Approachable free subsets and a question of Pereira

Gdansk, 5.v.2023,
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Free Sets

Definition (Free Sets)

A set $X \subseteq A$ where $\mathfrak{A} = (A, \langle f_n \rangle_{n<\omega}, \ldots)$ is an algebra is *free* if

$$\forall y \in X (y \notin SH^{\mathfrak{A}}[X \setminus \{y\}]).$$

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Definition (Free Sets with respect to a substructure)

A set $X \subseteq A$ for such an $\mathfrak{A}$ is *free over* N where $N \prec \mathfrak{A}$ if

$$\forall y \in X (y \notin SH^{\mathfrak{A}}[N \cup X \setminus \{y\}]).$$

Free Sets

• For full generality we consider $\mathfrak{A}$ as some $(H(\kappa), \in, \lhd, \langle F^n \rangle, \ldots)$ where $\lhd$ is a well order of $H(\kappa)$, and indeed the F^n include a set of skolem functions for $\mathfrak{A}$.

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• We shall assume the signature is always countable.

• Let $Fr(\theta, \lambda)$ be the assertion that every structure $\mathfrak{A}$ containing θ has a free subset $X \subseteq \theta$ with order type λ.

- (Baumgartner; $V = L$) $Fr(\theta, \omega) \Leftrightarrow \kappa \longrightarrow (\omega)_2^{<\omega}$.

- (Erdős-Hajnal; Devlin) (i) $Fr(\aleph_\alpha, n) \leftrightarrow \alpha \geq n)$;
  (ii) $\neg Fr(\aleph_\omega, \omega_1)$.

Let H_α be the least κ s.t. $Fr(\kappa, \omega_\alpha)$.

- (Shelah) $\neg Fr(\aleph_\alpha, |\alpha|^+)$ and hence $H_\alpha \geq \omega_{\omega_\alpha}$

- If λ is an infinite cardinal, then $\neg Fr(\kappa, \lambda) \Rightarrow \neg Fr(\kappa^+, \lambda)$.

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(*cf.* Jónsson cardinals; *n.b.* also $Fr(\kappa, \kappa) \Rightarrow \kappa$ Jónsson.

• If λ is an infinite cardinal, then $\neg Fr(\kappa, \lambda) \Rightarrow \neg Fr(\kappa^+, \lambda)$.

Theorem (Koepke)

The following are equiconsistent:
(i) ZFC+ "There exists a measurable cardinal".
(ii) $ZFC + Fr(\aleph_\omega, \omega)$.

Definition

N is an *internally approachable* substructure (of length τ) means $N = \bigcup_{\iota<\tau} N_\iota$ for some $\langle N_\iota | \iota < \tau \rangle$ which is a continuous chain of substructures of $N \prec \langle H(\theta), \in, \langle F_n \rangle_{n<\omega}, \ldots \rangle$ - continuous meaning in turn that $\langle N_\xi \mid \xi \leq \iota \rangle \in N_{\iota+1}$ and $\mathrm{Lim}(\zeta) \to \bigcup_{\iota<\zeta} N_\iota = N_\zeta$, for $\iota < \zeta < \tau$.

Our internally approachable substructures will always be of length some τ with $\mathrm{cf}(\tau) > \omega$.

AFSB - The Approachable Free Subset Property

Definition

(Pereira) The *Approachable Free Subset Property (AFSP)* for $\aleph_\omega$ states that for every internally approachable $N \prec \langle H(\theta), \in, \langle F_n \rangle_{n<\omega} \rangle$, the latter any extension of $\langle H(\theta), \in \rangle$, of length ω_l, for some $l < \omega$ and some large θ, if $\chi_N(m) =_{df} \sup(N \cap \omega_m)$ for $m < \omega$ then there is an infinite subsequence $\langle \aleph_{n_m} \rangle_{m<\omega}$ so that $C =_{df} \{\chi_N(n_m)\}_m$ is *free over* N. That is: for any $p < \omega$, then

$$\chi_N(n_m) \notin F_p``(N \cup C \setminus \{\chi_N(n_m)\}).$$

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- Isolated by Pereira (2007 thesis). He showed:

Theorem (Pereira, 2007)

$$ZFC \vdash \neg AFSP \textit{ for } \aleph_\omega \textit{ then } ZFC \vdash \neg(\textit{pcf-conjecture}).$$

I thought this had (2008) been shown:

(1) $\mathrm{Con}(\mathrm{ZFC} + \mathrm{AFSB}\ \textit{for}\ \aleph_\omega) \Rightarrow \mathrm{Con}(\mathrm{ZFC} + \textit{For any } k \geq 1 \textit{ and for arbitrarily large } m > k \{\alpha < \omega_m | o^K(\alpha) \geq \omega_k\} \textit{ is stationary})$.

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Theorem (Adolf, Ben-Neria)

$Con(ZFC + \exists \langle \tau_n \rangle_{n<\omega}\ with\ \sup o(\tau_n) = \sup_n \langle \tau_n \rangle_{n<\omega}) \Rightarrow$
$Con(ZFC + \exists \langle \aleph_{n_m} \rangle_{m<\omega} \wedge AFSP\ for\ \aleph_\omega)$.

ABSP - The Approachable Bounded Subset Property

Definition (A, B-N; Approachable Bounded Subset Property ABSP))

Let $\langle n_m \rangle_{m<\omega}$ be an ascending sequence from ω. The *ABSP* for $\langle \aleph_{n_m} \rangle_{m<\omega}$ states that for every internally approachable $N \prec \langle H(\theta), \in, \langle F_n \rangle_{n<\omega} \rangle$, of length ω_k for some $0 < k < \omega$ and some $\theta > \omega_\omega$, if $\chi_N(m) =_{df} \sup(N \cap \omega_{n_m})$, then for some $n_0 < \omega$, setting $C = \{\chi_N(m) \mid m \geq n_0\}$, for any $m \geq n_0$ $\chi_N(m) = \chi_{N[C \setminus \{\chi_N(m)\}]}(m)$.

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Remark: (i) *ABSP* for $\langle \aleph_{n_m} \rangle_{m<\omega}$ implies that for $m \geq n_0$ and C in the above definition, and for any $F \in N$, that:

(a) $F``(N \cup C \setminus \{\chi_N(m)\}) \cap [\chi_N(m), \omega_{n_m}) = \emptyset$ and in particular

(b) $\chi_N(m) \notin F``(N \cup C \setminus \{\chi_N(m)\})$.

Remark:

(ii) Thus if *ABSP* for $\langle \aleph_{n_m} \rangle_{m<\omega}$ holds then *a fortiori AFSP* for $\aleph_\omega$ holds.

(iii) Similarly we define *ABSP* in exactly the same way for $\langle \tau_m \rangle_{m<\omega}$ any ascending sequence of regular cardinals, rather than just an infinite subset of the $\aleph_n$. We shall use this in the sequel.

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Our previous 2008 argument for (1) actually showed (or can be read as having showed):

Theorem

$\mathrm{Con}(\mathrm{ZFC} + \mathrm{ABSP}$ *for* $\langle \aleph_{n_m} \rangle_{m<\omega}$, *for some* $\{n_m\}_m \subseteq \omega) \Rightarrow$
$\mathrm{Con}(\mathrm{ZFC} +$*For any* $k \geq 1$, *for arbitrarily large* $m > k$
$\{\alpha < \omega_{n_m} | o^K(\alpha) \geq \omega_k\}$ *is stationary*$)$.

Theorem (Adolf, Ben-Neria)

The following are equiconsistent:

(1) There exists an ascending sequence of regular cardinals $\langle \tau_n \rangle_{n<\omega}$ *for which the* ABSP *holds* $\langle \tau_n \rangle_{n<\omega}$.

(2) There exists an ascending sequence of regular cardinals $\langle \tau_n \rangle_{n<\omega}$ *for which the* AFSP *holds* $\langle \tau_n \rangle_{n<\omega}$.

(3) There exists an ascending sequence of regular cardinals $\langle \tau_n \rangle_{n<\omega}$ *for which the product* $\prod_n \tau_n$ *does not carry a continuous tree-like scale.*

(4) There exists a cardinal λ *such that the set of Mitchell orders* $\{o(\mu) \mid \mu < \lambda\}$ *is unbounded in* λ.

Theorem (W)

($\neg O^{\text{pistol}}$) *Let $\langle\tau_n\rangle_{n<\omega}$ be an increasing sequence of regular cardinals, for which* ABSP *holds.*

(i) If the τ_n are inaccessible cardinals in K then for all sufficiently large m either $\{\alpha < \tau_m | o^K(\alpha) \geq \tau_k\}$ is stationary below τ_m or there is $\lambda_m < \tau_m$ with $o^K(\lambda_m) \geq \tau_m$.

(ii) If additionally in (i), for all $\gamma < \tau =_{df} \sup_n \tau_n$ we have $\mathrm{cf}(\gamma) = \mathrm{cf}^K(\gamma)$ then the second alternative holds: for a tail of the τ_m, there is $\lambda_m < \tau_m$ with λ_m strong up to τ_m.

(iii) If the τ_n are successor cardinals in K, with $\tau_n = \lambda_n^{+K}$ for λ_n K-cardinals, then

$$\{\alpha \mid E^K_\alpha \textit{ is an extender with } crit(E^K_\alpha) < \lambda_m\}$$

is unbounded in τ_m for all sufficiently large τ_m.

Theorem $((\neg\exists IM(o(\kappa) = \kappa^{++}))$

If the τ_n *are inaccessible cardinals in K then for all sufficiently large m*

$$\{\alpha < \tau_m | o^K(\alpha) \geq \tau_k\} \text{ is stationary below } \tau_m.$$

Theorem $((\neg\exists IM(o(\kappa) = \kappa^{++}))$

If the τ_n are inaccessible cardinals in K then for all sufficiently large m $\{\alpha < \tau_m | o^K(\alpha) \geq \tau_k\}$ *is stationary below τ_m.*

Proof: For a contradiction suppose that in K and for an infinite set $Q \subseteq \omega$ we have for $m \in Q$ that we have cub $D_m \subseteq \tau_m$ and with no $\alpha \in D_m$ having $o^K(\alpha) \geq \tau_k$. Fix least such an Q and $\langle D_m | m \in Q \rangle$.
Fix an arbitrary $k > 0$. Let N be internally approachable of length τ_k. Let $\chi_N(m) =_{df} \sup(N \cap \tau_m)$.

(1) *For $k < m < \omega$, $\chi_N(m) \in \mathrm{Cof}_{\tau_k}$.*

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(1) *For $k<m<\omega$, $\chi_N(m)\in\mathrm{Cof}_{\tau_k}$.*

By ABSP for $\langle\tau_n\rangle_{n<\omega}$, let $n_0<\omega$ be such that the Goodness holds for N with respect to the set $X=_{df}\{\chi_N(m)\mid n_0<m<\omega\}$, thus for any $n\geq n_0$ $\chi_N(n)=\chi_{N[X\setminus\{\chi_N(n)\}]}(n)$.
For $r\in\omega$, we set $\pi_r:\mathrm{SH}^{\mathcal{K}}[X\setminus\chi_N(r)]\leftrightarrow K^r$ to be the transitive collapse map.

(2) *Claim*: $\exists p \in \omega \forall r, s \geq p$ *then* $K^r =^* K^s$.
Pf: By Dodd-Jensen and the wellfoundedness of the $\leq^*$ order. $\square$

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To facilitate our notation let $q =_{df} \min Q \setminus \max\{k+1, p\}$ and $D = \{\chi_N(n) \mid n \geq q\}$ and also let $x_0 < x_1 < \cdots$ enumerate D in ascending order. We set $\pi_D =_{df} \pi_q$, and $K_D =_{df} K^q$.

• We note that the ABSP implies, *a fortiori*, also that X is free for F, and then so is the above subset D. Moreover if we define $\bar{D}$ *via* $\pi_D^{-1}``\bar{D} = D$ then $\bar{D}$ is free for $\bar{F} =_{df} \pi_D^{-1}``F$ in K_D.

Let $\langle \bar{x}_I \rangle_{I<\omega}$ enumerate $\overline{D}$ with $\pi_D(\bar{x}_I) = x_I$.

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(3) *Claim*: *Let* $\alpha < x_0$, *then* $\sup N[\alpha \cup D \backslash \{x_0\}] \cap \tau_q = x_0$. $\square$

Define $\tilde{H} = \mathrm{SH}^{K_D}[\bar{x}_0 \cup \bar{D} \setminus \{\bar{x}_0\}]$. Let $\sigma : \bar{K} \leftrightarrow \tilde{H}$ again with $\bar{K}$ transitive. Let $\tau =_{df} \mathrm{crit}(\sigma)$ Then:

(4) (i) $\bar{K} \upharpoonright \tau = K_D \upharpoonright \tau$; (ii) $\tau = \bar{x}_0$; (iii) $K^{q+1} =^* K_D =^* \bar{K}$.

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(4) (i) $\bar{K} \restriction \tau = K_D \restriction \tau$; (ii) $\tau = \bar{x}_0$; (iii) $K^{q+1} =^* K_D =^* \bar{K}$.

(5) $\overline{K} \models$" $\bar{x}_0$ *is a regular cardinal*".
Proof: By (4)(ii). □

(6) $\bar{x}_0$ *is a* K_D*-singular.*
Proof: $\pi_D(\bar{x}_0) = x_0 \in X$ and has, by (1), V-cofinality τ_k, whilst at the same time the closed D_n is unbounded below it, where n is such that $x_0 \in (\tau_{n-1}, \tau_n)$; hence $x_0 \in D_n$ and has $o^K(x_0) < \tau_k$. By Cox's extension of the Covering Lemma we must have x_0 a K-singular. Hence (6) follows by elementarity. □ (6)

On coiterating $K^D =^* \overline{K}$ the only way to ensure the power sets of $\bar{x}_0$ to become equal, is for there to be an extender $E = E_\alpha$ in one of the models K^α of the coiteration with $(\text{crit}(E_\alpha)^+)^{K\alpha} \leq \bar{x}_0$ whilst $\alpha \geq \bar{x}_0$.

But that would imply in $\bar{K}$ that $\text{crit}(E)$ is strong up to the $\overline{K}$-inaccessible $\bar{x}_0$.

• Hence in K by elementarity we have $o^K(\pi_D \circ \sigma(\text{crit}(E_\alpha)) \geq \tau_q$.

But $q \in Q$ and so there is no such extender E_α. Hence the conjunction of (4), (5) and (6) is a contradiction, and our supposition that there was such a sequence of sets $\langle D_m \rangle_{m \in Q}$ was false. □ (i)