Universality properties of graph homomorphism: one construction to prove them all

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Graph homomorphism

$\mathcal{G}$ = all undirected graphs. Notations like $\mathcal{G}^{=\kappa}$, $\mathcal{G}^{<\kappa}$, ... are used for the restrictions of $\mathcal{G}$ to graphs of size $= \kappa$, or $< \kappa$, ...

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A homomorphism $h \colon G_1 \to G_2$ is a map $h \colon V(G_1) \to V(G_2)$ such that

$$v \; E(G_1) \; w \Rightarrow h(v) \; E(G_2) \; h(w).$$

We write $G_1 \preccurlyeq G_2$ if there is such a homomorphism, $G_1 \approx G_2$ if $G_1 \preccurlyeq G_2 \preccurlyeq G_1$, and $G_1 \prec G_2$ if $G_1 \preccurlyeq G_2$ but $G_2 \not\preccurlyeq G_1$.

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Empirical fact

The homomorphism relation $\preccurlyeq$ on $\mathcal{G}$ is **universal**, i.e. it can code most of the other mathematical relations.

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Category theory: The category $\mathfrak{G}$ of graphs with arrows given by homomorphisms is *alg-universal*, i.e. every concrete category can be fully embedded into it.

[A full embedding is an injective (on objects) fully faithful functor.]

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Complexity/computer science: Undecidability of the graph homomorphism problem (for finite graphs), etc...

Model theory: The theory of graphs interprets all first-order theories in a countable language.

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We are looking for a very strong yet flexible coding procedure...

The coding procedure

Connected sum

The (quotient) structure of graph homomorphism is a lattice: the meet is the (Cartesian) product $\times$, while the join is the disjoint sum $\oplus$.

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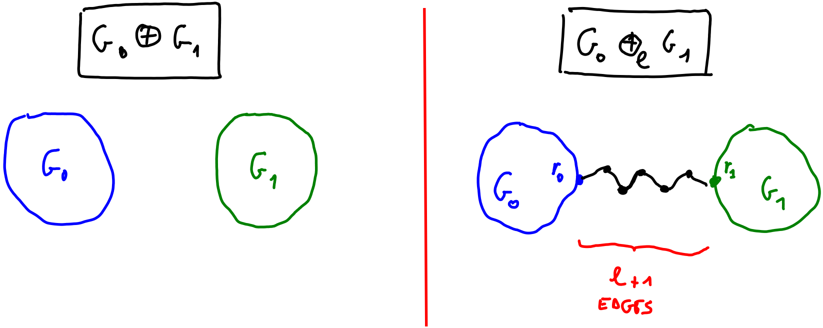
Definition (binary connected sum)

Given $\ell \in \omega$ and vertices $r_0 \in V(G_0)$ and $r_1 \in V(G_1)$, the (ℓ-)**connected sum** $G_0 \oplus_\ell G_1$ of G_0 and G_1 is the variant of $G_0 \oplus G_1$ where we add a brand new path of length $\ell + 1$ joining r_0 to r_1.



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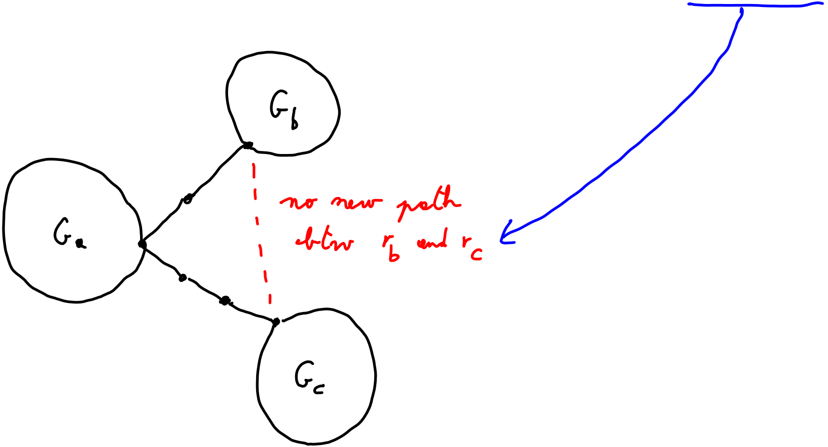
Definition (generalized connected sum)

$\{G_a\}_{a \in A}$ with $r_a \in V(G_a)$ $\qquad\qquad$ $\rho\colon [A]^2 \to \omega$

Then the (ρ-)**connected sum** $\bigoplus_\rho G_a$ of the G_a's is the graph obtained by adding to the disjoint sum of the G_a's a path of length $\rho(\{a, a'\}) + 1$ joining r_a to $r_{a'}$ for all $\{a, a'\} \in [A]^2$ such that $\rho(\{a, a'\}) \neq 0$.

Connected sum (picture)

$A = \{a, b, c\}$ $\rho(\{a,b\}) = 1$ $\rho(\{a,c\}) = 2$ $\rho(\{b,c\}) = 0$



Glossary from graph theory

(Finite) Path: Sequence of vertices $(v_i)_{i \leq n}$ with $v_i \; E(G) \; v_{i+1}$ for $i < n$

Closed path: Path with $v_n = v_0$

Cycle: Closed path with $v_i \neq v_j$ for $i \neq j$ (except for 1st and last element)

Odd closed path/cycle: Closed path/cycle with odd length

Bipartite: Graph without odd cycles (equiv.: 2-chromatic)

Rigid: Graph whose only endomorphism is the identity

Distance: $d(v,w) =$ length of the shortest path joining v and w

Diameter: $\mathrm{diam}(G) = \sup\{d(v,w) \mid v,w \in V(G)\}$

Connected: Graphs in which all pairs of vertices are joined by some path

Uniformly non-bipartite graphs

Definition

A graph G is **uniformly non-bipartite** if there is $n \in \omega$ such that every $v \in V(G)$ belongs to an odd closed path of length $\leq n$; the smallest such n, which is odd and at least 3, is denoted by n_G. We set

$$\delta(G) = \min\{n_G - 2, \operatorname{diam}(G)\}$$

if G is uniformly non-bipartite, and $\delta(G) = \infty$ otherwise.

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Remarks:

1. If G has finite diameter (e.g. G connected and finite), then this is equivalent to being non-bipartite.
2. Every non-bipartite finite graph is homomorphically equivalent to a uniformly non-bipartite graph.
3. For infinite graphs, this is more general then having finite diameter, yet there are non-bipartite graphs which are not homomorphically equivalent to a uniformly non-bipartite one.

Characteristic of graphs

Girth: $\gamma(G) =$ length of the shortest cycle

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Chromatic number: $\chi(G) =$ minimal cardinal I such that there is an I-**coloring** of G, i.e. a map $c \colon V(G) \to I$ such that $c(v) \neq c(w)$ whenever $v \; E(G) \; w$.

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Key fact

If $G \preccurlyeq H$ then $\chi(G) \leq \chi(H)$ and $\gamma_o(G) \geq \gamma_o(H)$.

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If $G \preccurlyeq H$ then $\chi(G) \leq \chi(H)$ and $\gamma_o(G) \geq \gamma_o(H)$.

Thus if $\chi(G) < \chi(H)$ and $\gamma_o(G) < \gamma_o(H)$, then G and H are $\preccurlyeq$-incomparable.

Connected sum (reprise)

Let I be a cardinal and $\mathcal{H} = \{H_i\}_{i \in I}$ be a family of connected uniformly non-bipartite pairwise $\preccurlyeq$-incomparable graphs with $r_i \in V(H_i)$.

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Let A be a nonempty set, and let $i \colon A \to I$ and $\rho \colon [A]^2 \to \omega$ be arbitrary functions. Then

$$\bigoplus_{\rho,i}^{A,\mathcal{H}} G_a$$

is the ρ-connected sum of $\{G_a\}_{a \in A}$, where each G_a is a (distinct) isomorphic copy of $H_{i(a)}$.

Key fact

Theorem (M.-Scamperti)

Suppose that ρ takes large enough values compared to the $\delta(H_i)$'s.

(E.g. $\rho(\{a, a'\}) \geq \delta(\mathcal{H})$ whenever $\rho(\{a, a'\}) \neq 0$, if $(\star_2)$ holds.)

Then for every $j \in I$ and every homomorphism $h \colon H_j \to \bigoplus_{\rho,i}^{A,\mathcal{H}} G_a$ there is a unique $\bar{a} \in A$ such that $h(H_j) \cap G_{\bar{a}} \neq \emptyset$, and $i(\bar{a}) = j$.

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Remark: It is known that there are enough $\preccurlyeq$-incomparable rigid graphs to start with...

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A variation of the interpretation procedure from model theory, shows that wlog we can assume that the structure is an I-**colored graph** $C = (G, c)$, i.e. a graph G together with a singled-out I-coloring $c \colon V(G) \to I$.

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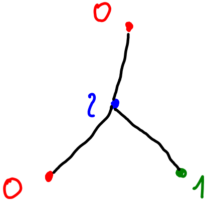
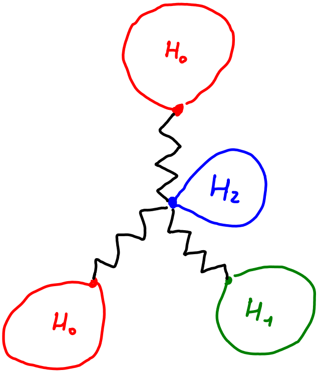
How to code a first-order structure (picture)

$$\mathcal{H}=\{H_0,H_1,H_2\}$$





$$C=(G,c)$$

$$c: \underset{\overset{\parallel}{A}}{V(G)} \to \{0,1,2\}$$

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Interesting fact: Once the full machinery is developed, for most of the applications it is enough to deal with the case $I = 3$.

Applications

Graphs with prescribed characteristics

Theorem (Erdős)

For every finite chromatic number n there are (finite) graphs G with $\chi(G) = n$ and arbitrarily high girth.

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- If G is uncountably chromatic, then either $\gamma(G) = \gamma_o(G) = 3$, or else $\gamma(G) = 4 < \gamma_o(G)$.

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- If G is uncountably chromatic, then either $\gamma(G) = \gamma_o(G) = 3$, or else $\gamma(G) = 4 < \gamma_o(G)$.
- For every $n > \aleph_0$, there are graphs G (of size n) with $\chi(G) = n$ and arbitrarily high odd girth.

Graphs with prescribed characteristics

More generally, fix a cardinal n and $m, k \in \omega \cup \{\infty\}$: we want to deal with the class $\mathcal{G}_{n,m,k}$ of graphs G with $\chi(G) = n$, $\gamma(G) = m$, and $\gamma_o(G) = k$.

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The answer was known for n finite and $m = k = 3$, or $n = 3$ and arbitrarily high $m = k$: indeed, the corresponding categories are alg-universal (Pultr-Trnková).

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Theorem (M.-Scamperti)

Let (n, m, k) be an acceptable triple. Then either $\mathcal{G}_{n,m,k}$ is trivial (if $n = 2$ or $n = m = k = 3$), or else the homomorphism category associated to it is alg-universal.

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4. If $n < \aleph_0$, then $\mathcal{G}^{<\aleph_0}_{n,m,k}$ is order-theoretically universal.

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The proof relies on the following results.

Lemma (M.-Scamperti)

Let $G = \bigoplus_{\rho,i}^{A,\mathcal{H}} G_a$ with $\rho(\{a,a'\}) \geq \delta(\mathcal{H})$ when different from 0. Then

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Let κ be a cardinal such that $\kappa^{<\kappa} = \kappa$. For all (n, m, k) as above and $n \leq \kappa$, the relation $\preccurlyeq \restriction \mathcal{G}^{=\kappa}_{n,m,k}$ is complete (in fact, invariantly universal) for κ-analytic quasi-orders with respect to κ-Borel reducibility.

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  It remains undecidable even in weaker models of computations, like the so-called "logic with atoms" (this answers a question by Lasota).
- The chromatic number can be replaced by the **circular chromatic number** χ_c, the **fractional chromatic number** χ_f, and so on.

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One of the best known results concerning the structure of $\mathrm{Forb}_{\mathcal{F}}$ was:

Theorem (Nešetřil-Rödl)

Every $\mathrm{Forb}_{\mathcal{F}}$, if not trivial, contains an infinite set of $\preccurlyeq$-incomparable (finite) graphs.

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   (All remaining cases.)

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Theorem (Hajnal-Komjáth)

Assuming the Continuum Hypothesis CH, there is a triangle-free graph (of size $2^{\aleph_0}$) having chromatic number $\aleph_1$ and omitting the graph $H_{\omega,\omega+2}$.

Uncountably chromatic graphs

It took 30 years to even just prove that the hypothesis CH can be removed.

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In particular, there are $2^{2^{\aleph_0}}$-many pairwise $\preccurlyeq$-incomparable (even rigid, connected) graphs of this kind.

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Moreover, we can control girth and odd girth (besides obvious limitations).

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This generalizes (with caution) to infinite graphs as well.

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1. Either all $\preccurlyeq$-antichains in $\mathcal{C}$ have size ≤ 2, or else there is a $\preccurlyeq$-preserving embedding of the full category of graphs into $\mathcal{C}$.
2. If there are three $\preccurlyeq$-incomparable graphs in $\mathcal{C}^{<\aleph_0}$, then such embedding respect cardinalities.

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1. Either all $\preccurlyeq$-antichains in $\mathcal{C}$ have size ≤ 2, or else there is a $\preccurlyeq$-preserving embedding of the full category of graphs into $\mathcal{C}$.
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Thank you for your attention!