Descriptive Set Theory in Generalized Baire Spaces

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Structure of the talk

- Introduction
- Part one: Trees and EF-games.
- Part two: Universally Baire sets.

Introduction.

Why generalize?

- Classical Baire space ω^ω: Limits of finite measurements, as in Natural Sciences.
- Generalized Baire space $\omega_1^{\omega_1}$: Limits of countable measurements, as in ?
- Cohen, L. W.; Goffman, Casper: A theory of transfinite convergence. (1949)
- Sikorski, Roman: Remarks on some topological spaces of high power. (1950)

Countable models I

- Countable models as elements of the Baire space ω^ω (Mostowski 1947, Kuratowski 1948).
- The orbit of a countable model is always Borel (Scott 1963).
- An invariant set is Borel iff it is $L_{\omega_1\omega}$-definable (Scott 1963).
- Useful tool: *EF*-game of length ω (Fraisse 1954, Ehrenfeucht 1957).
- Countable ordinals can work as “clocks".

Countable models II

- The Scott rank of a countable model.
- *The isomorphism of countable models of any complete consistent extension of Peano is Borel complete.* (Coskey-Kossak 2010)
- *The isomorphism of countable models of any complete consistent extension of ZFC + Global Choice is Borel complete.* (Clemens-Coskey-Dworetzky 2020)

Uncountable models

- *EF*-game of length ω with countable sequences, rather than single elements, as moves.
- Let η be the order type of the rationals. Let $\Phi(A)$, $A \subseteq \omega_1 \setminus \{0\}$, be the result of replacing α in $(\omega_1, <)$ by $1 + \eta$ if $\alpha \in A$, and by η if $\alpha \notin A$.
- $\Phi(A) \cong \Phi(B)$ iff $A = B \bmod \mathrm{NS}_{\omega_1}$. (Conway 1964)
- $\Phi(A) \equiv_{\infty\omega_1} \Phi(B)$. (Nadel-Stavi 1978)
- *Ergo*, the language $L_{\infty\omega_1}$ is not enough.

Generalized Baire Spaces

- Models of cardinality $\aleph_1$ as elements of $\omega_1^{\omega_1}$, 2^{ω_1}. (Mekler-V. 1993)
- Topology: $N(f, \alpha) = \{g : g \restriction \alpha = f \restriction \alpha\}$.
- Dense set of size 2^ω. A common assumption: CH.
- More generally κ^κ, 2^κ, κ^λ.

Even more generally: κ-spaces

There is a neighbourhood basis $\mathcal{U} = \{U_\alpha(x) : \alpha < \kappa, x \in S\}$, such that

1. $\bigcap_{\alpha<\kappa} U_\alpha(x) = \{x\}$.
2. $\beta < \alpha$ implies $U_\beta(x) \supseteq U_\alpha(x)$.
3. If $x, y \in S$, then for all $\alpha < \kappa$ there is $\beta < \kappa$ such that $\alpha < \beta$ and $U_\beta(x) \cap U_\beta(y) = \emptyset$ or $U_\beta(x) \subseteq U_\alpha(y)$.
4. If $\{U_{\delta_\beta}(x_\beta) : \beta < \alpha\}$, where $\alpha < \kappa$, is such that $\beta < \gamma < \alpha$ implies $U_{\delta_\beta}(x_\beta) \supseteq U_{\delta_\gamma}(x_\gamma)$, then $\bigcap_{\beta<\alpha} U_{\delta_\beta}(x_\beta)$ is open and non-empty.
5. Every κ-Cauchy sequence, i.e. sequence $(x_\alpha)_{\alpha<\kappa}$ such that

$$\forall\alpha < \kappa \exists\beta < \kappa \forall\gamma, \gamma'(\beta < \gamma, \gamma' < \kappa \Rightarrow x_\gamma \in U_\alpha(x_{\gamma'})),$$

converges.

What is it that we want?

- Topological properties of uncountable models, in analogy with countable models.
- How can we say that two uncountable models are very close to being isomorphic, without actually being isomorphic?
- Can we measure how close to being isomorphic two uncountable models are? This measure need not be in terms of ordinals?

Some history

1950-1976: Juhasz, Sikorski, Wang, Weiss: General topology.

1990-1993: Halko, Hyttinen, Mekler, Shelah, Tuuri, V.: Basic setup of descriptive set theory higher up. Trees as generalized ordinals - paradigm.

1999-2004: Dzamonja, Hyttinen, Shelah, Todorcevic, V., Velickovic: Structure of trees.

2012-2023: Dzamonja, Friedman, Hyttinen, Weinstein, Lücke, Montoya, Moreno, Motto Ros, Schlicht, Shelah, Sziraki, V. et al: Deeper into descriptive set theory higher up.

An example of a property of κ-spaces

Proposition (Baire Category Theorem)

Every κ-space satisfies the Baire Category Theorem i.e. the space itself is never κ-meager. (Cohen-Goffman 1949)

Proof.

Suppose we are given nowhere dense sets A_i, $i < \kappa$. We construct $f \notin \bigcup_{i<\kappa} A_i$. Since A_0 is not dense there is $U_{\delta_0}(x_0)$ such that $U_{\delta_0}(x_0) \cap A_0 = \emptyset$. Let us suppose we have constructed $U_{\delta_\xi}(x_\xi)$, $\xi < \alpha$, such that $U_{\delta_\xi}(x_\xi) \cap A_\xi = \emptyset$ and $\xi < \zeta < \alpha$ implies $U_{\delta_\xi}(x_\xi) \supseteq U_{\delta_\zeta}(x_\zeta)$. By the properties of the family $\mathcal{U}$, $\bigcap_{\beta<\alpha} U_{\delta_\beta}(x_\beta)$ is open and non-empty. Since A_α is nowhere dense, there is $U_{\delta_\alpha}(x_\alpha) \subseteq \bigcap_{\beta<\alpha} U_{\delta_\beta}(x_\beta)$ such that $U_{\delta_\alpha}(x_\alpha) \cap A_\alpha = \emptyset$. The sequence $\{x_i : i < \kappa\}$ is a κ-Cauchy sequence, hence converges to some f. This is the f we claimed exists. □

A Cantor-Bendixson Theorem

A set is ω_1-perfect if II wins the perfect set game of length ω_1. It is ω_1-scattered if I wins it.

Theorem (V. 1991)

Assume $I(\omega)$[1]*. Then every closed subset of $\omega_1^{\omega_1}$ is the disjoint union of an ω_1-perfect part and an ω_1-scattered part of cardinality $\leq \aleph_1$.*

Note that $I(\omega)$ implies CH.

[1] For some normal ideal I on ω_2, I^+ has a dense σ-closed subset. (Laver)

Part one: Trees and EF-games.

EF_{ω_1}

- To remedy the failure of $L_{\infty\omega_1}$ to describe (to any reasonable extent) models of cardinality $\aleph_1$, we introduce(d) the EF-game of length ω_1, denoted EF_{ω_1}.
- Players move countable sequences.
- There are ω_1 moves.
- Non-isomorphism player plays at limit stages.
- If $|M| = |N| = \aleph_1$, then $M \cong N$ iff isomorphism player has a winning strategy.

An unsurprising fact of life

- For models bigger than $\aleph_1$ the game may be non-determined. (Mekler-Shelah-V. 1993)
- Cardinality $\aleph_2$: $\Box$ implies EF_{ω_1} can be non-determined, $I^*(\omega)$[2] implies it is always determined.[3]
- Cardinality $\geq \aleph_3$: There are non-determined models, provably in ZFC.

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[2] $NS^+_{\omega_2}$ has a dense σ-closed subset K. (Laver)

[3] If there is a club of non-isomorphic initial segments, then I wins. Otherwise the set of isomorphic initial segments is stationary and II wins by playing moves which work for a set of initial segments in the dense set K.

Trees as clocks to make the game stop faster

- Hyttinen-V. 1990.
- Let T be a wide Aronszajn tree i.e. a tree of size and height $\aleph_1$ without uncountable branches.
- Approximated game: Non-isomorphism player has to go up the tree move by move: $EF^T_{\omega_1}$.
- Harder for non-isomorphism player but easier for isomorphism player.

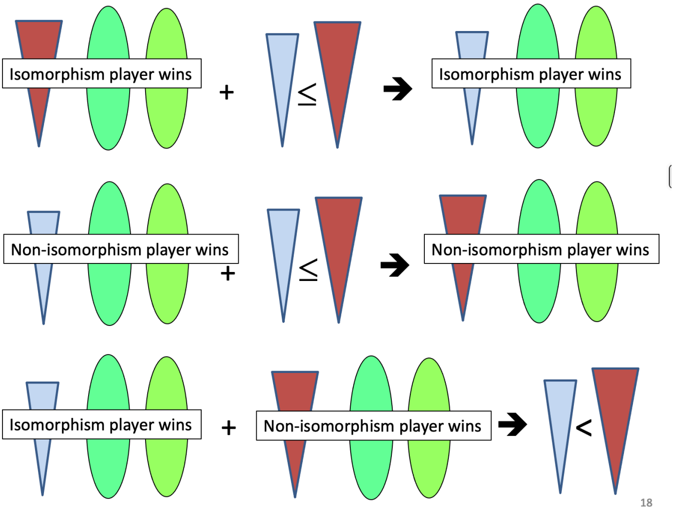
Comparing trees, or how long does the clock tick?

- $T \leq T'$ if there is $\pi : T \to T'$ such that always

$$t <_T t' \to \pi(t) <_{T'} \pi(t').$$

- $T \leq^* T'$ if additionally π is one-one.

Winning the game $EF^T_{\omega_1}$ with T as a clock.



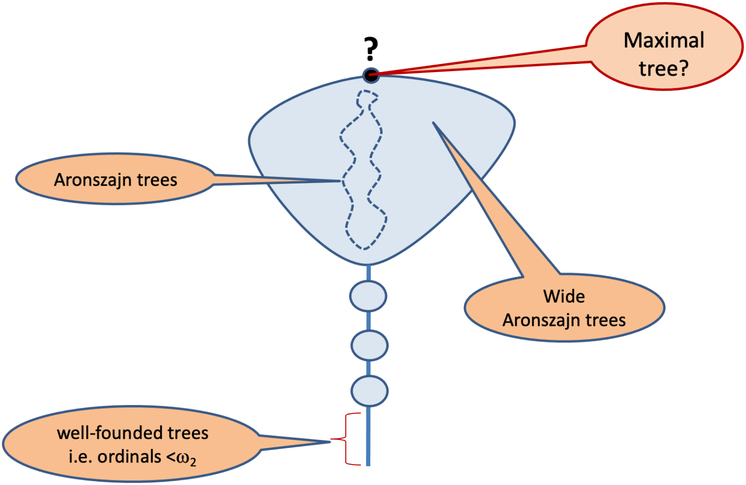
Examples

- Let B_α be the tree of descending sequences in α.
- Then $B_\alpha \leq B_\beta$ iff $\alpha \leq \beta$.
- If $A \subseteq \omega_1$, let $T(A)$ be the tree of increasing closed sequences in A.
- Then $T(A) \leq T(B)$ if and only if $A \subseteq B$ (mod NS_{ω_1}).
- Aronszajn trees, Souslin trees, etc.

From a tree to a bigger tree of the same height

- Let $\sigma(T)$ be the tree of increasing chains in T.
- Always $T < \sigma(T)$. (Kurepa)
- Typically $|\sigma(T)| = |T|^{<|T|}$.
- Example: If CH holds and T is a wide Aronszajn tree, then so is $\sigma(T)$.

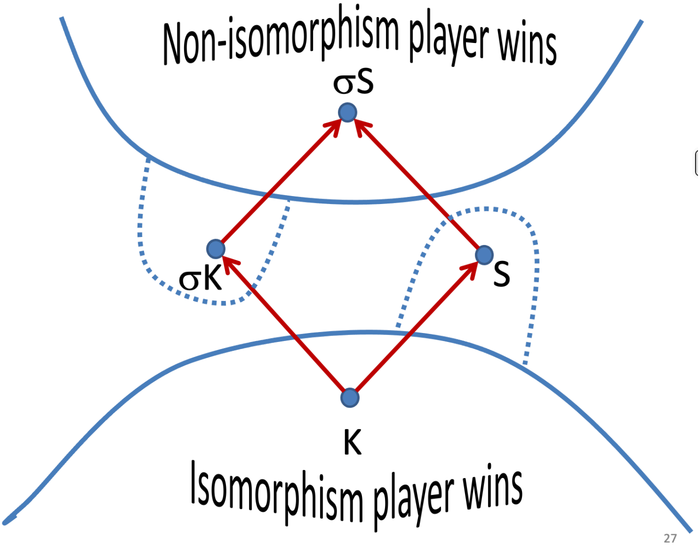
The structure of trees of size and height $\aleph_1$ under $\leq$



Approximating games with trees, trees as ordinals

- Are there for every wide Aronszajn tree T models M, N of cardinality $\aleph_1$ such that $M \not\cong N$ but isomorphism player has a winning strategy in $EF^{T}_{\omega_1}$?
- Yes, if we assume CH (Hyttinen-Tuuri 1991). Otherwise open, but Shelah has a sequence of partial results.
- An easier question: Are there for every wide Aronszajn tree T models M, N of cardinality $\aleph_1$ such that $M \not\cong N$ but non-isomorphism player does not have a winning strategy in $EF^{T}_{\omega_1}$? Yes. (Hyttinen-Tuuri 1991)

Boundary between the advantages of the players.



Two analogues for Scott rank

- A tree T without κ-branches is called a *universal non-equivalence tree* for a model M of cardinality κ if for all models N of cardinality κ in the same vocabulary, if $M \not\cong N$, then Player I has a winning strategy in $EF_\kappa^T(M, N)$.
- A Canary Trees[4] is (if it exists) a universal non-equivalence tree for the free Abelian group of cardinality $\aleph_1$.
- (CH) M has a universal non-equivalence tree of cardinality $\aleph_1$ iff $orb(M)$ is Δ_1^1. (Mekler-V. 1993)
- It is consistent relative to the consistency of ZF that CH holds and every non-classifiable theory has a model of cardinality $\aleph_1$ without a universal non-equivalence tree. (Hyttinen-Tuuri 1991)

[4] Mekler-Shelah 1993

Two analogues for Scott rank

- A tree T without κ-branches is called a *universal equivalence tree* for a model M of cardinality κ if for all models N of cardinality κ in the same vocabulary, if Player *II* has a winning strategy in $EF_\kappa^T(M, N)$, then $M \cong N$.
- If $\kappa = \omega$, every countable model has a universal non-equivalence tree $B_{\alpha+1}$ and a universal equivalence tree B_α, where α is the Scott height of the model.
- CH implies every unstable theory has a model of cardinality $\aleph_1$ without a universal equivalence tree. (Hyttinen-Tuuri 1991)

Let $B(\kappa, T)$ be the Borel rank of $\cong$ for models of T of cardinality κ, if it is Borel and $B(\kappa, T) = \infty$ otherwise.

Theorem (Mangraviti-Motto Ros 2021)

Let $\kappa^{<\kappa} = \kappa > 2^{\aleph_0}$ and T be a countable complete first-order theory.

1. *If T is classifiable shallow of depth α, then $B(\kappa, T) \leq 4\alpha$.*
2. *If T is not classifiable shallow, then $B(\kappa, T) = \infty$.*

Are there truly maximal trees?

- The role of universal trees in understanding of uncountable models raises the question of existence of maximal trees of size and height κ without κ-branches (i.e. wide κ-Aronszajn trees).
- The σ-operation shows that none exist if $\kappa^{<\kappa} = \kappa$.
- What if we have $\kappa^{<\kappa} > \kappa$?

Earlier results on maximal trees.

- Assuming MA_{ω_1}, the class of Aronszajn trees does not have a maximal tree. (Todorcevic 2007)
- Assuming MA_{ω_1}, the class of wide Aronszajn trees does not have a maximal tree. (Dzamonja-Shelah 2021)

The consistency of maximal trees

Theorem (Todorcevic-V. 2001, Ben-Neria - Magidor - V. 2023)

Assume $V = L$ and κ regular but not weakly compact. For every wide κ-Aronszajn tree T there is a κ-Souslin tree S such that $S \not\leq T$.

Theorem (Ben-Neria - Magidor - V. 2023)

Suppose λ is weakly compact and $\kappa < \lambda$ is regular. There is a forcing extension which does not collapse cardinals $\leq \kappa^+$ and in which there is a wide κ^+-Aronszajn tree T such that every wide κ^+-Aronszajn tree can be embedded into T.

Part two: Universally Baire sets.

Universally Baire sets in generalized Baire spaces

- Joint work with Menachem Magidor.
- Independently, similar but stronger results from stronger assumptions by Ikegami and Viale (unpublished).
- κ regular. We do not assume $\kappa^{<\kappa} = \kappa$.

κ-universally Baire

Definition

We call a model M *internally κ-closed*, if $M = \bigcup_{\alpha<\kappa} M_\alpha$ such that $|M_\alpha| < \kappa$ and $\langle M_\xi : \xi < \alpha \rangle \cup \{M_\alpha\} \subseteq M_{\alpha+1}$ for all $\alpha < \kappa$.

Definition

A set $A \subseteq \kappa^\kappa$ is *κ-universally Baire* if for every κ-strategically closed forcing $\mathbb{P}$ there is a term τ such that for any $\theta > 2^{|\mathbb{P}|}$, if $M \prec H_\theta$, $|M| = \kappa$, M internally κ-closed, $\mathbb{P}, \tau \in M$, and G $\mathbb{P}$-generic over M, then

$$[\tau]_G = A \cap M[G].$$

A more familiar formulation

Theorem

The following are equivalent for $A \subseteq \kappa^\kappa$:

(1) *A is κ-universally Baire*

(2) *If $f : E \to \kappa^\kappa$, where E is a κ-space, then $f^{-1}[A]$ is Baire in E.*

Proof idea:
(1)$\to$(2): Let $\mathbb{P}$ be the poset of non-empty basic open neighbourhoods of E.
(2)$\to$ (1): Let E be the space of descending chains of conditions in $\mathbb{P}$.

Bernstein Property

Theorem

If $A \subseteq \kappa^\kappa$ is κ-universally Baire, then either A or $\omega_1^{\omega_1} \setminus A$ contains a copy of 2^κ.

Proof.

Let us force a Cohen element μ of κ^κ. Suppose τ is the $\mathbb{P}$-term given by κ-universal Baireness. Let us first suppose there is a condition p such that $p \Vdash [\mu]_G \in \tau$. Otherwise there is a condition p such that $p \Vdash [\mu]_G \notin \tau$, which is a similar case. Using the universal Baireness of A we build a tree T of conditions which force different elements to τ. At the same time we build models $M_t \prec H_\theta$ of size $< \kappa$ (θ big) and $\mathbb{P} \cap M_t$-generic over M_t sets G_t. Each branch of the tree of height κ gives rise to an element of A in V, by virtue of the universal Baireness of A. □

Example (Halko-Shelah)

The Σ^1_1-set CLUB is not Baire, hence not κ-universally Baire.

Proof.

We can first use the proof of the Baire Category Theorem to show that CLUB is non-κ-meager. Similarly, $N(f, \alpha) \setminus$ CLUB is non-κ-meager for any f and α. From this the claim follows. □

$\Diamond$ implies SLN i.e. the Σ^1_1-set of $x \in \omega_1^{\omega_1}$ coding a Souslin tree, is not Baire.

Similarly for the Σ^1_1-set Tree of $x \in \omega_1^{\omega_1}$ coding a tree without an uncountable branch.

A wrong start?

- The concept “κ-universally Baire" seems very restrictive.
- Hardly any interesting sets are κ-universally Baire.

A step back

Suppose $\mathcal{P}$ is a class of forcing notions, $\mathcal{M}$ is a class of models, and $\mathcal{G}$ is a class of generics for forcing notions in $\mathcal{P}$ over models in $\mathcal{M}$.

Definition

A is $\mathrm{UB}(\mathcal{P}, \mathcal{M}, \mathcal{G})$ if for each forcing $\mathbb{P} \in \mathcal{P}$ there is a term τ such that for any $\theta > 2^{|\mathbb{P}|}$, if $M \prec H_\theta$, $M \in \mathcal{M}$, $\mathbb{P}, \tau \in M$, and $G \in \mathcal{G}$ $\mathbb{P}$-generic over M, then $[\tau]_G = A \cap M[G]$.

Definition

1. $\mathcal{P} = \mathrm{CC}$: σ-closed
2. SP: preserves stationarity of subsets of ω_1
3. $\mathcal{M} = \mathrm{IC}_{\omega_1}$: internally σ-closed.
4. $\mathcal{G} = \mathrm{SCO}$ = stationary correct: If $M[G] \models$ "$\dot{S} \subseteq \omega_1$ is stationary", then $[\dot{S}]_G$ is stationary.

Towards a more familiar formulation

Definition

A topological space E is stationary preserving (SP) if the poset of its non-empty open sets under the set inclusion is SP as a forcing notion.

Proposition

A space E is SP if and only if for every open B and sequence $(B_\alpha)_{\alpha<\omega_1}$ of non-empty open sets there is $C \subseteq B$ such that either eventually $C \cap B_\alpha$ is nowhere dense or there is a club $D \subseteq \omega_1$ such that for all $\alpha \in D$,

$$C \cap \Big(\bigcap_{\beta<\alpha} \bigcup_{\beta<\gamma<\alpha} B_\gamma\Big) \neq \emptyset.$$

A more familiar formulation

Theorem

Suppose $A \subseteq \omega_1^{\omega_1}$. TFAE:

1. *A is $\mathrm{UB}(\mathrm{SP}, \mathrm{IC}_{\omega_1}, \mathrm{SCO})$.*
2. *For every ω_1-space X that is SP and continuous $f : X \to \omega_1^{\omega_1}$ the set $f^{-1}(A)$ has the Baire property in X.*

Good news and bad news

Example

CLUB is $\mathrm{UB}(\mathrm{SP}, \mathrm{IC}_{\omega_1}, \mathrm{SCO})$.

Theorem

Assume $\diamondsuit$. Suppose there is a Woodin cardinal and a measurable cardinal above it. Then the sets SLN and Tree are not $\mathrm{UB}(\mathrm{CC}, \mathrm{IC}_{\omega_1}, \mathrm{SCO})$.

Corollary

Large cardinals cannot imply that all Σ^1_1-subsets of $\omega_1^{\omega_1}$ are $\mathrm{UB}(\mathrm{CC}, \mathrm{IC}_{\omega_1}, \mathrm{SCO})$.

Definition

1. $\mathrm{SP}(\mathrm{MM})$: SP and MM-inducing.
2. $\mathrm{SP}(\star)$: SP and $(\star)$-inducing.

Theorem

If there is a proper class of Woodin cardinals, then every Σ^1_1-subset of $\omega_1^{\omega_1}$ is $\mathrm{UB}(\mathrm{SP}(\mathrm{MM}), \mathrm{IC}_{\omega_1}, \mathrm{SCO})$.

Proof sketch:

- Suppose $\mathbb{P} \in \mathrm{SP}(\mathrm{MM})$. Let $\exists f \varphi(f, x)$ be a Σ^1_1-formula defining in the space $\omega_1^{\omega_1}$ a subset A. We choose the term τ canonically.
- Suppose now $\theta > 2^{|\mathbb{P}|}$ and $N \prec H_\theta$ internally ω_1-closed. We claim that for all stationary correct $\mathbb{P}$-generic G_N over N the equation $[\tau]_{G_N} = N[G_N] \cap A$ holds. Call N bad if this is not the case, i.e. there is a stationary correct $\mathbb{P} \cap N$-generic G_N over N such that there is $x_N \in N[G_N]$ with $N[G_N] \models \neg\exists f \varphi(f, x_N)$ although $\exists f \varphi(f, x_N)$ is true in V.
- We claim bad N do not exist. Suppose they do.

- W.l.o.g. $S = \{N \prec H_\theta : N \text{ is bad}\}$ is stationary in $\mathcal{P}_{\leq\omega_1}(H_\theta)$.
- We can now use a big Woodin cardinal to form a stationary tower forcing $\mathbb{P}^*$ with a generic H, a $\mathbb{P}$-generic G over V inside $V[H]$, such that $\mathbb{P}^*/\mathbb{P}$ is SP and for a suitable x we have $V[G] \models \neg\exists f \varphi(f,x)$ while $V[H] \models \exists f \varphi(f,x)$. This violates:
- Suppose MM, $f \in \omega_1^{\omega_1}$, $\Phi(x)$ is a Σ_1^1 formula, and $\mathbb{P}$ is an SP forcing such that $\mathbb{P} \Vdash$ "$\omega_1^{\omega_1} \models \Phi(f)$". Then $\Phi(f)$ is true.

Theorem

Assume $(\star)$ *and a proper class of Woodin cardinals (or PFA). Then every subset of* $\omega_1^{\omega_1}$ *which is definable over* H_{ω_2} *is* $\mathrm{UB}(\mathrm{SP}(\star), \mathrm{IC}_{\omega_1}, \mathrm{SCO})$.

Proof sketch:

- Suppose $\mathbb{P} \in \mathrm{SP}(\star)$. Let $\varphi(x)$ be a first order formula defining in H_{ω_2} a subset A of $\omega_1^{\omega_1}$. We choose the term τ canonically.
- Suppose now $\theta > 2^{|\mathbb{P}|}$, $M \prec H_\theta$ internally ω_1-closed, and G stationary correct $\mathbb{P}$-generic over N.
- Let $x \in [\tau]_G$. Since $M[G] \models (\star)$, there is a $\mathrm{P}_{\max}$-term μ in $L(\mathbb{R})^{M[G]}$ for x.
- It is a pity that $\mu \in M[G]$ rather than $\mu \in M$!
- We can use the universal Baireness of $\mathbb{R}^\sharp$ to eliminate the effect of G.

A Bernstein-type Property

Theorem

Assume MM^{++} *and a supercompact cardinal. If* $A \subseteq \omega_1^{\omega_1}$ *is* $\mathrm{UB}(\mathrm{SP}(\star), \mathrm{IC}_{\omega_1}, \mathrm{SCO})$*, then* A *or* $\omega_1^{\omega_1} \setminus A$ *contains an* $\aleph_1$*-rake.*

Proof idea

- Force a generic element of $\omega_1^{\omega_1}$ and then $(\star)$.
- Generate a full binary tree of height ω of conditions so that every branch determines a different element of $\omega_1^{\omega_1}$.
- Use MM^{++} to obtain for each branch a model $M \prec H_\theta$ and a generic.
- Use universal Baireness to conclude that all branches are in A or all branches are in $\omega_1^{\omega_1} \setminus A$. The stem decides this.
- Cannot construct a full binary tree of height ω_1 because the union of stationary correct generics is not stationary correct.

Bottom line

- The CLUB-filter is a stumbling block in generalizing descriptive set theory, at least universal Baireness, to generalized Baire spaces.
- We can overcome CLUB by restricting to stationary preserving forcings and stationary correct generics.
- UB-sets can be meaningfully defined (even if they are not even Baire) and they obey Bernstein-like properties.
- How ubiquitous is this (or other) UB in generalized Baire spaces?
- Is universal Baireness useful in a deeper understanding of uncountable models?

Thank you!