On countably perfectly meager sets

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based on joint papers with Roman Pol and Tomasz Weiss:

- R. Pol, P. Zakrzewski, Countably perfectly meager sets, J. Symbolic Logic (2021), 1-17.
- T. Weiss, P. Zakrzewski, Countably perfectly meager and countably perfectly null sets, arXiv:2304.07579 [math.LO].

Throughout this talk X is a Polish (i.e., separable, completely metrizable) space without isolated points.

Definition 1

A set $A \subseteq X$ is perfectly meager ($A \in \mathbf{PM}$), if for all perfect subsets P of X, the set $A \cap P$ is meager in P (equivalently, for every perfect subset P of X there exists an F_σ-set F in X such that $A \subseteq F$ and $F \cap P$ is meager in P).

Definition 2

We say that a set $A \subseteq X$ is countably perfectly meager in X ($A \in \mathbf{PM}_\sigma$), if for every sequence of perfect subsets $(P_n : n \in \mathbb{N})$ of X, there exists an F_σ-set F in X such that $A \subseteq F$ and $F \cap P_n$ is meager in P_n for each n.

A summary of basic facts about perfectly meager sets:

1. Many classical small (totally imperfect) sets are perfectly meager:
   - any λ'-set A in X (i.e., every countable set $D \subseteq X$ is relatively G_δ in $A \cup D$), in particular, any Hausdorff (ω_1, ω_1^*)-gap in $\mathcal{P}(\mathbb{N})$,
   - any totally imperfect set A in X with the Hurewicz property (i.e., every continuous image of A in $\mathbb{N}^\mathbb{N}$ is bounded in the ordering $\leq^*$ of eventual domination), in particular, any subset of X of cardinality less than $\mathfrak{b}$, any Sierpiński set in $2^\mathbb{N}$, any γ-set in $2^\mathbb{N}$.
2. (Lusin (1933)) If there is a Lusin set, then the class **PM** is not closed under continuous injective preimages (the image of any Lusin set under the Lusin function is perfectly meager).
3. (Recław (1991)) Assuming CH (or MA) there exist two perfectly meager sets whose product is not perfectly meager.

Definition 3

A set $A \subseteq X$ is universally meager ($A \in \mathbf{UM}$), if A is meager with respect to any perfect Polish topology τ ($A \in \mathcal{M}(X, \tau)$) on X giving the original Borel structure of X.

Equivalently, $A \in \mathbf{UM}$ if A does not contain any injective Borel image of a non-meager subset of any perfect Polish space.

Remark 1

Universally meager sets were earlier recognised and studied as two apparently different classes of absolutely of the first category (every Borel isomorphic image of A in X is meager) and $\overline{AFC}$ (A does not contain any Borel one-to-one image of a non-meager set) by Grzegorek.

A summary of basic facts about universally meager sets:

1. **UM** $\subseteq$ **PM**.
2. All classical perfectly meager sets are universally meager; in particular, there are ZFC examples of uncountable universally meager sets.
3. (Grzegorek (1981)) There is a universally meager set of cardinality non($\mathcal{M}$), the smallest cardinality of a non-meager subset of $2^{\mathbb{N}}$ (or any Polish space without isolated points).
4. Consistently (eg., under CH or MA): **UM** $\subsetneq$ **PM** (a witness: the image of any non($\mathcal{M}$)-Lusin set under the Lusin function).
5. (PZ (2001)) Any product of two universally meager sets is universally meager.
6. (Bartoszyński (2001)) Consistently (under Axiom P: For every non-meager set A in $\mathbb{N}^{\mathbb{N}}$ there exists a perfect compact subset P in $\mathbb{N}^{\mathbb{N}}$ such that $A \cap P$ is non-meager in P): **UM** $=$ **PM**.
7. (PZ (2008)) The following are equivalent:
   - $A \in$ **UM**.
   - For every continuous bijection $f : \mathbb{N}^{\mathbb{N}} \to X$ there are sets F_n in X such that $A \subseteq \bigcup_n F_n$ and $f^{-1}(F_n)$ is closed and nowhere dense in $\mathbb{N}^{\mathbb{N}}$ for each $n \in \mathbb{N}$.

A reminder:

We say that a set $A \subseteq X$ is countably perfectly meager in X ($A \in \mathbf{PM}_\sigma$), if for every sequence of perfect subsets $(P_n : n \in \mathbb{N})$ of X, there exists an F_σ-set F in X such that $A \subseteq F$ and $F \cap P_n$ is meager in P_n for each n.

Remark 2

The class $\mathbf{PM}_\sigma$ first appeared (without any specific name) in a paper by Bartoszyński where it was claimed that this property characterizes universally meager sets in the Cantor space $2^{\mathbb{N}}$. Unfortunately, there was a flaw in the part of the argument showing the inclusion $\mathbf{UM} \subseteq \mathbf{PM}_\sigma$.

A summary of basic facts about countably perfectly meager sets:

1. (Bartoszyński (2003)) $\mathbf{PM}_\sigma \subseteq \mathbf{UM}$.
2. Many classical perfectly meager sets are countably perfectly meager in X including:
   - λ'-sets in X, in particular, any Hausdorff (ω_1, ω_1^*)-gap in $\mathcal{P}(\mathbb{N})$,
   - totally imperfect sets in X with the Hurewicz property in particular, subsets of X of cardinality less than $\mathfrak{b}$, Sierpiński sets and γ-sets in $2^{\mathbb{N}}$,
   - subsets of $2^{\mathbb{N}}$ perfectly meager in the transitive sense, i.e., such $A \subseteq 2^{\mathbb{N}}$ that for every perfect subset P of $2^{\mathbb{N}}$, there exists an F_σ-set F in X such that $A \subseteq F$ and $F \cap (P + t)$ is meager in $P + t$ for each $t \in 2^{\mathbb{N}}$, in particular meager-additive and strongly meager sets.
3. Consistently (eg., under CH or MA): $\mathbf{PM}_\sigma \subsetneq \mathbf{PM}$ (the image of any non($\mathcal{M}$)-Lusin set under the Lusin function is not even in $\mathbf{UM}$).
4. (PZ, R. Pol (2021)) Any product of two countably perfectly meager sets is countably perfectly meager: if A and B are $\mathbf{PM}_\sigma$-sets in perfect Polish spaces X and Y, respectively, then $A \times B$ is a $\mathbf{PM}_\sigma$-set in $X \times Y$.

A summary of basic facts about countably perfectly meager sets:

5. (Bartoszyński) For $A \subseteq 2^{\mathbb{N}}$ the following are equivalent:
   - $A \in \mathbf{PM}_\sigma$,
   - for every perfect subset P of $2^{\mathbb{N}}$, there exists an F_σ-set F in $2^{\mathbb{N}}$ such that $A \subseteq F$ and $F \cap (P + q)$ is meager in $P + q$ for every $q \in \mathbb{Q}$, where $\mathbb{Q}$ consists of all eventually zero binary sequences.

6. (PZ, R. Pol (2021)) Consistently, there is a countably perfectly meager (even a λ') set A in $2^{\mathbb{N}}$ which is not perfectly meager in the transitive sense (i.e., it is not the case that for every perfect subset P of $2^{\mathbb{N}}$, there exists an F_σ-set F in X such that $A \subseteq F$ and $F \cap (P + t)$ is meager in $P + t$ for every $t \in 2^{\mathbb{N}}$).

A summary of basic facts about countably perfectly meager sets:

7. (PZ, R. Pol (2021)) The following are equivalent:
   - $A \in \mathbf{PM}_\sigma$.
   - For any perfect Polish topology τ on X giving the original Borel structure of X there are closed in the original topology, τ-meager sets $F_n \subseteq X$ with $A \subseteq \bigcup_n F_n$.
   - For every continuous bijection $f : \mathbb{N}^\mathbb{N} \to X$ there are closed sets F_n in X such that $A \subseteq \bigcup_n F_n$ and $f^{-1}(F_n)$ is nowhere dense in $\mathbb{N}^\mathbb{N}$ for each $n \in \mathbb{N}$.

To be compared with:

7'. (PZ (2001)) The following are equivalent:
   - $A \in \mathbf{UM}$, i.e., for any perfect Polish topology τ on X giving the original Borel structure of X there are τ-closed, τ-meager sets $F_n \subseteq X$ with $A \subseteq \bigcup_n F_n$.
   - For every continuous bijection $f : \mathbb{N}^\mathbb{N} \to X$ there are sets F_n in X such that $A \subseteq \bigcup_n F_n$ and $f^{-1}(F_n)$ is closed and nowhere dense in $\mathbb{N}^\mathbb{N}$ for each $n \in \mathbb{N}$.

Main results

Theorem 1 (PZ, R. Pol (2021))

Consistently: **UM** $\neq$ **PM**$_\sigma$*: there is a universally meager set in* $2^{\mathbb{N}}$ *which is not countably perfectly meager in* $2^{\mathbb{N}}$*.*
More precisely, if there exists a universally meager set in $2^{\mathbb{N}}$ *of cardinality of the continuum, then there is also one which is not countably perfectly meager (in particular, this holds if non*$(\mathcal{M}) = 2^{\aleph_0}$*, so it is true under CH or MA).*

Theorem 2 (T. Weiss, PZ (202?))

If $2^{\aleph_0} \leq \aleph_2$*, then there is a universally meager set in* $2^{\mathbb{N}}$ *which is not countably perfectly meager in* $2^{\mathbb{N}}$*.*

Sketch of a proof of Theorem 1.

It suffices to prove the following

Theorem 3

Let T be a subset of $2^{\mathbb{N}}$ of cardinality $2^{\aleph_0}$. There exist a set $H \subseteq T \times 2^{\mathbb{N}}$ intersecting each vertical section $\{t\} \times 2^{\mathbb{N}}$, $t \in T$, in a singleton and a homeomorphic copy E of H in $2^{\mathbb{N}}$ which is not a $\mathbf{PM}_\sigma$-set in $2^{\mathbb{N}}$. In particular, T is a continuous injective image of E.

Indeed, if T is universally meager, then so is E. But $E \notin \mathbf{PM}_\sigma$.

Let $C_0, C_1, \ldots$ be pairwise disjoint meager Cantor sets in $2^{\mathbb{N}}$ such that:

(1) each non-empty open set in $2^{\mathbb{N}}$ contains some C_n.

Let $P = 2^{\mathbb{N}} \setminus \bigcup_n C_n$.

Claim 1. There exists a set $H \subseteq T \times P$ intersecting each vertical section $\{t\} \times P$, $t \in T$, in a singleton, such that each F_σ-set in $2^{\mathbb{N}} \times 2^{\mathbb{N}}$ containing H contains also $\{t\} \times V$ for some $t \in T$ and a non-empty open set V in $2^{\mathbb{N}}$.

This is proved by the following diagonalization argument.

Let $\{F_t : \ t \in T\}$ be a parametrization on T of all F_σ-sets in $2^{\mathbb{N}} \times 2^{\mathbb{N}}$.

For each $t \in T$, we pick $(t, \varphi(t)) \in (\{t\} \times P) \setminus F_t$, whenever this is possible, and we let $\varphi(t)$ be an arbitrary fixed element of P, otherwise.

Then the graph $H = \{(t, \varphi(t)) : t \in T\}$ has the required property.

For any $s \in 2^{<\mathbb{N}}$ let $N_s = \{x \in 2^{\mathbb{N}} : s \subseteq x\}$ be the standard basic open set in $2^{\mathbb{N}}$ determined by s.

Let $\sim$ be the equivalence relation on $2^{\mathbb{N}} \times 2^{\mathbb{N}}$, whose equivalence classes are given by:

$$[(x,y)]_{\sim} = \begin{cases} N_{x|n} \times \{y\}, & \text{if } y \in C_n, \\ \{(x,y)\}, & \text{if } y \in P \end{cases}$$

Let $\pi(x,y) = [(x,y)]_{\sim}$ be the quotient map onto the quotient space

$$K = (2^{\mathbb{N}} \times 2^{\mathbb{N}})/\sim$$

(whose topology consists of sets $U \subseteq K$ such that $\pi^{-1}(U)$ is open in $2^{\mathbb{N}} \times 2^{\mathbb{N}}$).

Claim 2. The space K is homeomorphic to $2^{\mathbb{N}}$.

Indeed, K is compact, Hausdorff, second countable, zero-dimensional topological space without isolated points.

Finally, let

$$E = \pi(H)$$

(cf. Claim 1).

Clearly, E is a homeomorphic copy of H in K and T is the injective image of E under the continuous function $\mathrm{proj}_1 \circ \pi^{-1}|E$, where proj_1 is the projection of $2^{\mathbb{N}} \times 2^{\mathbb{N}}$ onto the first axis.

The proof is completed by showing that E is not a $\mathbf{PM}_\sigma$-set in K.

It is done by showing that if F^* is an F_σ-set in K such that $E \subseteq F^*$, then F^* contains one of the members of the countable collection $\{P_s : \ s \in 2^{<\mathbb{N}}\}$ of perfect subsets of K defined by letting

$$P_s = \pi(N_s \times C_n).$$

To that end, let F^* be an F_σ-set in K such that $E \subseteq F^*$.

Then $F = \pi^{-1}(F^*)$ is an F_σ-set in $2^{\mathbb{N}} \times 2^{\mathbb{N}}$ containing H, so there are $t \in T$ and a non-empty open set V in $2^{\mathbb{N}}$ such $\{t\} \times V \subseteq F$, cf. Claim 1.

Let us fix $C_n \subseteq V$ (cf. (1)) and let $s = t|n$.

We have $\{t\} \times C_n \subseteq F$, so

(2) $\pi(\{t\} \times C_n) \subseteq F^*$.

But for any $y \in C_n$

$$\pi(t, y) = N_s \times \{y\}.$$

so it follows that:

$$\pi(\{t\} \times C_n) = \{N_s \times \{y\} : y \in C_n\} = \pi(N_s \times C_n) = P_s.$$

Consequently, $P_s \subseteq F^*$ by (2).

The idea of a proof of Theorem 2.

We shall split the argument into two cases.

Case (A): $\mathrm{non}(\mathcal{M}) = \aleph_2$.
Then, by a result of Grzegorek, there exists a universally meager set in $2^{\mathbb{N}}$ of cardinality $\aleph_2 = 2^{\aleph_0}$ and the result follows from Theorem 1.

Case (B): $\mathrm{non}(\mathcal{M}) = \aleph_1$. Then, since $\mathfrak{b} \leq \mathrm{non}(\mathcal{M})$, we also have $\mathfrak{b} = \aleph_1$.
An idea:

Let T be a universally meager set in $2^{\mathbb{N}}$ of cardinality $\aleph_1$ with some additional properties strong enough for the argument to work.

More precisely, the assumption $\mathfrak{b} = \aleph_1$ gives us a set $X = \{x_\alpha : \alpha < \omega_1\} \subseteq [\mathbb{N}]^{\aleph_0}$ which is unbounded and increasing in the sense of the ordering of eventual domination (when viewed as the subset of $\mathbb{N}^{\mathbb{N}}$ consisting of increasing enumerations of sets x_α) and such that x_β is almost contained in x_α whenever $\alpha < \beta < \omega_1$.

We let $T = X \cup \mathbb{Q} \subseteq 2^{\mathbb{N}}$ be the union of X (identified with a subset of $2^{\mathbb{N}}$ via the characteristic functions) and $\mathbb{Q}$, the copy of rationals in $2^{\mathbb{N}}$ consisting of all eventually zero binary sequences.

Then X is concentrated on $\mathbb{Q}$ and by a result of Scheepers, T has property $S_1(\Gamma, \Gamma)$,

i.e., for every sequence $\mathcal{U}_0, \mathcal{U}_1 \dots$ of point-cofinite open covers of T there are sets $U_0 \in \mathcal{U}_0, U_1 \in \mathcal{U}_1, \dots$ such that $\{U_n : n \in \mathbb{N}\}$ is a point-cofinite open cover of T.

In particular, T is universally meager.

The assumption $\mathrm{non}(\mathcal{M}) = \aleph_1$ gives us a non-meager set $M \subseteq 2^{\mathbb{N}}$ of cardinality $\aleph_1$.

Finally, let H be the graph of any bijection between T and M.

Since T is the injective continuous image of H under the projection onto the first axis and T is universally meager, so is H.

The proof is completed by showing that H is not a $\mathbf{PM}_\sigma$-set in $2^{\mathbb{N}} \times 2^{\mathbb{N}}$.

Some more details.

Let C and D be disjoint copies of the Cantor set in $2^{\mathbb{N}}$ such that

(1) the operation $+$ of addition is a homeomorphism between $C \times D$ and $C + D$.

The idea is to repeat the above construction in $C \times D$.

More precisely, abusing the notation, let $T = X \cup \mathbb{Q} \subseteq C$, where $X \cap \mathbb{Q} = \emptyset$, $|X| = \aleph_1$, $|\mathbb{Q}| = \aleph_0$, X is concentrated on $\mathbb{Q}$ and T has property $S_1(\Gamma, \Gamma)$.

Let M be a non-meager set in D of cardinality $\aleph_1$.

Let H be the graph of a bijection $f : T \to M$.

Finally, let

$$Z = \{x + y : (x, y) \in H\} = \{x + f(x) : x \in T\}. \tag{2}$$

Clearly, Z is universally meager as the image of H under the homeomorphism $+$ between $C \times D$ and $C + D$ (cf. (1)).

The proof is completed by showing that Z is not a $\mathbf{PM}_\sigma$-set in $2^{\mathbb{N}}$ (which is actually equivalent to H not being $\mathbf{PM}_\sigma$-set in $C \times D$).

Namely, for the countable collection of perfect sets of the form $P_q = q + D$, where $q \in \mathbb{Q}$, there is no F_σ-set F with $Z \subseteq F$ and $F \cap P_q$ meager in P_q.

The point is that otherwise $(\mathbb{Q} + F) \cap D$ is relatively meager in D,

from which with the help of properties of T one shows that $(T' + F) \cap D$ is meager in D for some $T' \subseteq T$ with $|T \setminus T'| \leq \aleph_0$.

This, however, is impossible as already $T' + Z$ almost contains M which is non-meager in D.

Indeed, let $N = T \setminus T'$.

If $y \in M \setminus f(N)$, then $y = f(x)$ for some $x \in T'$ hence $y = (x + (x + f(x))) \in x + Z$.

Problems:

- Is $\mathbf{PM}_\sigma = \mathbf{UM}$ consistent ?
- Is $\mathbf{PM}_\sigma = \mathbf{PM}$ consistent ?

Thank you for your attention!