Games for chromatic numbers of analytic graphs

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Γ ... analytic edge relation, $\Gamma \subseteq [X]^2$

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$\mathrm{ch}(G) = \min\{\, \kappa \mid \forall L\colon X \to [V]^\kappa \; \exists c\colon X \to V \text{ proper } L\text{-selector} \,\}$

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$E\colon 2^{\omega} \leftrightarrow \{ g\colon 2^{<\omega} \to \omega, \forall t \in 2^{<\omega}\ g(t) > |t| \}$

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Let c be a L-selector. There is $x \in 2^{\omega}$ such that $c \restriction 2^{<\omega} = E(x)$.

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Let c be a L-selector. There is $x \in 2^{\omega}$ such that $c \restriction 2^{<\omega} = E(x)$.
$c(x) \in L(x) \quad \Rightarrow \quad$ there is $n \in \omega$ such that
$c(x) = E(x)(x \restriction n) = c(x \restriction n)$; c is not proper.

Let $\Delta_0 = \Delta$, where 2^ω is isomorphic to the Cantor space and $2^{<\omega}$ is a closed set of isolated points.

Theorem 0 (Adams, Zapletal)

Let $G = (X, \Gamma)$ be a graph, X a Polish space, Γ an analytic edge relation. Then one of the following holds

1. $\Delta_0 \hookrightarrow G$ continuously, or
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Game $\mathcal{G}_0(X, \Gamma)$

Player 1	...	$x_n \in X$, B_n basic open of diameter $<2^{-n}$	...
Player 2	...	$y_n \in X$	...

Player 1 wins if

- $x_n \neq x_m \quad$ if $n \neq m$,
- $y_n \notin \overline{B_{n+1}} \subset B_n \quad$ for every $n \in \omega$,
- $(\forall n \in \omega)\ z \Gamma x_n$, where $z = \bigcap \{ B_n \mid n \in \omega \}$.

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Let $G = (X, \Gamma)$ be a graph, X a Polish space, Γ an analytic edge relation.

1. The game $\mathcal{G}_0(X, \Gamma)$ is determined.
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Proof of 1.

Fix a continuous function $k\colon \omega^\omega \to X^{\omega+1}$ such that

$$k[\omega^\omega] = \{ \langle \langle x_n \restriction n \in \omega \rangle, z \rangle \text{ such that } (\forall n)\, x_n \Gamma z \}.$$

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Unraveled game $\mathcal{G}_0^{\mathrm{U}}(X, \Gamma)$

Player 1	...	$x_n \in X$, B_n basic open of diameter $<2^{-n}$, $t_n \in \omega^n$	...
Player 2	...	$y_n \in X$	...

Player 1 wins if he wins $\mathcal{G}_0(X, \Gamma)$, and additionally $k(\bigcup\{ t_n \mid n \in \omega \}) = \langle\langle x_n \mid n \in \omega\rangle, z\rangle$.

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If Player 1 has a winning strategy for $\mathcal{G}_0^{\mathrm{U}}(X, \Gamma)$, then he has a winning strategy for $\mathcal{G}_0(X, \Gamma)$.

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Proof of 2.
(Hand-waving)

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Proof of 3 ($\Leftarrow$).

If $\mu(G) \leq \omega$, then Δ_0 can not embed, and Player 1 does not have a winning strategy.

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Lemma

If $b \subseteq X$ is closed with respect to σ, and $z \in X \setminus b$, then

$$|\{ x \in b \mid x \Gamma z \}| < \omega.$$

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Proof.

By induction on $|b|$.

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$\{ x \prec y \mid x \Gamma y \} = \{ x \in b_\alpha \mid x \Gamma y \} \cup \{ x \prec_{\alpha+1} y \mid x \Gamma y \}$ for $y \in b_{\alpha+1}$

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Corollary (Adams, Zapletal)

Let $G = (X, \Gamma)$ be a graph. If Γ is analytic, then $\mathrm{ch}(G) \leq \omega$ if and only if $\mu(G) \leq \omega$.

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Corollary (Komjáth)

If $G = (\mathbb{R}^2, d(x, y) \in \mathbb{Q})$, then $\mu(G) = \omega$.

Neighborhood assignments for $G = (X, \Gamma)$

A function $O \colon X \to \tau(x)$ is a *neighborhood assignment* if $x \in O(x)$ for each $x \in X$.

A neighborhood assignment is *proper* if $x \Gamma y$ implies $x \notin O(y)$ or $y \notin O(x)$ for every $x, y \in X$.

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We say that G is *left-separated* if there is a well order $\prec$ on X such that $x \notin \overline{\{\, y \in X \mid y \prec x, y \Gamma x \,\}}$ for each $x \in X$.

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Proof $\Leftarrow$.

Let O be a proper neighborhood assignment.

For each $x \in X$ choose $n(x) \in \omega$ such that $B_{2^{-n(x)}}(x) \subseteq O(x)$.

Any well order $\prec$ satisfying $(n(x) < n(y)) \Rightarrow (x \prec y)$ works.

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Corollary

$$\chi(G) \leq \omega \quad \Leftarrow \quad G \text{ is left-separated} \quad \Leftarrow \quad \mu(G) \leq \omega \Leftrightarrow \mathrm{ch}(G) \leq \omega$$

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Game $\mathcal{G}_1(X, \Gamma)$

Player 1	$\dots$	$x_n \in X$	$\dots$
Player 2	$\dots$	$y_n \in X$	$\dots$

Player 1 wins if

- $\lim x_n = z \in X$,
- $(\forall n \in \omega)\ z \Gamma x_n$, and
- $(\forall n \in \omega)\ z \neq y_n$.

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Corollary (Schmerl)

If $G = (\mathbb{R}^3, d(x, y) \in \mathbb{Q})$, then G is left-separated. I.e. $\chi(G) = \omega$.