Was Ulam right?

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Bibliography

I'll be talking about a series of papers with Tanmay Inamdar:

[47] T.I. and A.R., *Was Ulam right? I: Basic theory and subnormal ideals*, Topology Appl., 323(C), 2023.

[53] T.I. and A.R., *Was Ulam right? II: Small width and general ideals*, submitted March 2022.

[59] T.I. and A.R., *Was Ulam right? III: Indecomposable ideals*, in preparation.

Some conventions

- κ denotes a regular uncountable cardinal;
- NS_κ denotes the ideal of nonstationary subsets of κ;
- $J^{\mathrm{bd}}[\kappa]$ denotes the ideal of bounded subsets of κ;
- An ideal J over κ is *uniform* if it covers $J^{\mathrm{bd}}[\kappa]$;
- $E^\kappa_\theta := \{\alpha < \kappa \mid \mathrm{cf}(\alpha) = \theta\}$, and $E^\kappa_{>\theta} := \{\alpha < \kappa \mid \mathrm{cf}(\alpha) > \theta\}$;
- For $S \subseteq \kappa$, $\mathrm{Tr}(S) := \{\beta \in E^\kappa_{>\omega} \mid S \cap \beta \text{ is stationary in } \beta\}$.

Introduction

Our starting point is a very basic and useful fact of set theory:

Theorem (Solovay, 1971)

Every stationary subset of κ *may be decomposed into* κ *many stationary sets.*

Solovay's decomposition theorem has numerous applications. Some variations of it have further potential applications.

To mention a few variations

Variation 1

Decomposing a stationary set into nonreflecting stationary subsets.

Application

By Shelah (1975), a nonreflecting stationary subset of E^{κ}_{θ} is sufficient to construct a graph of size κ of coloring number θ^+ all of whose small subgraphs have coloring number $< \theta^+$.

Fact

If $\square_\lambda$ holds, then $\lambda^+ = \biguplus_{\tau<\lambda} S_\tau$ such that, for every $\tau < \lambda$, S_τ is stationary and $\mathrm{Tr}(S_\tau) = \emptyset$.

Fact

If κ is weakly compact, then every stationary subset of κ reflects.
In fact, $\mathrm{Tr}(S)$ contains a regular cardinal for every stat. $S \subseteq \kappa$.

To mention a few variations

Variation 2

Decomposing a reflecting stationary set into stationary subsets that reflect simultaneously.

Application

By Cummings and Foreman (2010), for a singular cardinal λ, the existence of a decomposition $E^{\lambda^+}_{\mathrm{cf}(\lambda)} = \biguplus_{\tau<\mathrm{cf}(\lambda)} S_\tau$ such that $\bigcap_{\tau<\mathrm{cf}(\lambda)} \mathrm{Tr}(S_\tau)$ is stationary implies that any product of cardinals that admits a scale for λ, admits one which is not *very good*.

Compare this with the fact that if a product $\prod_{i<\mathrm{cf}(\lambda)} \lambda_i$ admits a *good scale*, then all scales in the same product are good.

Fact (with Levine [38])

Suppose that $\mu < \theta < \lambda$ are infinite cardinals with μ, θ regular. Then $E^{\lambda^+}_{\mu} = \biguplus_{\tau<\theta} S_\tau$ such that $\bigcap_{\tau<\theta} \mathrm{Tr}(S_\tau)$ is stationary.

To mention a few variations

Variation 3

Decomposing a fat set into two or more fat subsets.

Application

Consider the inner model $C(aa)$ for stationary-logic introduced by Kennedy, Magidor and Väänänen.
Ur Ya'ar (2022) used club shooting through pairwise disjoint fat sets to get a strictly decreasing sequence $\langle W^\gamma \mid \gamma < \delta \rangle$ of universes of set theory such that $W^{\gamma+1} = (W^\gamma)^{C(aa)}$.

Fact (with Brodsky [29])

If $\square(\kappa)$ holds, then any fat subset of κ admits a decomposition into κ-many fat sets.

Fact (Magidor, 1982)

Modulo a weakly compact cardinal, it is consistent that $\aleph_2$ cannot be decomposed into two fat sets.

To mention a few variations

Variation 4 (Woodin)

κ is ω-strongly measurable in HOD if there exists $\theta < \kappa$ such that:

1. $(2^\theta)^{\mathsf{HOD}} < \kappa$, and
2. There is no partition $\langle S_\tau \mid \tau < \theta \rangle$ of E^κ_ω into stationary sets such that $\langle S_\tau \mid \tau < \theta \rangle \in \mathsf{HOD}$.

Application

By Woodin's HOD dichotomy theorem (2010), if δ is an extendible cardinal, then one of the following hold:

1. No regular cardinal $\kappa \geq \delta$ is ω-strongly measurable in HOD;
2. Every regular cardinal $\kappa \geq \delta$ is ω-strongly measurable in HOD.

Fact (Ben-Neria and Hayut, 2023)

It is consistent from large cardinals that every successor of a regular cardinal is ω-strongly measurable in HOD.

Solovay through Ulam

Let us point out that for a successor cardinal $\kappa = \lambda^+$,
Solovay's theorem follows from the existence of an Ulam matrix.

Theorem (Ulam, 1930)

For every infinite cardinal λ, there exists a matrix $\langle A_{\eta,\tau} \mid \eta < \lambda, \tau < \lambda^+ \rangle$ such that:

1. *For every $\eta < \lambda$, $\langle A_{\eta,\tau} \mid \tau < \lambda^+ \rangle$ consists of pairwise disjoint subsets of λ^+;*
2. *For every $\tau < \lambda^+$, $\bigcup_{\eta<\lambda} A_{\eta,\tau}$ is co-bounded in λ^+.*

Corollary

If J is a uniform κ-complete ideal over a successor cardinal κ, then every $B \in J^+$ may be decomposed into κ many sets in J^+.

Solovay through Ulam (cont.)

We now give a proof of the corollary from the previous slide

Lemma

Suppose J is a uniform κ-complete ideal over $\kappa = \lambda^+$.
Then, for every $B \in J^+$, there exists some $\eta < \lambda$ such that the following set has size κ:

$$\{\tau < \kappa \mid A_{\eta,\tau} \cap B \in J^+\}.$$

Proof. Suppose that $B \in J^+$ is a counterexample. This means that for every $\eta < \lambda$, the set $T_\eta := \{\tau < \kappa \mid A_{\eta,\tau} \cap B \in J^+\}$ has size $\leq \lambda$, and hence $|\bigcup_{\eta<\lambda} T_\eta| \leq \lambda$. Pick $\tau \in \lambda^+ \setminus \bigcup_{\eta<\lambda} T_\eta$. As $\bigcup_{\eta<\lambda} A_{\eta,\tau}$ is co-bounded, it is in the dual of the uniform ideal J, so that $B \cap \bigcup_{\eta<\lambda} A_{\eta,\tau}$ is in J^+. Since J is λ^+-complete, there must exist some $\eta < \lambda$ such that $A_{\eta,\tau} \cap B \in J^+$. This means that $\tau \in \bigcup_{\eta<\lambda} T_\eta$, contradicting the choice of τ. □

A better Ulam matrix

Curiously, a theorem of Sierpiński from around the same time may be rephrased as follows:

Theorem (Sierpiński, 1932)

For every infinite cardinal λ such that $2^\lambda = \lambda^+$, there is an Ulam matrix $\langle A_{\eta,\tau} \mid \eta < \lambda, \tau < \lambda^+ \rangle$ such that for every uniform κ-complete ideal J over $\kappa = \lambda^+$, for every $B \in J^+$, there is an $\eta < \lambda$ for which the following set is equal to κ:

$$\{\tau < \kappa \mid A_{\eta,\tau} \cap B \in J^+\}.$$

This is more in the spirit of variation 4 (two-universe partition).

What about inaccessibles?

In 1969, in an attempt to prove what is now known as Solovay's decomposition theorem, Hajnal introduced triangular Ulam matrix, a notion that is also applicable to inaccessible cardinals.

$$
\begin{bmatrix}
\cdot & A_{0,1} & A_{0,2} & A_{0,3} & \cdots & A_{0,\eta+1} & \cdots & A_{0,\tau} & \cdots \\
 & & A_{1,2} & A_{1,3} & \cdots & A_{1,\eta+1} & & & \\
 & & & \ddots & \ddots & A_{2,\eta+1} & & \vdots & \\
 & & & & \ddots & \vdots & & & \\
 & & & & & A_{\eta,\eta+1} & \cdots & {\color{red}A_{\eta,\tau}} & \cdots \\
 & & & & & & \ddots & \vdots & \\
 & & & & & & & \ddots & \cdots \\
 & & & & & & & &
\end{bmatrix}
$$

Upper triangular matrix

What about inaccessibles?

In 1969, in an attempt to prove what is now known as Solovay's decomposition theorem, Hajnal introduced triangular Ulam matrix, a notion that is also applicable to inaccessible cardinals.

Theorem (Hajnal, 1969)

If κ admits a stationary set that does not reflect at regulars then there exists a matrix $\langle A_{\eta,\tau} \mid \eta < \tau < \kappa \rangle$ such that:

1. *For every $\eta < \kappa$, $\langle A_{\eta,\tau} \mid \eta < \tau < \kappa \rangle$ consists of pairwise disjoint subsets of κ;*
2. *For stationarily many $\tau < \kappa$, $\bigcup_{\eta < \tau} A_{\eta,\tau}$ is co-bounded in κ.*

Lemma

Suppose J is a uniform κ-complete ideal over a cardinal κ that admits a triangular Ulam matrix. Then, for every $B \in J^+$, there exists some $\eta < \kappa$ such that the following set has size κ:

$$\{\tau \in \kappa \setminus (\eta + 1) \mid A_{\eta,\tau} \cap B \in J^+\}.$$

What about inaccessibles?

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1. *For every $\eta < \kappa$, $\langle A_{\eta,\tau} \mid \eta < \tau < \kappa \rangle$ consists of pairwise disjoint subsets of κ;*
2. *For stationarily many $\tau < \kappa$, $\bigcup_{\eta<\tau} A_{\eta,\tau}$ is co-bounded in κ.*

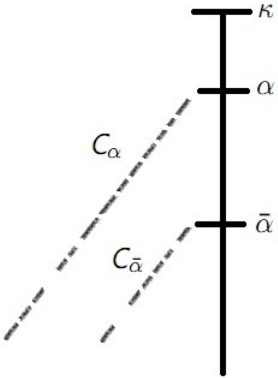
A dead end

Hajnal's extension of Ulam's approach cannot be pushed further to yield Solovay's theorem, as one can show that the existence of a triangular Ulam matrix over κ is in fact equivalent to the existence of a stationary subset of κ that does not reflect at regulars.

Dead end at successor cardinals, as well

Definition

For a subset $S \subseteq \kappa$, a C-sequence over S is $\vec{C} = \langle C_\alpha \mid \alpha \in S \rangle$ s.t. for every $\alpha \in S$, C_α is a closed subset of α with $\sup(C_\alpha) = \sup(\alpha)$.



Dead end at successor cardinals, as well

Definition

For a subset $S \subseteq \kappa$, a C-sequence over S is $\vec{C} = \langle C_\alpha \mid \alpha \in S \rangle$ s.t. for every $\alpha \in S$, C_α is a closed subset of α with $\sup(C_\alpha) = \sup(\alpha)$.

Fact

One can construct (in ZFC) a C-sequence $\langle C_\alpha \mid \alpha \in S \rangle$ and a function $f : \aleph_3 \to \aleph_1$ such that:

- *S is a stationary subset of $E^{\aleph_3}_{\aleph_1}$;*
- *for every $\alpha \in S$, $f \restriction C_\alpha$ is strictly increasing;*
- *for every club $D \subseteq \aleph_3$, there exists $\alpha \in S$ such that $C_\alpha \subseteq D$.*

The following is a countably-complete uniform proper ideal on $\aleph_3$:

$$J := \{X \subseteq \aleph_3 \mid \exists \text{club } D \subseteq \aleph_3 \forall \alpha \in S\, [\sup(C_\alpha \cap D \cap X) < \alpha]\}.$$

$\langle f^{-1}[i] \mid i < \aleph_1 \rangle$ is a sequence of sets in J whose union is not in J.

So, J is not $\aleph_2$-closed, let alone $\aleph_3$-closed, meaning that we cannot use an Ulam matrix to find $\aleph_3$-many pairwise disjoint J^+-sets.

Matrices are colorings in disguise

Note 1

A triangular Ulam matrix in particular gives a coloring $c : [\kappa]^2 \to \kappa$ such that for every uniform κ-complete ideal J over κ and every $B \in J^+$, there exists some $\eta < \kappa$ such that, for κ-many $\tau < \kappa$,

$$\{\beta \in B \mid c(\eta, \beta) = \tau\} \in J^+.$$

Note 2

A Sierpiński-type matrix in particular gives a coloring $c : [\kappa]^2 \to \kappa$ such that for every uniform κ-complete ideal J over κ and every $B \in J^+$, there exists some $\eta < \kappa$ such that, for every $\tau < \kappa$,

$$\{\beta \in B \mid c(\eta, \beta) = \tau\} \in J^+.$$

Matrices are colorings in disguise

Definition

For an ideal J over κ, $\text{unbounded}^+(J,\theta)$ asserts the existence of a coloring $c : [\kappa]^2 \to \theta$ such that for every $B \in J^+$, there is an $\eta < \kappa$ such that, for θ-many $\tau < \theta$,

$$\{\beta \in B \mid c(\eta,\beta) = \tau\} \in J^+.$$

Definition

For an ideal J over κ, $\text{onto}^+(J,\theta)$ asserts the existence of a coloring $c : [\kappa]^2 \to \theta$ such that for every $B \in J^+$, there is an $\eta < \kappa$ such that, for every $\tau < \theta$,

$$\{\beta \in B \mid c(\eta,\beta) = \tau\} \in J^+.$$

Matrices are colorings in disguise

Definition

For an ideal J over κ, unbounded$^+(J,\theta)$ asserts the existence of a coloring $c:[\kappa]^2\rightarrow\theta$ such that for every $B\in J^+$, there is an $\eta<\kappa$ such that, for θ-many $\tau<\theta$,

$$\{\beta\in B\mid c(\eta,\beta)=\tau\}\in J^+.$$

Remark

For a class $\mathcal{J}$ of ideals over κ, we may write unbounded$^+(\mathcal{J},\theta)$ to assert the existence of a coloring $c:[\kappa]^2\rightarrow\theta$ simultaneously witnessing unbounded$^+(J,\theta)$ for all $J\in\mathcal{J}$.

*Same goes to onto$^+$.

Matrices are colorings in disguise

Definition

For an ideal J over κ, unbounded$^+(J,\theta)$ asserts the existence of a coloring $c : [\kappa]^2 \to \theta$ such that for every $B \in J^+$, there is an $\eta < \kappa$ such that, for θ-many $\tau < \theta$,

$$\{\beta \in B \mid c(\eta,\beta) = \tau\} \in J^+.$$

Theorem (Larson, 2007)

onto$^+(\mathrm{NS}_{\aleph_1}, \aleph_1)$ *may consistently fail.*

By Ulam, unbounded$^+(\mathcal{J}, \aleph_1)$ holds for the class $\mathcal{J}$ of all countably-closed uniform ideals on $\aleph_1$.

Lemma ([53])

If unbounded$^+(\mathcal{J}, \theta)$ *holds for a given collection* $\mathcal{J}$ *of uniform ideals over* κ*, then* onto$^+(\mathcal{J}, \mu)$ *holds for all regular or finite* $\mu < \theta$*.*

Matrices are colorings in disguise

Definition

For an ideal J over κ, unbounded$^+(J,\theta)$ asserts the existence of a coloring $c : [\kappa]^2 \to \theta$ such that for every $B \in J^+$, there is an $\eta < \kappa$ such that, for θ-many $\tau < \theta$,

$$\{\beta \in B \mid c(\eta,\beta) = \tau\} \in J^+.$$

Solovay's decomposition theorem

For every stationary $S \subseteq \kappa$, there exists a stationary $S^- \subseteq S$ such that unbounded$^+(\mathrm{NS}_\kappa \restriction S^-, \kappa)$ *holds.*

Motivation: Encoding a partition modulo a coloring

Any coloring $c : [\kappa]^2 \to \theta$ naturally induces a list $\langle \vec{A}_\eta \mid \eta < \kappa \rangle$ of κ many partitions of κ into θ-many pieces.

▶ If c were to witness $\text{onto}^+(J, \theta)$, then for every $B \in J^+$, there exists $\eta < \kappa$ such that $\vec{A}_\eta$ shatters B into θ many positive sets. Here, **a code for a partition of** B is nothing but an ordinal $\eta < \kappa$. Thus, if S is a stationary subset (of some large enough regular cardinal) reflecting stationarily often at points of cofinality κ, then there there will be stationarily many $\delta \in \text{cof}(\kappa)$ for which the same η encodes a partition of $S \cap \delta$ into θ many pieces. Running a global-to-local argument, one could then decompose S into θ many sets that reflect simultaneously.

▶ If c were to witness $\text{unbounded}^+(J, \theta)$, then for every $B \in J^+$, there exists $\eta < \kappa$ such that θ-many cells of $\vec{A}_\eta$ have a positive intersection with B.

Here, **a code for a partition of** B is a pair $(\eta, T) \in \kappa \times [\theta]^\theta$.

One coloring to fit them all (Ulam was right)

Suppose $\theta \leq \kappa$ is infinite, and p is either unbounded$^+$ or onto$^+$.

Lemma ([53])

The following are equivalent:

- $\mathrm{p}(J^{\mathsf{bd}}[\kappa], \theta)$ *holds;*
- $\mathrm{p}(\mathcal{J}, \theta)$ *holds for the class* $\mathcal{J}$ *of all uniform* κ*-complete ideals over* κ*.*

Lemma ([47])

For every stationary $S \subseteq \kappa$*, the following are equivalent:*

- $\mathrm{p}(\mathsf{NS}_\kappa \restriction S, \theta)$ *holds;*
- $\mathrm{p}(\mathcal{J}, \theta)$ *holds for the class* $\mathcal{J}$ *of all uniform normal ideals over* S*.*

Ulam and Hajnal were wrong, after all

Definition (Todorcevic, 1987)

A C-sequence $\vec{C} = \langle C_\alpha \mid \alpha < \kappa \rangle$ is trivial if there exists a club $D \subseteq \kappa$ such that, for every $\epsilon < \kappa$, there is $\alpha < \kappa$ with $D \cap \epsilon \subseteq C_\alpha$.

Theorem ([53])

The following are equivalent:

- $\mathrm{unbounded}^+(J^{\mathrm{bd}}[\kappa], \kappa)$ *holds;*
- *There is a nontrivial C-sequence over κ.*

Ulam and Hajnal were wrong, after all

To compare

If κ does not admit an Ulam-Hajnal matrix, then every stationary subset of κ reflects at an inaccessible.

Theorem ([53])

The following are equivalent:

- $\mathrm{unbounded}^+(J^{\mathrm{bd}}[\kappa], \kappa)$ *holds;*
- *There is a nontrivial C-sequence over κ.*

Theorem (with Lambie-Hanson [35])

If all C-sequences over κ are trivial, then κ is weakly compact in L, and every sequence $\langle S_\tau \mid \tau < \kappa \rangle$ of stationary subsets of κ reflects diagonally at some inaccessible. In particular, κ is greatly Mahlo.

Ulam and Hajnal were wrong, after all

Let $S \subseteq \kappa$ be stationary.

Definition (with Brodsky [29])

A C-sequence $\vec{C} = \langle C_\alpha \mid \alpha \in S \rangle$ is not amenable if there exists a club $D \subseteq \kappa$ such that $\{\alpha \in S \mid D \cap \alpha \subseteq C_\alpha\}$ is stationary.

Theorem ([47])

The following are equivalent:

- $\text{unbounded}^+(\text{NS}_\kappa \restriction S, \kappa)$ *holds;*
- *There is an amenable C-sequence over S.*

It is clear that if S is ineffable, then no C-seq. over S is amenable.

Conjecture

If there is no amenable C-sequence over κ, then κ is ineffable in L.

No Ulam-type proof of Solovay's theorem

Recall: Solovay's decomposition theorem

For every stationary $S \subseteq \kappa$, there exists a stationary $S^- \subseteq S$ such that $\text{unbounded}^+(\mathsf{NS}_\kappa \restriction S^-, \kappa)$ *holds.*

We have seen that if κ is ineffable, then $\text{unbounded}^+(\mathsf{NS}_\kappa, \kappa)$ fails. In this case, the number of codes needed to represent all partitions of all stationary subsets of κ is as large as 2^κ.

Dealing with gaps

Theorem (Foreman, 1998)

Modulo a huge cardinal, there consistently exists a uniform proper countably-complete ideal J over $\aleph_2$ that is weakly $\aleph_2$-saturated (that is, $\aleph_2$ cannot be decomposed into $\aleph_2$-many J^+-sets.)

Recall (Club-guessing ideal)

Starting with a club-guessing sequence $\langle C_\alpha \mid \alpha \in S\rangle$ over a stationary subset $S \subseteq E^{\aleph_3}_{\aleph_1}$, the following is a countably-complete uniform proper ideal over $\aleph_3$:

$$J := \{X \subseteq \aleph_3 \mid \exists \text{club } D \subseteq \aleph_3 \forall \alpha \in S\,[\sup(C_\alpha \cap D \cap X) < \alpha]\}.$$

It is not necessarily closed under unions of length $\aleph_1$,
but it is closed under linear unions of length $\aleph_2$.

Dealing with gaps

Definition

An ideal J is θ-indecomposable if for every $\subseteq$-increasing sequence $\langle X_i \mid i < \theta\rangle$ of sets from J, the union $\bigcup_{i<\theta} X_i$ is in J.

Recall (Club-guessing ideal)

Starting with a club-guessing sequence $\langle C_\alpha \mid \alpha \in S\rangle$ over a stationary subset $S \subseteq E^{\aleph_3}_{\aleph_1}$, the following is a countably-complete, $\aleph_2$-indecomposable uniform proper ideal over $\aleph_3$:

$$J := \{X \subseteq \aleph_3 \mid \exists \text{club } D \subseteq \aleph_3 \forall \alpha \in S\, [\sup(C_\alpha \cap D \cap X) < \alpha]\}.$$

Dealing with gaps

Definition

An ideal J is *θ-indecomposable* if for every $\subseteq$-increasing sequence $\langle X_i \mid i < \theta\rangle$ of sets from J, the union $\bigcup_{i<\theta} X_i$ is in J.

Theorem ([59])

If there exists a nonreflecting stationary subset of E^{κ}_{θ}, then $\mathrm{unbounded}^{+}(\mathcal{J},\kappa)$ *holds for the class $\mathcal{J}$ of all countably-complete θ-indecomposable uniform ideals over κ.*

Recall (Club-guessing ideal)

Starting with a club-guessing sequence $\langle C_\alpha \mid \alpha \in S\rangle$ over a stationary subset $S \subseteq E^{\aleph_3}_{\aleph_1}$, the following is a countably-complete, $\aleph_2$-indecomposable uniform proper ideal over $\aleph_3$:

$$J := \{X \subseteq \aleph_3 \mid \exists \text{club } D \subseteq \aleph_3 \forall \alpha \in S\, [\sup(C_\alpha \cap D \cap X) < \alpha]\}.$$

Dealing with gaps

Definition

An ideal J is *θ-indecomposable* if for every $\subseteq$-increasing sequence $\langle X_i \mid i < \theta \rangle$ of sets from J, the union $\bigcup_{i<\theta} X_i$ is in J.

Corollary (Improved Ulam matrix for successors of regulars)

For every infinite regular cardinal λ, unbounded$^+(\mathcal{J}, \lambda^+)$ holds for the class $\mathcal{J}$ of all countably-complete λ-indecomposable uniform ideals over λ^+.

Recall (Club-guessing ideal)

Starting with a club-guessing sequence $\langle C_\alpha \mid \alpha \in S \rangle$ over a stationary subset $S \subseteq E^{\aleph_3}_{\aleph_1}$, the following is a countably-complete, $\aleph_2$-indecomposable uniform proper ideal over $\aleph_3$:

$$J := \{X \subseteq \aleph_3 \mid \exists \text{club } D \subseteq \aleph_3 \forall \alpha \in S\, [\sup(C_\alpha \cap D \cap X) < \alpha]\}.$$

So, unbounded$^+(J, \aleph_3)$ does hold! Therefore, onto$^+(J, \aleph_2)$ holds.

Dealing with gaps (cont.)

The following result is motivated by an open problem of Shelah from 1997. Its proof used generic ultrapowers, and it was not clear whether it can be obtained using an Ulam-type matrix.

Theorem (with Zhang [52])

Suppose λ is a regular uncountable cardinal and $\square(\lambda^+, <\lambda)$ holds. Then any λ-complete uniform ideal on λ^+ isn't weakly λ-saturated.

Note it does not contrast with Foreman's theorem ($\lambda = \aleph_1$).

Recall (Foreman, 1998)

Modulo a huge cardinal, there consistently exists a uniform proper countably-complete ideal over $\aleph_2$ that is weakly $\aleph_2$-saturated.

Dealing with gaps (cont.)

The following result is motivated by an open problem of Shelah from 1997. Its proof used generic ultrapowers, and it was not clear whether it can be obtained using an Ulam-type matrix.

Theorem (with Zhang [52])

Suppose λ is a regular uncountable cardinal and $\square(\lambda^+, {<}\lambda)$ holds. Then any λ-complete uniform ideal on λ^+ isn't weakly λ-saturated.

Recently, by incorporating walks on ordinals and a *wide club-guessing* theorem from [29], we obtained the following improvement:

Theorem ([59])

Suppose $\lambda < \kappa$ is a regular uncountable card. and $\square(\kappa, {<}\lambda)$ holds. Then $\mathrm{unbounded}^+(\mathcal{J}, \kappa)$ holds for the class $\mathcal{J}$ of all λ-complete uniform ideals on κ.

Connection to scales. The regular case

Suppose $\theta < \kappa$ are infinite and regular. Let:

- $\mathcal{J}$ denote the class of all θ^+-complete uniform ideals over κ,
- $\mathcal{J}'$ denote the class of all θ-indecompos. uniform ideals over κ.

Note that $\mathcal{J} \subseteq \mathcal{J}'$.

Theorem ([53],[59])

1. *If* $\mathfrak{b}_\theta = \kappa$, *then* unbounded$^+(\mathcal{J}, \theta)$ *holds;*
2. *If* $\mathfrak{d}_\theta = \kappa$, *then* onto$^+(\mathcal{J}, \theta)$ *holds;*
3. *If* $\mathfrak{b}_\theta = \mathfrak{d}_\theta = \kappa$, *then* onto$^+(\mathcal{J}', \theta)$ *holds.*

Fodor's question on stationary refinements

In the 1970's, Fodor asked whether for any $\vec{S} = \langle S_\tau \mid \tau < \kappa \rangle$ of stationary subsets of κ there exists a sequence of pairwise disjoint stationary sets $\langle S'_\tau \mid \tau < \kappa \rangle$ such that $S'_\tau \subseteq S_\tau$ for every $\tau < \kappa$.
Solovay's theorem is the special case in which $\vec{S}$ is constant.

The general answer to Fodor's question is negative. For instance, if $\mathrm{NS}_{\aleph_1}$ is $\aleph_1$-dense and $\vec{S}$ enumerates a dense subset of $\mathrm{NS}_{\aleph_1}$.
(A dense family in NS_{ω_1} cannot consist of pairwise disjoint sets.)

Fodor's question on stationary refinements

In the 1970's, Fodor asked whether for any $\vec{S} = \langle S_\tau \mid \tau < \kappa \rangle$ of stationary subsets of κ there exists a sequence of pairwise disjoint stationary sets $\langle S'_\tau \mid \tau < \kappa \rangle$ such that $S'_\tau \subseteq S_\tau$ for every $\tau < \kappa$.
Solovay's theorem is the special case in which $\vec{S}$ is constant.

In [47], we proved that for every $\theta < \kappa$ and every $\vec{S} = \langle S_\tau \mid \tau < \theta \rangle$ of stationary subsets of κ there exists a sequence of pairwise disjoint stationary sets $\langle S'_\tau \mid \tau < \theta \rangle$ such that $S'_\tau \subseteq S_\tau$ for every $\tau < \theta$.
But can we get it in a way that requires a small number of codes?

Fodor's question on stationary refinements

Definition

For an ideal J over κ, $\text{onto}^{++}(J,\theta)$ asserts the existence of a coloring $c:[\kappa]^2\rightarrow\theta$ such that for every sequence $\langle B_\tau \mid \tau<\theta\rangle$ of sets in J^+, there is an $\eta<\kappa$ such that, for every $\tau<\theta$,

$$\{\beta\in B_\tau \mid c(\eta,\beta)=\tau\}\in J^+.$$

Theorem ([47])

$\text{onto}^{++}(J,\kappa)$ *fails for every uniform proper ideal* J *over* κ.

Under Hajnal's hypothesis, we do get a universal refinement matrix:

Theorem ([53])

If there is a stationary subset of κ *that does not reflect at regulars, then* $\text{onto}^{++}(\mathcal{J},\theta)$ *holds for every* $\theta<\kappa$*, where* $\mathcal{J}$ *is the class of all* κ*-complete uniform ideals over* κ*.*

An open problem

A well-known longstanding open problem asks whether the successor of a singular cardinal λ may be Jónsson.

By work of Eisworth and Shelah, a negative answer holds provided that a certain club-guessing ideal J over λ^+ admits sequences of pairwise disjoint J^+-sets $\langle B_\tau \mid \tau < \theta \rangle$ for arbitrarily large $\theta < \lambda$.