (Steel, Woodin)

Suppose there is a proper class of Woodin cardinals. Let $V[g] \subseteq V[g * h]$ be set generic extensions of V. Then

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2. *for any universally Baire set* A, $L(A, \mathbb{R}) \models \mathrm{AD}$ *and there is an elementary embedding*
$$j\colon L(A_g, \mathbb{R}_g) \to L(A_{g*h}, \mathbb{R}_{g*h}).$$

How about all uB sets?

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- Is it a strong model of determinacy, e.g., does it satisfy $\mathrm{AD}_\mathbb{R}$ or "$\mathrm{AD}_\mathbb{R} + \Theta$ is regular"?
- Is there a generic absoluteness theorem for the theory of $L(\Gamma^\infty_g, \mathbb{R}_g)$?

Sealing

For any g generic over V, write $\mathbb{R}_g = \mathbb{R}^{V[g]}$ and

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Definition (Woodin)

Sealing is the conjunction of the following statements.

1. For every set generic g over V, $L(\Gamma^\infty_g, \mathbb{R}_g) \models \mathrm{AD}^+$ and $\mathcal{P}(\mathbb{R}_g) \cap L(\Gamma^\infty_g, \mathbb{R}_g) = \Gamma^\infty_g$.
2. For every set generic g over V and set generic h over $V[g]$, there is an elementary embedding

$$j\colon L(\Gamma^\infty_g, \mathbb{R}_g) \to L(\Gamma^\infty_{g*h}, \mathbb{R}_{g*h})$$

such that for every $A \in \Gamma^\infty_g$, $j(A) = A_h$.

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↳ This contradicts Clause (1) of Sealing,

so Sealing cannot hold in mice

(the way we know them)

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Is there a large cardinal that implies Sealing?

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- We would like to add the stack of all of these.
- In the presence of Mouse Capturing and letting

$$j_g : L(\Gamma^\infty, \mathbb{R}) \to L(\Gamma_g^\infty, \mathbb{R}_g)$$

be the canonical embedding with $j_g(A) = A_g$ for every $A \in \Gamma^\infty$, this can be represented as $Lp^{j_g``\Gamma^\infty}(X)$.

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This motivates the definition of the uB-powerset.

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Main results: Determinacy and generic absoluteness ("Sealing") for the uB-powerset of the universally Baire sets.

Definition of the uB-powerset

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Suppose X is a set and let $\iota_X = \max(|X|, |\Gamma^\infty|)$. Then we define $\wp_{uB}(X)$ to be the set of those Y such that whenever $g \subseteq \mathrm{Col}(\omega, \iota_X)$ is V-generic,

$$Y \text{ is ordinal definable in } L(\Gamma^\infty_g, \mathbb{R}_g)$$
$$\text{from parameters in } \{X, j_g\text{''}\Gamma^\infty\} \cup j_g\text{''}\Gamma^\infty,$$

where $j_g : L(\Gamma^\infty, \mathbb{R}) \to L(\Gamma^\infty_g, \mathbb{R}_g)$ is the canonical embedding with the property that $j_g(A) = A_g$ for every $A \in \Gamma^\infty$.

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Sealing implies that instead of $g \subseteq \mathrm{Col}(\omega, \iota_X)$ above we can consider any V-generic h with the property that there is $k \in V[h]$ which is V-generic for $\mathrm{Col}(\omega, \iota_X)$.

Determinacy for the uB-powerset: the statement

Write $\mathcal{A}_h^\infty = (\wp_{uB}(\Gamma^\infty))^{V[h]}$.

Theorem

Suppose κ is a supercompact cardinal, there is a proper class of inaccessible limits of Woodin cardinals and λ is an inaccessibe limit of Woodin cardinals above κ. Suppose $h \subseteq \mathrm{Col}(\omega, {<}\lambda)$ is V-generic. Then $L(\mathcal{A}_h^\infty) \models \mathsf{AD}^+$.

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The proof uses Sealing and ideas from Steel's stationary-tower-free proof of Woodin's Derived Model Theorem.

Weak Sealing for the uB-powerset: the statement

Write $\mathcal{A}^\infty = \wp_{uB}(\Gamma^\infty)$ and, if g is V-generic, $\mathcal{A}^\infty_g = (\mathcal{A}^\infty)^{V[g]}$. If Sealing holds, let

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be the canonical Sealing embedding.

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Definition

We say Weak Sealing holds for the uB-powerset if

1. Sealing holds,

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$$j_{g,g'} : L(\Gamma^\infty_g, \mathbb{R}_g) \to L(\Gamma^\infty_{g*g'}, \mathbb{R}_{g*g'})$$

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This implies that the theory of the model $L(\mathcal{A}^\infty)$ cannot be changed by forcing.

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Theorem (M-Sargsyan)

Suppose κ is a supercompact cardinal and there is a proper class of inaccessible limits of Woodin cardinals. Suppose $g \subseteq \mathrm{Col}(\omega, 2^{2^\kappa})$ is V-generic. Then Weak Sealing *for the uB-powerset holds in $V[g]$.*

Weak Sealing for the uB-powerset: the proof

The key technical lemma is a useful derived model representation of $L(\Gamma^\infty, \mathbb{R})$.

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$$L(\Gamma^\infty_{g*h}, \mathbb{R}_{g*h}) = (L(\mathrm{Hom}^*, \mathbb{R}^*))^{M[G]}.$$

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We, in fact, use a slightly more general version of this.

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Conjecture

Suppose κ is a supercompact cardinal and there is a proper class of inaccessible limits of Woodin cardinals. Suppose $g \subseteq \mathrm{Col}(\omega, 2^{2^\kappa})$ is V-generic. Let $\eta_g^\infty = (\Theta_{\Gamma_g^\infty})^{L(\Gamma_g^\infty, \mathbb{R}_g)}$. Then

1. $\mathrm{cf}(\eta_g^\infty) = \omega$.
2. The Sealing Theorem holds for $\mathcal{B}^\infty$ in $V[g]$.